Modulation and water waves – Part 2

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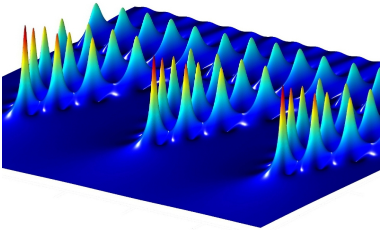


Outline

Examples

- defocussing NLS to KdV
- KP equation via modulation
- Whitham modulation theory from the viewpoint of modulation!
- Modulation in 3(2+1) dimensions + “dimension breaking”
- modulation of elliptic PDEs: “KdV planforms”
- multi-dimensional conservation laws: e.g. modulation equations for interfacial waves

Classical NLS dark solitary waves

Defocussing NLS,

$$\mathrm{i}\Psi_t + \Psi_{xx} - |\Psi|^2\Psi = 0\,,$$

has an exact DSW solution

$$\Psi(x,t) = \sqrt{2}\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\big(k + \mathrm{i}\beta\tanh(\beta x)\big)\,,$$

when $\beta^2 = \frac{1}{2}(\omega - 3k^2) > 0$

Bi-asymptotic to a periodic state with a phase shift

$$\Psi(x,t) \to \sqrt{2}\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\big(k \pm \mathrm{i}\beta\big) \quad \text{as } x \to \pm\infty\,.$$

Gen solitary waves: Wahlén (on Tuesday)

DSWs in Euler: Milewski, Vanden-broeck, Wang (2013) PRSLA

Emergence of DSWs

$$\Psi(x,t) = \sqrt{2}\, e^{i(kx-\omega t)}\big(k + i\beta \tanh(\beta x)\big)$$

Periodic state at infinity exists for

$$k^2 < \omega\,,$$

but the DSW exists only for

$$k^2 < \frac{1}{3}\omega\,.$$

Kivshar's theory: emergence of the DSW at $k^2 = \omega/3$ governed by a KdV equation.

Y. Kivshar [1990] Phys. Rev. A **42** 1757–1761

KIVSHAR's theory

Kivshar proposes a solution of NLS the form

$$\Psi(x,t) = (\Psi_0 + \varepsilon^2 q(X,T,\varepsilon))\mathrm{e}^{\mathrm{i}(kx+\varepsilon\phi(X,T,\varepsilon))}\,,$$

where

$$X = \varepsilon x \quad \text{and} \quad T = \varepsilon^3 t\,,$$

and shows that $q(X,T,0)$ satisfies (to leading order) a KdV equation

$$c_0 q_T + c_1 q q_X + c_2 q_{XXX} = 0\,.$$

NB. Works, but better to modulate wavenumber, not amplitude.

$(\mathscr{A}, \mathscr{B})$ calculation for NLS DSWs

Defocussing NLS (setting $\omega = 1$),

$$\mathrm{i}\boldsymbol{\Psi}_t + \boldsymbol{\Psi}_{xx} + \boldsymbol{\Psi} - |\boldsymbol{\Psi}|^2\boldsymbol{\Psi} = 0\,.$$

Basic spatially periodic state

$$\boldsymbol{\Psi}(x) = A_0 \mathrm{e}^{\mathrm{i}\theta}\,, \quad \theta = kx + \theta_0\,, \quad A_0^2 + k^2 = 1\,.$$

The components of the conservation law are

$$A = \tfrac{1}{2}|\boldsymbol{\Psi}|^2 \quad \text{and} \quad B = \mathrm{Im}(\overline{\boldsymbol{\Psi}}\boldsymbol{\Psi}_x)\,,$$

and so $\mathscr{A}(k) = \frac{1}{2}(1-k^2)$ and $\mathscr{B}(k) = k(1-k^2)$. Differentiating

$$\mathscr{A}'(k) = -2k\,, \quad \mathscr{B}'(k) = 1 - 3k^2\,, \quad \mathscr{B}''(k) = -6k\,.$$

and $\mathscr{K} = -\frac{1}{2}$. Hence the emergent KdV is

$$-2kq_T - 6kqq_X - \tfrac{1}{2}q_{XXX} \quad \text{(same as Kivshar)}\,.$$

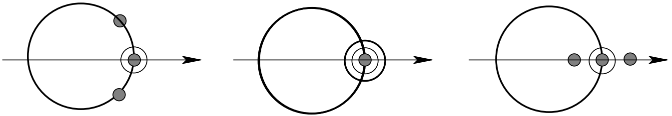
Periodic states $\rightarrow$ DSWs via KdV

Given a 1+1 PDE generated by a Lagrangian ...

- ... and a branch of periodic solutions, or periodic travelling waves, e.g. branch of Stokes waves
- modulate the periodic solution; look for $\mathscr{B}'(k) = 0$
- in this case $q(X, T, \varepsilon)$ is the perturbation wave number,
- with appropriate conditions q satisfies KdV
- $\mathscr{A}(k)$ is the wave action evaluated on periodic state,
- $\mathscr{B}(k)$ is the wave action flux
- $\mathscr{K}$ is a Krein index of the periodic orbit

Finding bifurcation points for KdV

Parameterize using $(\mathscr{A}, \mathscr{B})$ and look for points where $\mathscr{B}_k = 0$, or look for eigenvalues meeting at zero (or Floquet multipliers at $+1$).



This picture occurs along branches of Stokes waves

Vanden-Broeck (1983), Zufuria (1987– finite depth)

Baesens & MacKay (1992) – infinite depth

TJB & Donaldson (2006) – finite depth with mean flow

NB. in this view, c is fixed and wavelength varies along a branch.

NLS $\rightarrow$ shallow water equations: Madelung transformation*

Madelung transformation: in NLS let

$$\mathbf{\Psi} = \rho e^{i\phi} \,,$$

and define

$$gh = 2\rho^2 \quad \text{and} \quad u = 2\phi_x \,,$$

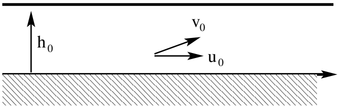
then h and u satisfy

$$\begin{aligned} h_t + uh_x + hu_x &= 0 \\ u_t + uu_x + gh_x &= h^{-1} h_{xxx} + \cdots \,. \end{aligned}$$

Emergence of DSWs in NLS $\Leftrightarrow$ KdV emerges in shallow water

$\star$ pointed out by Roger Grimshaw

Modulation of uniform flows and KP



$$Q_1 = h_0 u_0 \quad \text{and} \quad Q_2 = h_0 v_0$$
$$R = g h_0 + \tfrac{1}{2}\left(u_0^2 + v_0^2\right).$$

Consider the generalized criticality conditions

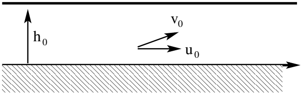
$$\left.\frac{\partial Q_1}{\partial u_0}\right|_{R\ \text{fixed}} = 0 \quad \text{and} \quad \left.\frac{\partial Q_2}{\partial u_0}\right|_{R\ \text{fixed}} = 0\,,$$

and the coefficients

$$\left.\frac{\partial^2 Q_1}{\partial u_0^2}\right|_{R\ \text{fixed}} \quad \text{and} \quad \left.\frac{\partial Q_2}{\partial v_0}\right|_{R\ \text{fixed}}.$$

Modulation of uniform flows and KP

Modulate the uniform flow:



$$u_0 \mapsto u_0 + \varepsilon^2 q(X,Y,T,\varepsilon)\,,\quad X=\varepsilon x\,,\ Y=\varepsilon^2 x\,,\ T=\varepsilon^3 t\,.$$

Claim: q satisfies the KP equation with

$$\left(2\mathcal{M}_u \frac{\partial q}{\partial T} + \frac{\partial^2 Q_1}{\partial u_0^2}\bigg|_{R\ \text{fixed}} q\frac{\partial q}{\partial X} + \mathscr{K}\frac{\partial^3 q}{\partial X^3}\right)_X = \frac{\partial Q_2}{\partial v_0}\bigg|_{R\ \text{fixed}} q_{YY}\,.$$

It is a modulation of the mass CLAW for the full WW problem

$$M_t + (Q_1)_x + (Q_2)_y = 0\,,$$

where $\mathcal{M}$ is M evaluated on the uniform flow.

Symmetry, modulation, KdV

- Lagrangian $\int\int\int L\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t$ (e.g. Luke's Lagr)
- One-parameter symmetry group $\Rightarrow$ CLAW $A_t + B_x + C_y = 0$
- basic state: $\widehat{Z}(\theta, k, \ell)$ with $\theta = kx + \ell y + \theta_0$
- modulate: $Z(x, y, t) = \widehat{Z}(\theta + \varepsilon\psi, k + \varepsilon^2 q, \ell) + \varepsilon^3 W(\theta, X, Y, T)$
- Evaluate CLAW on basic state: $\mathscr{A}(k, \ell)$, $\mathscr{B}(k, \ell)$ and $\mathscr{C}(k, \ell)$
- If $\mathscr{B}_k = 0$ and $\mathscr{C}_k = 0$ (gen criticality) then KP emerges

$$(2\mathscr{A}_k q_T + \mathscr{B}_{kk} q q_X + \mathscr{K} q_{XXX})_X = \mathscr{C}_\ell q_{YY}\,.$$

- $\mathscr{K}$: Krein signature, sign of momentum flux, dispersion relation, ... (same as KdV)

Substitute and expand in powers of ε

- ε^2: $q = \phi_X$
- ε^3: $\mathscr{B}_k = 0$
- ε^4: $\mathscr{C}_k = 0$
- ε^5: $\mathscr{C}_\ell \neq 0$, $\mathscr{A}_k \neq 0$ and $\mathscr{B}_{kk} \neq 0$. Then

$$2\mathscr{A}_k q_T + \mathscr{B}_{kk} q q_X + \mathscr{K} q_{XXX} + \mathscr{C}_\ell \phi_{YY} = 0 \,.$$

or, by differentiating with respect to X,

$$(2\mathscr{A}_k q_T + \mathscr{B}_{kk} q q_X + \mathscr{K} q_{XXX})_X + \mathscr{C}_\ell q_{YY} = 0 \,.$$

NLS → KP-I

NLS 2+1 model for moderate values of surface tension in deep water

$$\mathrm{i}\Psi_t + \Psi_{xx} + \Psi_{yy} + \Psi - |\Psi|^2\Psi = 0\,,$$

$$\text{RE:} \quad \Psi = \Psi_0 \mathrm{e}^{\mathrm{i}\theta},\ \theta = kx + \ell y + \theta_0$$

$$\text{CL:} \quad A_t + B_x + C_y = 0$$

$$\vdots$$

$$-2kq_T - 6kqq_X - \tfrac{1}{2}q_{XXX} + (1-k^2)\phi_{YY} = 0\,,$$

Necessary conditions: $k^2 = \frac{1}{3}$ and $\ell = 0$

KP is a model for water waves on infinite depth!

Ansatz

Given a basic state $\widehat{Z}(\theta, k)$ the modulation is an ansatz

$$Z(x,t) = \widehat{Z}(\theta + \varepsilon^a \phi, k + \varepsilon^b q, \omega + \varepsilon^c \Omega) + \varepsilon^d W(\theta, X, T, \varepsilon)\,,$$

with $X = \varepsilon^\alpha x$ and $T = \varepsilon^\beta t$.
Whitham modulation theory

$$Z(x,t) = \widehat{Z}(\theta + \phi, k + \varepsilon q, \omega + \varepsilon \Omega) + \varepsilon^2 W(\theta, X, T, \varepsilon)\,,$$

with $X = \varepsilon x$ and $T = \varepsilon t$. $\mathscr{A}_\omega \mathscr{B}_k - \mathscr{A}_k \mathscr{B}_\omega \neq 0$.
Emergence of the Boussinesq equation

$$Z(x,t) = \widehat{Z}(\theta + \varepsilon \phi, k + \varepsilon^2 q, \omega) + \varepsilon^3 W(\theta, X, T, \varepsilon)\,,$$

with $X = \varepsilon x$ and $T = \varepsilon^2 t$. Also need $\mathscr{B}'(k) = 0$.

Whitham modulation theory

Consider a PDE generated by a Lagrangian $\mathscr{L}(Z)$, and suppose there is a steady (RE) solution of the form

$$Z(x,t) = \widehat{Z}(\theta,\omega,k)\,, \quad \theta = kx - \omega t + \theta_0\,,$$

associated with a conservation law

$$A_t + B_x = 0\,.$$

In Whitham theory the basic state is a periodic TW and the conservation law is conservation of wave action. Modulate

$$Z(x,t) = \widehat{Z}(\theta+\phi,\omega+\varepsilon\Omega,k+\varepsilon q) + \varepsilon^2 W(\theta,X,T,\varepsilon)\,,$$

with $\phi(X,T,\varepsilon)$, $q(X,T,\varepsilon)$, $\Omega(X,T,\varepsilon)$ and scaling

$$T = \varepsilon t \quad \text{and} \quad X = \varepsilon x\,.$$

Need $\mathscr{A}_\omega\mathscr{B}_k - \mathscr{A}_k\mathscr{B}_\omega \neq 0$. Modulation generates the Whitham modulation equations which are quasilinear hyperbolic or elliptic – no dispersion.

Generalizing Whitham modulation theory

Consider a PDE generated by a Lagrangian $\mathscr{L}(Z)$. The ansatz

$$Z(x,t) = \widehat{Z}(\theta+\phi, k+\varepsilon q, \omega+\varepsilon\Omega) + \varepsilon^2 W(\theta, X, T, \varepsilon)\,,$$

with $T = \varepsilon t$ and $X = \varepsilon x$ gives

$$\mathscr{A}_\omega \Omega_T + (\mathscr{A}_k + \mathscr{B}_\omega) q_T + \mathscr{B}_k q_X = 0\,, \quad \mathscr{A}_\omega \mathscr{B}_k - \mathscr{A}_k \mathscr{B}_\omega \neq 0\,.$$

The ansatz

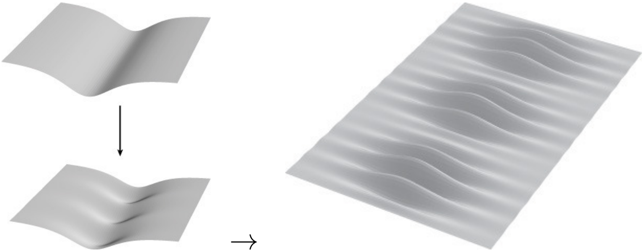
$$Z(x,t) = \widehat{Z}(\theta+\varepsilon\phi, k+\varepsilon^2 q, \omega+\varepsilon^4\Omega) + \varepsilon^3 W(\theta, X, T, \varepsilon)\,,$$

with $T = \varepsilon^3 t$ and $X = \varepsilon x$ and $\mathscr{B}_k = 0$ gives

$$(\mathscr{A}_k + \mathscr{B}_\omega) q_T + \mathscr{B}_{kk} q q_X + \mathscr{K} q_{XXX} = 0\,.$$

Dimension breaking via modulation

The *dimension breaking of patterns*



- bif from localised 1D pattern to periodic in y
- Kirchgässner, Hărăguş, Groves, B & Derks, etc

NB. Figure credit: Mark Groves

Converse dimension breaking problem

- look for mechanisms for multi-dimensional patterns
- begin with a 1D spatially periodic pattern
- advantage: spatially periodic patterns can be *modulated*
- derive modulation equation(s)

Modulation: from rolls $\to$ planforms

- Swift Hohenberg equation

$$u_t + \Delta^2 u + p\Delta u + f(u) = 0\,,$$

for (real scalar-valued) $u(x, y, t)$ and some given $f(u)$, and

- real Ginzburg-Landau equation

$$\boldsymbol{\Psi}_t = \Delta\boldsymbol{\Psi} + \boldsymbol{\Psi} - |\boldsymbol{\Psi}|^2\boldsymbol{\Psi}\,,$$

for (complex scalar-valued) $\boldsymbol{\Psi}(x, y, t)$ where

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,.$$

Rolls

Swift-Hohenberg equation

$$u_t + \Delta^2 u + p\Delta u + f(u) = 0\,.$$

1D steady SH: $u_{xxxx} - Pu_{xx} + u - u^2 = 0$, has a family of roll solutions, $\widehat{u}(\theta, k)$, with a sequence of points where $\mathscr{B}'(k) = 0$.

Real Ginzburg Landau equation

$$\mathbf{\Psi}_t = \Delta\mathbf{\Psi} + \mathbf{\Psi} - |\mathbf{\Psi}|^2\mathbf{\Psi}\,,$$

1D steady real GL: rolls: $\mathbf{\Psi}(x) = \Psi_0 \mathrm{e}^{\mathrm{i}\theta}$, $\theta = kx + \theta_0$.
Can compute everything explicitly.

Modulate rolls

$$u(x,y,t) = \widehat{u}(\theta + \varepsilon\phi, k + \varepsilon^2 q, \ell) + \varepsilon^3 w(\theta, X, Y, T, \varepsilon)\,,$$

with

$$X = \varepsilon x\,, \quad Y = \varepsilon^2 y \quad \text{and} \quad T = \varepsilon^4 t\,.$$

Substitute into equation and expand. Fifth order terms give

$$q_T = \mathscr{B}_\ell q_{YY} + (\tfrac{1}{2}\mathscr{A}_{kk} q^2 + \mathscr{K}\, q_{XX})_{XX}\,,$$

a gradient-like KP equation (replace q_T by q_{XT} to get KP). The steady part is a Boussinesq equation (completely integrable).

The condition $\mathscr{B}'(k) = 0$ is a generalization of the Ekhaus instability threshold in Swift-Hohenberg.

TJB PhysicaD (2014)

Gradient in time elliptic in space Boussinesq

$$q_T = \mathscr{B}_\ell q_{YY} + (\tfrac{1}{2}\mathscr{A}_{kk} q^2 + \mathscr{K} q_{XX})_{XX} \,.$$

The PDE is gradient in time (can show that the right-hand side is the derivative of a functional), and is well posed if $\mathscr{B}_\ell > 0$ and $\mathscr{K} < 0$ in which case the steady part is an elliptic Boussinesq equation (the “bad” Boussinesq equation).
Related to KP and (2+1) Boussinesq.

$$q_{XT} = (aqq_X + bq_{XXX})_X + cq_{YY} \,,$$

and

$$q_{TT} = (aqq_X + bq_{XXX})_X + cq_{YY} \,,$$

(both of which can be derived via modulation).

Explicit Multi-bump solutions

Scale the steady part

$$v_{YY} - v_{XX} - 3(v^2)_{XX} - v_{XXXX} = 0\,. \qquad (1)$$

This is the canonical form for the “bad” Boussinesq equation and it has an exact solution DAI ET AL.[1],

$$v(X,Y) = v_0 + 2\phi_{XX}\,, \quad \phi(X,Y) = \ln F(X,Y)\,, \qquad (2)$$

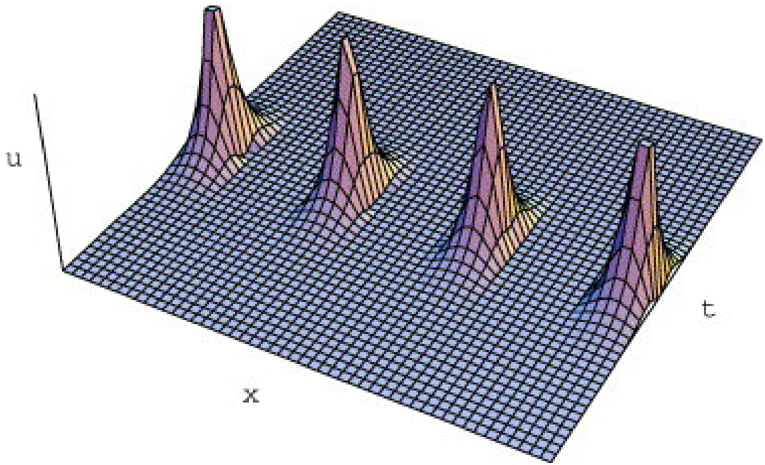
with

$$F(X,Y) = 1 + 2a\cos(pX)\mathrm{e}^{\Omega Y + \theta_0} + b\mathrm{e}^{2\Omega Y + 2\theta_0}\,,$$

The solution is periodic in the $X-$direction and localized in the $Y-$direction. Are these solutions stable?

1. Dai et al, Chaos Solitons & Fractals **26** (2005)

Multi-bump solutions of Dai et al



N-pulse web-like multi-dimensional patterns

For

$$w_{tt} - w_{xx} - 6(w^2)_{xx} - w_{xxxx} = 0\,,$$

Hirota[2] shows that there are explicit multi-pulse $N-$web solutions for every natural number N

$$w = \frac{\partial^2}{\partial x^2}\log f(x,t)\,,$$

with explicit expressions for f for every N in terms of sums of exponentials. Replacing time in Hirota by space, they become steady web-like multi-pulse patterns. Biondini et al[3] call these Boussinesq solutions *web solitons*.

2. Hirota, J Math Phys **14** (1973)

3. Biondini et al, Stud Appl Math. **122** (2009)

Canonical form for the steady part

$$\mathbf{C}(Z_{xx} + Z_{yy}) + \nabla F(Z) = 0\,, \quad Z \in \mathbb{R}^n\ (n \geq 2)\,,$$

where **C** is a symmetric (not necessarily positive definite) matrix, and $F(Z)$ is a specified function.

Both real Ginzburg Landau and Swift-Hohenberg fit into this framework.

Time can be brought in by adding a term $\mathbf{M}Z_t$ where **M** is a positive semi-definite matrix

$$\mathbf{M}Z_t + \mathbf{C}(Z_{xx} + Z_{yy}) + \nabla F(Z) = 0\,, \quad Z \in \mathbb{R}^n\ (n \geq 2)\,,$$

Conservation law on loops

$$\mathbf{C}(Z_{xx} + Z_{yy}) + \nabla F(Z) = 0\,, \quad Z \in \mathbb{R}^n\ (n \geq 2)\,,$$

Suppose there exists a family of solutions, parameterised by μ,

$$Z(x, y, \mu) \quad \text{with} \quad Z(x, y, \mu + 2\pi) = Z(x, y, \mu)\,.$$

A “loop” of solutions (which may or may not exist). Then this family satisfies the conservation law

$$A_x + B_y = 0\,,$$

with

$$A = \overline{\langle \mathbf{C}Z_\mu, Z_x \rangle} \quad \text{and} \quad B = \overline{\langle \mathbf{C}Z_\mu, Z_y \rangle}\,.$$

(A steady version of “wave action” conservation.)

$q-$Boussinesq – an example

For real GL, $\mathbf{\Psi}_t = \Delta\mathbf{\Psi} + \mathbf{\Psi} - |\mathbf{\Psi}|^2\mathbf{\Psi}$, take the simple family of steady rolls $\mathbf{\Psi} = \Psi_0 e^{i\theta}$ with $\theta = kx + \ell y + \theta_0$ giving $\Psi_0^2 + k^2 + \ell^2 = 1$.

Evaluate the loop conservation law on this two-parameter family

$$\mathscr{A}(k,\ell) = k\Psi_0^2 = k(1-k^2-\ell^2)\,, \quad \mathscr{B}(k,\ell) = \ell\Psi_0^2 = \ell(1-k^2-\ell^2)\,.$$

The conditions $\mathscr{A}_k = 0$ and $\mathscr{B}_k = 0$ are both satisfied with $\ell = 0$. For the second derivative

$$\mathscr{A}_{kk}(k,0) = -6k\,.$$

The coefficient $\mathscr{K} = -\frac{1}{2}$, giving the parabolic PDE

$$q_T = \Psi_0^2 q_{YY} - \left(3kq^2 + \tfrac{1}{2}q_{XX}\right)_{XX}$$

Multi-dimensional conservation laws

- N-component (abelian) symmetry group
- N CLAWS $(A_j)_t + (B_j)_x = 0$, $j = 1, \ldots, N$
- RE: $\widehat{Z}(\theta_1, \ldots, \theta_N, k_1, \ldots, k_N)$, $\theta_j = k_j x + \theta_j^o$.
- criticality: define

$$\mathrm{D}\mathbf{B}(\mathbf{k}) = \begin{bmatrix} \frac{\partial \mathscr{B}_1}{\partial k_1} & \cdots & \frac{\partial \mathscr{B}_1}{\partial k_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathscr{B}_N}{\partial k_1} & \cdots & \frac{\partial \mathscr{B}_N}{\partial k_N} \end{bmatrix} .$$

- non-degenerate $\Rightarrow$ multi-CWA;
  critical: $\det[\mathrm{D}\mathbf{B}(\mathbf{k})] = 0$ with simple zero eigenvalue

$$[\mathrm{D}\mathbf{B}(\mathbf{k})]\,\mathbf{n} = 0\,.$$

Multi-dimensional conservation laws

$$\mathrm{D}\mathbf{B}(\mathbf{k}) = \begin{bmatrix} \frac{\partial \mathscr{B}_1}{\partial k_1} & \cdots & \frac{\partial \mathscr{B}_1}{\partial k_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathscr{B}_N}{\partial k_1} & \cdots & \frac{\partial \mathscr{B}_N}{\partial k_N} \end{bmatrix} .$$

Singularities of this matrix generate modulation equations.

- simple zero eigenvalue $\Rightarrow$ generates KdV
- double zero eigenvalue $\Rightarrow$ generates coupled KdV
- simple zero + secondary singularity (Thom-Boardman classification) $\Rightarrow$ KdV with cubic nonlinearity

Strategy: modulate and see what happens ....

Multi-dimensional CLAWS at criticality

Modulate ansatz at criticality

$$Z(x,t) = \widehat{Z}(\theta_1+\varepsilon\phi_1,\ldots,\theta_N+\varepsilon\phi_N,k_1+\varepsilon^2 q_1,\ldots,k_N+\varepsilon^2 q_N)+\varepsilon^3 W\,.$$

Linear theory: 0 has geom mult N and algebraic mult $2N+2$

Modulation equation for $\hat{q}(X,T,\varepsilon)$ defined by $\mathbf{q} = \hat{q}\mathbf{n} + \widehat{\mathbf{p}}$

$$a\hat{q}_T + \kappa\hat{q}\hat{q}_X + \mathscr{K}\hat{q}_{XXX} = 0\,.$$

where

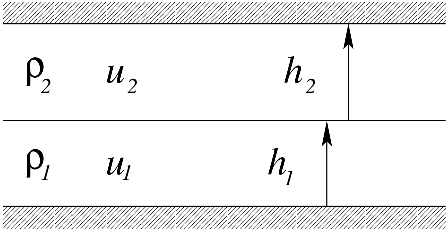
$$a = \mathbf{n}^T\left[\mathrm{D}\mathbf{A}(\mathbf{k}) + \mathrm{D}\mathbf{A}(\mathbf{k})^T\right]\mathbf{n}\,,$$

and κ is the "intrinsic second derivative" of the map $\mathbf{B}(\mathbf{k})$ in the direction $\mathbf{n}$

$$\kappa = \frac{d^2}{ds^2}\langle\mathbf{n},\mathbf{B}(\mathbf{k}+s\mathbf{n})\rangle\Big|_{s=0}\,.$$

$\mathscr{K}$ is as before (Krein signature, dispersion relation, excess Momentum flux, top of the Jordan chain)

Example: two fluids with a rigid lid



Uniform flow: h_1, h_2 and u_1, u_2, with $h_1 + h_2 = d$.
Three conserved quantities (R, Q_1, Q_2) where R is the Bernoulli energy and Q_j are the mass flux in each layer.

Emergence of KdV at criticality from the uniform flow

Criticality of two layer flow

Let $\mathbf{c} = (h_1, u_1, u_2)$ and $\mathbf{B}(\mathbf{c}) = (R(\mathbf{c}), Q_1(\mathbf{c}), Q_2(\mathbf{c}))$, with

$$\begin{aligned} R(\mathbf{c}) &= \tfrac{1}{2}\rho_1 u_1^2 - \tfrac{1}{2}\rho_2 u_2^2 + (\rho_1 - \rho_2) g h_1 \\ Q_1(\mathbf{c}) &= \rho_1 h_1 u_1 \\ Q_2(\mathbf{c}) &= \rho_2 (d - h_1) u_2 \,. \end{aligned}$$

The derivative of the mapping is

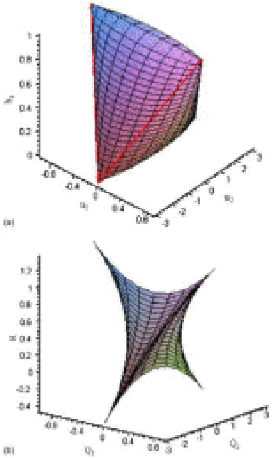
$$\mathrm{D}\mathbf{B}(\mathbf{c}) = \begin{bmatrix} (\rho_1 - \rho_2) g & \rho_1 u_1 & -\rho_2 u_2 \\ \rho_1 u_1 & \rho_1 h_1 & 0 \\ -\rho_2 u_2 & 0 & \rho_2 (d - h_1) \end{bmatrix} ,$$

and there exists $\mathbf{n}$ satisfying $\mathrm{D}\mathbf{B}(\mathbf{c})\mathbf{n} = 0$ when $f(\mathbf{c}) = 0$ where

$$f(\mathbf{c}) = \det(\mathrm{D}\mathbf{B}(\mathbf{c})) = \rho_1 \rho_2 (\rho_1 - \rho_2) g h_1 (d - h_1) \left[1 - F_1^2 - r F_2^2 \right] ,$$

where $F_j^2 = u_j^2 / ((1 - r) g h_j)$ and $r = \rho_2 / \rho_1$.

Mapping $(h_1, u_1, u_2) \mapsto (R, Q_1, Q_2)$



Criticality and $\mathrm{d}f(\mathbf{c}) \cdot \mathbf{n}$

Now

$$f(\mathbf{c}) := \det[\mathrm{D}\mathbf{B}(\mathbf{c})] = \mathrm{C}\left[(1-r) - \frac{u_1^2}{gh_1} - r\frac{u_2^2}{gh_2}\right], \quad \mathrm{C} = \rho_1^2\rho_2 g h_1 h_2 .$$

The criticality surface in (h_1, u_1, u_2) space is defined by $f^{-1}(0)$ and a vector $\mathbf{v}$ is tangent to this surface if $\mathrm{d}f \cdot \mathbf{v} = 0$. Now,

$$\nabla f = \frac{\mathrm{C}}{g}\left(\frac{u_1^2}{h_1^2} - \frac{ru_2^2}{h_2^2}, -\frac{2u_1}{h_1}, -\frac{2ru_2}{h_2}\right),$$

and so

$$\langle \nabla f, \mathbf{n}\rangle = \frac{3\mathrm{C}}{\rho_1 g}\left(\rho_1\frac{u_1^2}{h_1^2} - \rho_2\frac{u_2^2}{h_2^2}\right) = \frac{\mathrm{C}}{\rho_1 g}\frac{d^2}{ds^2}\langle \mathbf{n}, \mathbf{B}(\mathbf{c}+s\mathbf{n})\rangle\Big|_{s=0} .$$

Criticality surfaces: TJB & Donaldson [2007] Phys. Fluids

Remarks

- Start with a Lagrangian with symmetry
- Construct families of relative equilibria
- Given a relative equilibrium it depends on a phase (or phases) and each phase has a parameter (speed, wavenumber, uniform flow): $\widehat{Z}(\theta, k)$
- Modulation ansatz

$$Z(x,t) = \widehat{Z}(\theta + \varepsilon^a \phi, k + \varepsilon^b q) + \varepsilon^c W(\theta, X, T, \varepsilon)\,,$$

  with $X = \varepsilon^\alpha x$ and $T = \varepsilon^\beta t$.
- Substitute and determine modulation equation via a sequence of solvability conditions
- generalizations: higher space dimension, higher dimensional group, non-abelian groups