Quantum Dissipation and Control through Feedback Networks

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Quantum Control Engineering:
Mathematical Principles and Applications
Isaac Newton Institute for Mathematical Sciences Cambridge, UK
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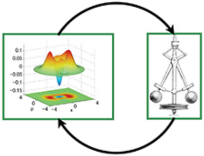


Outline

- Classical Feedback Networks and Control
- The SLH Formalism
- Quantum Feedback Networks

We will investigate linear systems. Here a input x_{in} is transformed into an output x_{out} via a linear relation

$$x_{\mathrm{out}} = G\, x_{\mathrm{in}}$$

which is depicted below as a block diagram.

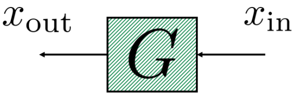
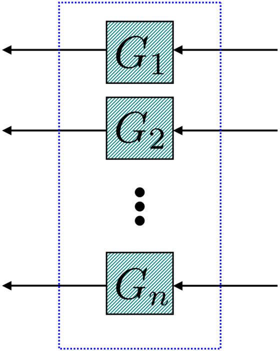


Figure : The linear relation $x_{\mathrm{out}} = G\, x_{\mathrm{in}}$.

The main characteristic of the input-output equation is that it is linear:

$$G(c_1\, x_1 + c_2\, x_2) = c_1\, Gx_1 + c_2\, Gx_2,$$

for all scalars c_1 and c_2.



$$\begin{bmatrix} x_1^{\mathrm{out}} \\ \vdots \\ x_n^{\mathrm{out}} \end{bmatrix} = \begin{bmatrix} G_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G_n \end{bmatrix} \begin{bmatrix} x_1^{\mathrm{in}} \\ \vdots \\ x_n^{\mathrm{in}} \end{bmatrix},$$

We may feed the output of G in as input to H:

$$\begin{aligned} x_{\rm out} &= G\, x_{\rm in} \\ y_{\rm out} &= H\, y_{\rm in}. \end{aligned}$$

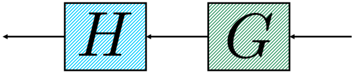


Figure : Cascaded systems.

We set $x_{\rm out} = y_{\rm in}$, then

$$y_{\rm out} = H\, G\, x_{\rm in}.$$

Figure : Cascaded systems.

In order to build more sophisticated networks, we need to be able to connect the inputs and outputs of various systems in a more flexible manner.

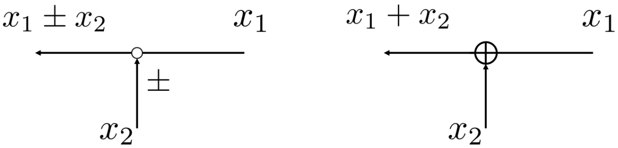


Figure : Summing points.

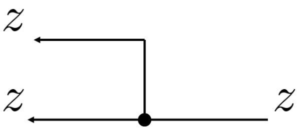


Figure : A take-off point.

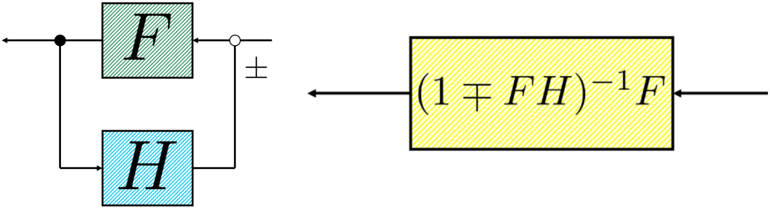


Figure : A simple feedback arrangement, and its equivalent system.

The output then satisfies the algebraic identity

$$x_{\text{out}} = F(x_{\text{in}} \pm H x_{\text{out}})$$

which can be rearranged to give

$$x_{\text{out}} = (1 \mp FH)^{-1} F \, x_{\text{in}}.$$

The term $1 \mp FH$ must be invertible and in such cases the network is said to be **well-posed**.

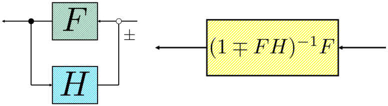


Figure : A simple feedback arrangement, and its equivalent system.

The result is that we obtain the **effective transfer function**

$$G = (1 \mp FH)^{-1} F \equiv F(1 \mp HF)^{-1}.$$

We may think of G as the "*renormalized*" transfer function! Formally we have the geometric series expansion:

$$\begin{aligned} G &\equiv F \mp FHF + FHFHF \mp FHFHFHF + \cdots \\ &= F \sum_{n=0}^{\infty} (\mp HF)^n = \sum_{n=0}^{\infty} (\mp FH)^n F. \end{aligned}$$

Let us consider a more generic model as sketched in below.

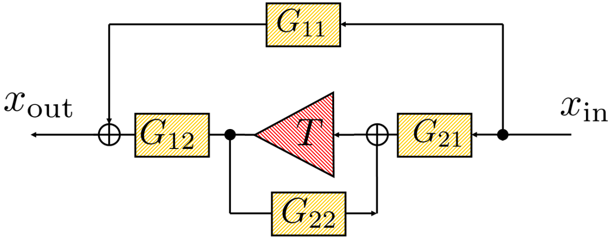
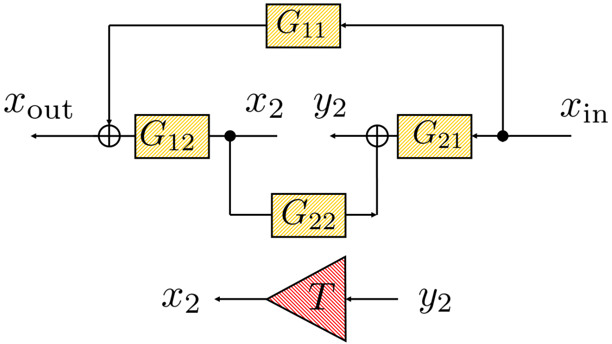


Figure : General feedback arrangement.

Here the feedback component is through the component G_{22}.

It is convenient to split up the network into two subnetworks:



Setting $x_1 = x_{\text{in}}$ and $y_1 = x_{\text{out}}$ we see that the first of these subnetworks is in fact just a linear system with the block matrix gain

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

while the second network implies the supplementary relation

$$x_2 = T\, y_2 .$$

We eliminate the feedback loop through a "*renormalization*" as before. We rearrange the input-output relation $x_2 = T(z + G_{22} x_2)$ to get $x_2 = R\, z$, where

$$R = (1 - T G_{22})^{-1} T.$$

This eliminates the feedback loop!

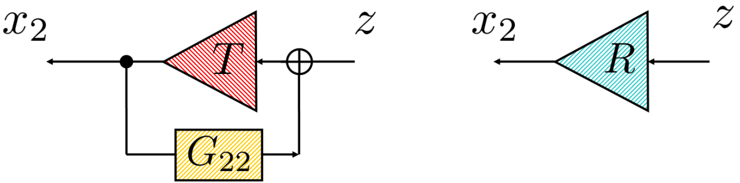
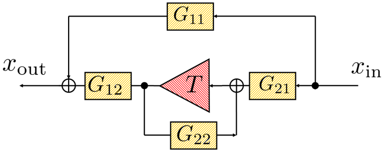


Figure : Feedback loop "renormalization": $R = (1 - T G_{22})^{-1} T$.

This leads to the effective input-output model with transfer function

$$\boxed{G(T) = G_{11} + G_{12}(1 - TG_{22})^{-1} TG_{21}.}$$

The transfer function G is a fractional linear transformation of the gain T. The network is well-posed whenever the inverse $(1 - TG_{22})^{-1}$ exists.



We note that the direct feedthrough is $G_0 = G_{11}$, while the asymptotic gain is a Schur complement

$$\boxed{G_\infty = G_{11} - G_{12} G_{22}^{-1} G_{21},}$$

(whenever G_{22} is invertible).

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

or

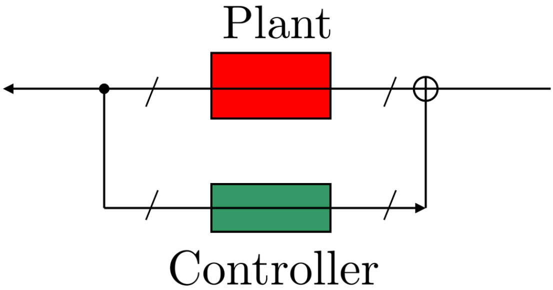
$$\begin{bmatrix}\dot{x} \\ y\end{bmatrix} = \mathbf{V}\begin{bmatrix}x \\ u\end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix}A & B \\ C & D\end{bmatrix}.$$

The function $T(s) = D + C(sI - A)^{-1}B$ is called the *transfer function*.

We define the **parallel sum**

$$\begin{aligned}\mathbf{V}_1 \boxplus \mathbf{V}_2 &= \begin{bmatrix}A_1 & B_1 \\ C_1 & D_1\end{bmatrix} \boxplus \begin{bmatrix}A_2 & B_2 \\ C_2 & D_2\end{bmatrix} \\ &\triangleq \begin{bmatrix}A_1 + A_2 & [B_1, B_2] \\ \begin{bmatrix}C_1 \\ C_2\end{bmatrix} & \begin{bmatrix}D_1 & 0 \\ 0 & D_2\end{bmatrix}\end{bmatrix}.\end{aligned}$$

The classical plant-controller setup:



We have position and momentum observables satisfying the CCR $[q, p] = 2i$, and introduce

$$b \triangleq \frac{1}{2}(q + ip), \quad b^\dagger \triangleq \frac{1}{2}(q - ip)$$

so that

$$\boxed{\left[b, b^\dagger\right] = 1,}$$

and define the number operator

$$N \triangleq b^\dagger b.$$

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The Schrödinger repesentation on $L^2(\mathbb{R}, dx)$:

$$(q\phi)(x) = x\phi(x), \quad (p\phi)(x) \equiv -2i\frac{\partial}{\partial x}\phi(x).$$

A complete orthonormal basis is given by the Hermite polynomials

$$\langle x|n\rangle = \frac{1}{\sqrt{n!}}H_n(x)\sqrt{\rho(x)}, \quad (n = 0, 1, 2, \cdots)$$

with $\rho(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. We have

$$\begin{aligned} b^*\,|n\rangle &= \sqrt{n+1}|n+1\rangle, \quad b\,|n\rangle = \begin{cases} \sqrt{n}|n-1\rangle, & n \geq 1; \\ 0, & n = 0. \end{cases} \\ N\,|n\rangle &= n|n\rangle. \end{aligned}$$

Definition

For $\beta \in \mathbb{C}$ we define the exponential vector (Bargmann state) as

$$|e(\beta)\rangle = e^{\beta b^{\dagger}}|0\rangle = \sum_{n} \frac{\beta^{n}}{\sqrt{n!}}|n\rangle.$$

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The situation easily generalizes to n modes:

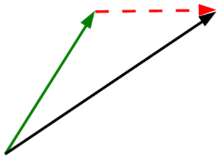
$$\left[b_j, b_k^*\right] = \delta_{jk}$$

and set $|e(\vec{\beta})\rangle = \otimes_k |e(\beta_k)\rangle$ on $L^2(\mathbb{R}^n, dx^n) \cong \otimes^n L^2(\mathbb{R}, dx)$.

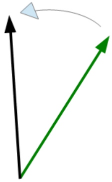
We consider three basic operations on exponential vectors

$$\begin{array}{ll} \text{phase:} & |e(\vec{\alpha})\rangle \mapsto e^{-i\theta}|e(\vec{\alpha})\rangle \\ \text{translation:} & |e(\vec{\alpha})\rangle \mapsto |e(\vec{\alpha}+\vec{\beta})\rangle \\ \text{rotation:} & |e(\vec{\alpha})\rangle \mapsto |e(R\vec{\alpha})\rangle \end{array}$$

where $\vec{\beta} \in \mathbb{C}^n$ and $R \in U(n)$.



Translation



Rotation

Displacement

Introduce the Weyl unitary $D(\vec{\beta}) \triangleq e^{\overrightarrow{b}^{\dagger}.\vec{\beta}-\overrightarrow{\beta^{*}}.\vec{b}}$, then

$$\begin{aligned} D(\vec{\beta})\,|e(\vec{\alpha})\rangle &= e^{-\overrightarrow{\beta^{*}}.\vec{\alpha}-\frac{1}{2}|\vec{\beta}|^{2}}\,|e(\vec{\alpha}+\vec{\beta})\rangle, \\ D(\vec{\beta})^{\dagger}\,\vec{b}\,D(\vec{\beta}) &= \vec{b}+\vec{\beta}. \end{aligned}$$

But

$$D(\vec{\beta}_2)\,D(\vec{\beta}_1) = e^{-i\mathrm{Im}\{\vec{\beta}_2^{*}.\vec{\beta}_1\}}D(\vec{\beta}_1+\vec{\beta}_2).$$

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Rotation

Let $R = e^{-i\Omega} \in U(n)$, so that $\Omega^{\dagger} = \Omega$. Set

$$\Gamma(R) \triangleq e^{-i\,\overrightarrow{b}^{\dagger}\,\Omega\,\vec{b}},$$

then

$$\begin{aligned} \Gamma(R)\,|e(\vec{\alpha})\rangle &= |e(R\vec{\alpha})\rangle, \\ \Gamma(R)^{\dagger}\,\vec{b}\,\Gamma(R) &= R\,\vec{b}. \end{aligned}$$

And $\Gamma(R_2)\,\Gamma(R_1) = \Gamma(R_2R_1)$.

The Euclidean group $U(n) \rtimes \mathbb{C}^n$

The group product is

$$(R_2, \vec{\beta}_2) \star (R_1, \vec{\beta}_1) = (R_2 R_1, \vec{\beta}_2 + R_2 \vec{\beta}_1)$$

i.e., we have

$$\vec{\alpha} \mapsto R_1 \vec{\alpha} + \vec{\beta}_1 \mapsto R_2 (R_1 \vec{\alpha} + \vec{\beta}_1) + \vec{\beta}_2$$

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More generally define the family of unitary operators $W(R, \vec{\beta}, \theta)$ by

$$W(R, \vec{\beta}, \theta) \, |e(\vec{\alpha})\rangle \triangleq e^{-i\theta} e^{-\overrightarrow{\beta^*}.R\,\vec{\alpha} - \frac{1}{2}|\vec{\beta}|^2} |e(R\vec{\alpha} + \vec{\beta})\rangle.$$

Then

$$W(R_2, \vec{\beta}_2, \theta_2) \, W(R_1, \vec{\beta}_1, \theta_1) \, = W(R_2 R_1, \vec{\beta}_2 + R_2 \vec{\beta}_1, \theta_1 + \theta_2 + \mathrm{Im} \left\{ \vec{\beta}_2^*.R_2 \beta_1 \right\})$$

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Canonical Transformation and their Composition

$$W(R, \vec{\beta}, \theta)^\dagger \, \vec{b} \, W(R, \vec{\beta}, \theta) = R \, \vec{b} + \vec{\beta}$$

with the cascade rule

$$(R_2, \vec{\beta}_2, \theta_2) \lhd (R_1, \vec{\beta}_1, \theta_1) = (R_2 R_1, \vec{\beta}_2 + R_2 \vec{\beta}_1, \theta_1 + \theta_2 + \mathrm{Im} \left\{ \vec{\beta}_2^* . R_2 \vec{\beta}_1 \right\}).$$

Steps to the "SLH" Formalism

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1. Go from $\mathbb{C}^n$ to $\mathbb{C}^n \otimes L^2(\mathbb{R}, dt)$, that is replace modes b_k with quantum white noises $b_k(t)$,

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2. Introduce a system with Hilbert space $\mathfrak{h}$ and replace
$$\begin{aligned} R &\hookrightarrow S = \left[S_{jk} \right] \text{ with } S^\dagger S = SS^\dagger = I; \\ \vec{\beta} &\hookrightarrow \vec{L} = \left[L_j \right]; \\ \theta &\hookrightarrow H = H^\dagger; \end{aligned}$$
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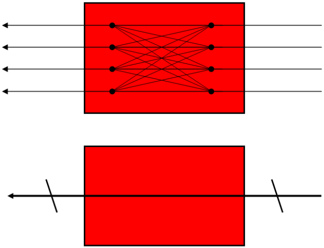
The "series product" is
$$(S_2, \vec{L}_2, H_2) \triangleleft (S_1, \vec{L}_1, H_1) = (S_2 S_1, \vec{L}_2 + S_2 \vec{L}_1, H_1 + H_2 + \mathrm{Im} \left\{ \vec{L}_2^\dagger S_2 \vec{L}_1 \right\})$$

The SLH formalism for quantum Markov models deals with the category of models

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}, L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}, H$$

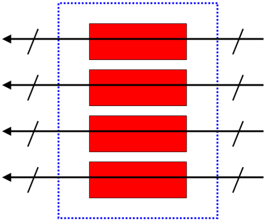


These may be assimilated into the *model matrix*

$$\mathbf{V} = \begin{bmatrix} -\frac{1}{2}L^*L - iH & -L^*S \\ L & S \end{bmatrix}$$

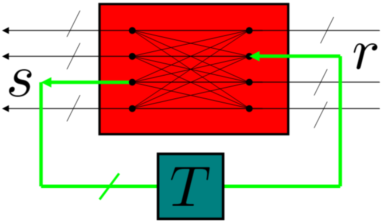
$$= \begin{bmatrix} -\frac{1}{2}\sum_j L_j^* L_j - iH & -\sum_j L_j^* S_{j1} & \cdots & -\sum_j L_j^* S_{jm} \\ L_1 & S_{11} & \cdots & S_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ L_n & S_{n1} & \cdots & S_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{V}_{00} & \mathbf{V}_{01} & \cdots & \mathbf{V}_{0n} \\ \mathbf{V}_{10} & \mathbf{V}_{11} & \cdots & \mathbf{V}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{V}_{n0} & \mathbf{V}_{n1} & \cdots & \mathbf{V}_{nn} \end{bmatrix}.$$



The **parallel sum** of models is defined by

$$
\mathbf{V}_1 \boxplus \mathbf{V}_2 = \left[\begin{array}{ccc} \sum_{k=1,2}(-\frac{1}{2}L^{(k)*}L^{(k)} - iH^{(k)}) & -L^{(1)*}S^{(1)} & -L^{(2)*}S^{(2)} \\ L^{(1)} & S^{(1)} & 0 \\ L^{(2)} & 0 & S^{(2)} \end{array} \right]
$$



The feedback reduction is

$$\left[\mathscr{F}_{(r,s)}(\mathbf{V},T)\right]_{\alpha\beta}=\mathbf{V}_{\alpha\beta}-\mathbf{V}_{\alpha r}T(1-\mathbf{V}_{rs}T)^{-1}\mathbf{V}_{s\beta}$$

for $\alpha\neq r$ and $\beta\neq s$.

Proposition

As long as the network is well-posed we will have that the FLT $\mathscr{F}_{(r,s)}(\mathbf{V}, T)$ *is again the model matrix of an SLH system.* [a]

[a] *It is a unexpected triumph over non-commutativity! It is essentially due to C.L. Siegel and was discovered in the 1940's.* N. Young An Introduction to Hilbert Space, Cambridge Math. Textbook.

Proposition

As long as the network is well-posed we will have that the FLT $\mathscr{F}_{(r,s)}(\mathbf{V}, T)$ is again the model matrix of an SLH system. [a]

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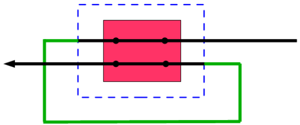
The series product is a special case



$$\mathscr{F}_{(2,1)}\left(\mathbf{V} \equiv \left(\begin{bmatrix} S^{(1)} & 0 \\ 0 & S^{(2)} \end{bmatrix}, \begin{bmatrix} L^{(1)} \\ L^{(2)} \end{bmatrix}, H\right), T = I\right)$$