Interconnected Balance Sheets, Market Liquidity, and the Amplification Effects in a Financial System

Nan Chen
The Chinese University of Hong Kong

Workshop on Systemic Risk: Models and Mechanism
Issac Newton Institute for Mathematical Sciences
Cambridge University
Aug. 28, 2014
Systemic Risk

- Systemic risk: the risk of collapse of an entire financial system due to its ill structure.
  - The risk starts from the failure of single entity or a cluster of entities
  - Interconnectedness among entities propagates such idiosyncratic risk to the whole system
- The objective of our work is to understand the amplification effect of the financial system, especially the contribution of two major factors to the systemic risk:
  - interconnected balance sheets of the banking system (Network Effect)
  - market liquidity (Liquidity Effect)
This Paper

▶ Method:
  ▶ Equilibrium-constrained optimization formulation
  ▶ Partition algorithm
  ▶ Sensitivity analysis

▶ Insights:
  ▶ Two multipliers, network multiplier and liquidity amplifier, are identified to characterize analytically the amplification effects.
  ▶ Capital and systemic resilience
  ▶ Policy assessment: direct asset purchase and capital injection
Talk Outline

2. The general model with liquidity effect
   ▶ Formulation and algorithm
   ▶ Contagion probability estimation
   ▶ Liquidity amplifier
3. Numerical experiments based on the data from the 2011 EU-wide stress test
Timeline:

- **Time 0**: An interbank lending-and-borrowing system is established
- **Time 1 (Armageddon, maybe?)**: All debts mature; the system reaches a market-clear repayment equilibrium
Review of E-N: Bank’s Balance Sheets at Time 0

- There are $N$ banks, indexed by $1, 2, \ldots, N$, and a non-leveraged sector in the economy.
- For bank $i$, its balance sheet at time 0 is given by

<table>
<thead>
<tr>
<th>The Balance Sheet of Bank $i$ at Date 0</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Assets</strong></td>
</tr>
<tr>
<td>External Assets: $\bar{y}_i$</td>
</tr>
<tr>
<td>Interbank Loans: $\bar{x}_{i,k}$ (some $k \neq i$)</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

- Let $l_i$ indicate the nominal amount of total liabilities of bank $i$:

$$l_i = \sum_j \bar{x}_{j,i} + \bar{x}_i^E.$$  

Let $\Pi$ denote the relative liability matrix in which $\pi_{ji} = \bar{x}_{ij}/l_j$. 
Review of E-N: Illustration

Date 0

Banking system
Bank 1
Bank 2
Bank 3

Non-leveraged Sector

Date 1

Banking system
Bank 1
Bank 2
Bank 3

Non-leveraged Sector
Loan repayment at time 1 obeys the following rules:

A. Residual claimant: debt claims have priority than equity.
B. Limited liabilities: each banks pays at most its available assets.
C. In the default of a bank, all the interbank loans and the external debt claims are with the same seniority.

Given the values of external assets at time 1 are \( \hat{y} = (\hat{y}_1, \cdots, \hat{y}_N) \), \( \hat{y}_i \leq \bar{y}_i \), the interbank repayments in equilibrium for each bank is then determined by

\[
p_i = \min \left\{ l_i, \hat{y}_i + \sum_{j \neq i} p_j \pi_{ji} \right\}.
\]
Multiplicity of Market Equilibria and the Fix-Point Formulation

- In a vectorial form, the equilibrium repayment $p$ is the solution to the following fixed-point problem:

$$p = \min \{ l, \hat{y} + p\Pi \}.$$ 

- Equilibrium existence and multiplicity:
In light of this self-fulfilling complexity, we focus on the largest equilibrium in this talk. Equivalently, we are interested in the following optimization problem:

\[
\max \sum_i p_i \\
\text{s.t. } \quad p = \min \{l, \hat{y} + p\Pi\}
\]

The above problem can be further reduced to a linear programming:

**Primal**: \(\max p \mathbf{1}\) \\
\[\text{s.t. } \quad p \leq l \ (\eta) \quad \Leftrightarrow \quad (l - \Pi)\delta + \eta \geq 1 \]
\[p \leq \hat{y} + p\Pi \ (\delta) \quad \delta \geq 0, \ \eta \geq 0\]

**Dual**: \(\min \hat{y}\delta + l\eta\)
Our Proposal: Optimization with Equilibrium Constraints

- The optimal solution to the LP takes the following form: there exist a partition of the set of banks \(\{1, 2, \cdots, n\}\) into two complementary subsets \(\mathcal{D}\) and \(\mathcal{N}\), \(\mathcal{D} = \{i : p_i < \ell_i\}\), \(\mathcal{N} = \{i : p_i = \ell_i\}\), such that

  Primal sol: \(p_{\mathcal{N}} = 1_{\mathcal{N}};\quad p_{\mathcal{D}} = (\hat{y}_{\mathcal{D}} + 1_{\mathcal{N}} \Pi_{\mathcal{N}, \mathcal{D}})(l_{\mathcal{D}} - \Pi_{\mathcal{D}})^{-1}\).

  Dual sol: \(\delta_{\mathcal{D}} = (l_{\mathcal{D}} - \Pi_{\mathcal{D}})^{-1}1_{\mathcal{D}},\quad \delta_{\mathcal{N}} = 0_{\mathcal{N}};\)
  \(\eta_{\mathcal{D}} = 0_{\mathcal{D}},\quad \eta_{\mathcal{N}} = 1_{\mathcal{N}} + \Pi_{\mathcal{N}, \mathcal{D}}(l_{\mathcal{D}} - \Pi_{\mathcal{D}})^{-1}1_{\mathcal{D}}\)

- Network multiplier:

  \(\delta_{\mathcal{D}} = (l_{\mathcal{D}} - \Pi_{\mathcal{D}})^{-1}1_{\mathcal{D}} = (l_{\mathcal{D}} + \Pi_{\mathcal{D}} + \Pi^2_{\mathcal{D}} + \cdots)1_{\mathcal{D}}\)
Partition Algorithm

Round 1

- Partition Candidate: \( \mathcal{N} = \mathcal{D}^c, \mathcal{D} = \emptyset \)
- Equilibrium Candidate:
  \( p = 1 \)
- Feasibility Checking:
  \[ l \leq \hat{y} + l \Pi? \]
  (Equity := \( \hat{y} + l \Pi - l \geq 0? \))
  - Yes: terminate
  - No: Include the violators into \( \mathcal{D} \)
Partition Algorithm

Round 2

- Partition Candidate: $\mathcal{N}' = \mathcal{D}^c$, $\mathcal{D} = \mathcal{I}$
- Equilibrium Candidate: $p_{\mathcal{N}} = l_{\mathcal{N}}$, $p_{\mathcal{D}} = \hat{y}_{\mathcal{D}} + l_{\mathcal{N}}\Pi_{\mathcal{N},\mathcal{D}} + p_{\mathcal{D}}\Pi_{\mathcal{D}}$
- Feasibility Checking: $l_{\mathcal{N}} \leq \hat{y}_{\mathcal{N}} + l_{\mathcal{N}}\Pi_{\mathcal{N}} + p_{\mathcal{D}}\Pi_{\mathcal{D},\mathcal{N}}$? (Equity := $\hat{y}_{\mathcal{N}} + l_{\mathcal{N}}\Pi_{\mathcal{N}} + p_{\mathcal{D}}\Pi_{\mathcal{D},\mathcal{N}} - l_{\mathcal{N}} \geq 0$)
  - Yes: terminate
  - No: Include the violators into $\mathcal{D}$
Partition Algorithm

Round 3

- Partition Candidate: $\mathcal{N} = \mathcal{D}^c$, $\mathcal{D} = \mathcal{I} + \mathcal{II}$
- Equilibrium Candidate: $p_{\mathcal{N}} = l_{\mathcal{N}}$, $p_{\mathcal{D}} = \hat{y}_{\mathcal{D}} + l_{\mathcal{N}}\Pi_{\mathcal{N},\mathcal{D}} + p_{\mathcal{D}}\Pi_{\mathcal{D}}$
- Feasibility Checking: $l_{\mathcal{N}} \leq \hat{y}_{\mathcal{N}} + l_{\mathcal{N}}\Pi_{\mathcal{N}} + p_{\mathcal{D}}\Pi_{\mathcal{D},\mathcal{N}}$? (Equity := $\hat{y}_{\mathcal{N}} + l_{\mathcal{N}}\Pi_{\mathcal{N}} + p_{\mathcal{D}}\Pi_{\mathcal{D},\mathcal{N}} - l_{\mathcal{N}} \geq 0$?)
  - Yes: terminate
  - No: Include the violators into $\mathcal{D}$
An Extended Model with Market Liquidity

Now we consider an extended model to incorporate the influence of market liquidity. For bank $i$, its balance sheet at time $0$ is given by

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<td>Liquid Assets: $\bar{a}_i$</td>
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<td>Illiquid Assets: $\bar{s}_i$</td>
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The selling price of illiquid assets is influenced by the market demand. Its inverse demand function is

$$ q = Q\left(\sum_{i=1}^{N} s_i\right), $$

where $q$ is the market price of one unit illiquid asset, $\sum_{i=1}^{N} s_i$ is the aggregate supply, and $Q(0) = 1$, $Q(\cdot)$ monotonically decreasing.

Example: $Q(y) = \exp(-\gamma y)$, $\gamma$: market depth

Liquid asset can be sold at its face value.
Repayment Vector

Given a realization of external repayment \( \hat{y} = (\hat{y}_1, \cdots, \hat{y}_N) \), the liquidation amounts of liquid and illiquid assets \( a = (a_1, \cdots, a_N) \) and \( s = (s_1, \cdots, s_N) \), the income for each bank is

\[
\hat{y}_i + \sum_{j \neq i} p_j \pi_{ji} + a_i + s_i q
\]

where \( q = Q(\sum_{i=1}^{N} s_i) \).
Market Equilibrium

Definition

A quadruplet \((p^*, a^*, s^*, q^*)\) is called a market equilibrium if for all \(i\),

1. \((Limited \ liability)\) \(p_i = \min\{l_i, \hat{y}_i + \sum_{j \neq i} p_j^* \pi_{ji} + a_i^* + s_i^* q^*\}\);
2. \((No \ short \ sale \ of \ assets)\) \(a_i^* \in [0, \bar{a}_i]\) and \(s_i^* \in [0, \bar{s}_i]\);
3. \((Sale \ of \ liquid \ asset \ occurs \ first)\)

\[
a_i^* = \min \left\{ \max \left\{ p_i^* - \left( \hat{y}_i + \sum_{j \neq i} p_j^* \pi_{ji} \right), 0 \right\}, \bar{a}_i \right\}
\]

and

\[
s_i^* = \min \left\{ \frac{\max \left\{ p_i^* - \left( \hat{y}_i + \sum_{j \neq i} p_j^* \pi_{ji} \right) - \bar{l}_i, 0 \right\}}{q^*}, \bar{s}_i \right\}.
\]

4. \((Market \ clearing)\) \(q^* = Q(\sum_{i=1}^{N} s_i^*)\).

A variation of Cifuentes et al. (2005)
Optimization Formulation

- Multiple equilibria caused by limited liquidity
- We can formulate the following optimization problem with equilibrium constraints to solve for the largest equilibrium \((p^*, a^*, s^*, q^*)\):

\[
\max \quad q \\
\text{s.t.} \quad (p, a, s, q) \text{ satisfy constraints 1-4.}
\]

- An extended partition algorithm is developed to solve the above optimization problem. We have, \(p^* \geq p\), \(a^* \leq a\), \(s^* \leq s\), and \(q^* \geq q\), for any other equilibrium \((p, a, s, q)\).
Structure of the Largest Equilibrium

- The algorithm can help us classify the whole banking system in equilibrium into four subsets:
  - $\mathcal{N} = \{i : p_i = l_i, \ a_i = 0, \ s_i = 0\}$ (Banks which can repay their liabilities without liquidating assets).
  - $\mathcal{M} = \{i : p_i = l_i, \ 0 < a_i < \bar{a}_i, \ s_i = 0\}$ (Banks which have to liquidate part of their liquid asset to repay the liabilities).
  - $\mathcal{L} = \{i : p_i = l_i, \ a_i = \bar{a}_i, \ 0 < s_i < \bar{s}_i\}$ (Banks which have to liquidate part of their illiquid asset to repay the liabilities).
  - $\mathcal{D} = \{i : p_i < l_i\}$ (Banks in default)

- The repayment vectors of these three subsets of banks satisfy the following equations:

  $$
  \mathbf{p}_D = [\hat{\mathbf{y}}_D + \bar{a}_D + q\bar{s}_D + l_{Dc} \Pi_{Dc,D}][I_D - \Pi_D]^{-1},
  \mathbf{p}_L = \hat{\mathbf{y}}_L + \bar{a}_L + q\bar{s}_L + \mathbf{p}_D \Pi_{D,D} \mathcal{L} + l_{Dc} \Pi_{Dc,L} = I_L,
  \mathbf{p}_M = \hat{\mathbf{y}}_L + a_L + \mathbf{p}_D \Pi_{D,M} + l_{Dc} \Pi_{Dc,M} = I_M,
  \mathbf{p}_N = I_N.
  $$
Round 1

- Partition Candidate: $\mathcal{N} = \mathcal{D}^c$, $\mathcal{D} = \emptyset$

- Equilibrium Candidate: $\mathbf{p} = \mathbf{l}$, $q^1 = 1$

- Feasibility Checking:
  \[
  \text{Equity} \coloneqq \hat{y} + \bar{a} + \bar{s} \cdot q^1 + l\Pi - l \geq 0?
  \]
  
  - Yes: terminate
  
  - No: Include the violators into $\mathcal{D}$
Partition Algorithm in the Presence of Liquidity

Round 2

- Partition Candidate: $\mathcal{N} = \mathcal{D}^c$, $\mathcal{D} = \mathcal{I}$
- Equilibrium Candidate: $p_\mathcal{N}, p_\mathcal{D}, q^2$
- Feasibility Checking:
  Equity :=
  \[
  \hat{y}_\mathcal{N} + \bar{a}_\mathcal{N} + q^2 \bar{s}_\mathcal{N} + l_\mathcal{N} \cdot \Pi_\mathcal{N} + p_\mathcal{D} \cdot \Pi_{\mathcal{D},\mathcal{N}} - l_\mathcal{N} \geq 0?
  \]

  - Yes: terminate
  - No: Include the violators into $\mathcal{D}$
Partition Algorithm in the Presence of Liquidity

Round 3

- Partition Candidate: $\mathcal{N} = \mathcal{D}^c$, $\mathcal{D} = \mathcal{I} \cup \mathcal{II}$
- Equilibrium Candidate: $p_\mathcal{N}, p_\mathcal{D}, q^3$
- Feasibility Checking:
  
  $\text{Equity} := \hat{y}_\mathcal{N} + \bar{a}_\mathcal{N} + q^3 \bar{r}_\mathcal{N} + \mathbf{l}_\mathcal{N} \cdot \Pi_\mathcal{N} + p_\mathcal{D} \Pi_{\mathcal{D},\mathcal{N}} - \mathbf{l}_\mathcal{N} \geq 0$?
  
  - Yes: terminate
  - No: Include the violators into $\mathcal{D}$
Consider a network with the initial condition that the book value of every bank’s capital is positive:

\[ e_i^{(0)} := \bar{y}_i + \bar{a}_i + \bar{s}_i + \sum_{j=1}^{n} \ell_j p_{ji} - \ell_i > 0, \quad \text{for all } i. \]

Suppose that a shock hits bank 1, and its size is \( \Delta y_1 = \bar{y}_1 - \hat{y}_1 \).

The partition algorithm helps us develop estimates for the contagion magnitude.
Contagion Estimation and Capital

- Round 1 (Fundamental default):

\[ P(\text{Bank 1 Defaults}) \geq P(\Delta y_1 \geq e_1^{(0)}). \]
Contagion Estimation and Capital

Round 2 (First-order default):

\[ E(\text{\# of Defaults in this round}|\text{Bank 1 defaults}) \geq \sum_{j \sim 1} P \left( \Delta y_1 - e_1^{(1)} > \frac{e_j^{(1)}}{\pi_{1j}} \right), \]

where

\[ e_j^{(1)} := \hat{y}_j + \bar{a}_j + \bar{s}_j q^1 + \sum_{j=1}^n l_k \pi_{k1} - l_1 \]

and

\[ q^1 = Q(\bar{s}_1). \]
All rounds:

\[ E(\# \text{ of defaults in the equilibrium} | \text{Bank 1 defaults}) \geq \sum_{j \neq 1} P \left( \Delta y_1 - e_1^{(1)} > \frac{1}{z_{1j}} \sum_{i=2}^{n} e_i^{(1)} z_{ij} \right), \]

where \( Z = (z_{ij})_{i,j \in \{1, \ldots, n\}} = (I - P)^{-1} \).

Resilience index:

\[ \frac{\sum_{i=2}^{n} e_i^{(1)} z_{ij}}{z_{1j}} = \frac{[e^{(1)} + e^{(1)} P + e^{(1)} P^2 + \cdots]_j}{[I + P + P^2 + \cdots]_1j} \]
Liquidity Amplifier

For $i \in \mathcal{L}$,

$$\frac{\partial q^*}{\partial \hat{y}_i} = \frac{\gamma}{1 - \gamma(s_L 1 + \bar{s}_D [l_D - \Pi_D]^{-1}\Pi_{D,L} 1)}$$

We refer to the above quantity as the **liquidity amplifier**. To understand this, note

$$\frac{\gamma}{1 - \gamma S} = \gamma + \gamma(\gamma S) + \gamma(\gamma(\gamma S))S + \cdots$$
Policy Assessment and Liquidity Amplifier

Take two intervention policies as an example:

- Direct asset purchase:
- Capital injection

**Figure**: Direct Asset Purchase

**Figure**: Capital Injection
Policy Assessment and Liquidity Amplifier

- Asset Purchase at Market price

\[
\frac{\partial q^*}{\partial \Delta} \bigg|_{\text{Market}} = \frac{\gamma}{1 - \gamma (s_L 1 + s_D [I_D - \Pi_D]^{-1} \Pi_D, L 1)} ^{\text{Liquidity Amplifier}}
\]

\[
\frac{\partial p^*}{\partial \Delta} \bigg|_{\text{Market}} = LA \cdot s_D^* (I_D^* - P_D^*)^{-1}.
\]

- Capital Injection:

\[
\frac{\partial q^*}{\partial \Delta} \bigg|_{\text{Capital}} = LA \cdot e_i (I_D^* - P_D^*)^{-1} P_D^* , L^* 1,
\]

\[
\frac{\partial p^*}{\partial \Delta} \bigg|_{\text{Market}} = e_i (I_D^* - P_D^*)^{-1} + LA \cdot e_i (I_D^* - P_D^*)^{-1} P_D^* , L^* 1 \cdot s_D^* (I_D^* - P_D^*)^{-1}.
\]

- Asset Purchase at Face Value
Policy Assessment and Liquidity Amplifier

- Policy implication:
  - Asset purchase focuses on liquidity improvement
    - Total asset value
      \[
      \text{Total Asset Value (TAV) After Purchase} = \text{TAV Before Purchase} + \Delta - q^* \cdot \frac{\Delta}{q^*}
      \]
      \[
      = \text{TAV Before Purchase}.
      \]
  - Liquidity ratio:
    \[
    LR \text{ After Purchase} = \frac{\text{Liquidity} + \Delta}{TAV}
    \]
- Capital injection eyes more on network effect
  - Total asset value increases by \( \Delta \)
  - Liquidity ratio:
    \[
    LR \text{ After Purchase} = \frac{\text{Liquidity} + \Delta}{TAV + \Delta}
    \]
Numerical Example: Data

- We use the data set from the European Banking Authority’s 2011 stress test.

<table>
<thead>
<tr>
<th>Bank Number and Name</th>
<th>Total Asset</th>
<th>Interbank Asset $\sum_j L_{ji}$</th>
<th>Non-Interbank Asset</th>
<th>Equity</th>
<th>Liability $\ell_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DE017 DEUTSCHE BANK AG</td>
<td>1905630</td>
<td>194399</td>
<td>1711231</td>
<td>30361</td>
<td>1875269</td>
</tr>
<tr>
<td>DE018 COMMERZBANK AG</td>
<td>771201</td>
<td>138190</td>
<td>633011</td>
<td>26728</td>
<td>744473</td>
</tr>
<tr>
<td>DE019 LANDESBANK BADEN</td>
<td>374413</td>
<td>133906</td>
<td>240507</td>
<td>9838</td>
<td>364575</td>
</tr>
<tr>
<td>DE020 DZ BANK AG DT. ZEN</td>
<td>323578</td>
<td>135860</td>
<td>187718</td>
<td>7299</td>
<td>316279</td>
</tr>
<tr>
<td>DE021 BAYERISCHE LANDES</td>
<td>316354</td>
<td>97336</td>
<td>219018</td>
<td>11501</td>
<td>304853</td>
</tr>
<tr>
<td>DE022 NORDDEUTSCHE LAN</td>
<td>228586</td>
<td>91217</td>
<td>137369</td>
<td>3974</td>
<td>224612</td>
</tr>
<tr>
<td>DE023 HYPO REAL ESTATE I</td>
<td>328119</td>
<td>29084</td>
<td>299035</td>
<td>5539</td>
<td>322580</td>
</tr>
<tr>
<td>DE024 WESTLB AG, DÜSSELDORF</td>
<td>191523</td>
<td>58128</td>
<td>133395</td>
<td>4218</td>
<td>187305</td>
</tr>
<tr>
<td>DE025 HSH NORDBANK AG, HAMBURG</td>
<td>150930</td>
<td>9532</td>
<td>141398</td>
<td>4434</td>
<td>146496</td>
</tr>
<tr>
<td>DE027 LANDESBANK BERLIN</td>
<td>133861</td>
<td>49253</td>
<td>84608</td>
<td>5162</td>
<td>128699</td>
</tr>
<tr>
<td>DE028 DEKABANK DEUTSCH</td>
<td>130304</td>
<td>41255</td>
<td>89049</td>
<td>3359</td>
<td>126945</td>
</tr>
</tbody>
</table>
Numerical Example: Three Network Configuration

- The detailed specification of network structures are not available in the raw data.
- We recover it using the relative-entropy based algorithm in Upper and Worms (2004, EER).
  - To find the bilateral exposures between banks, let
    
    $L = \begin{bmatrix}
    0 & \cdots & l_{1j} & \cdots & l_{1N} \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    l_{i1} & \cdots & 0 & \cdots & l_{iN} \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    l_{N1} & \cdots & l_{Nj} & \cdots & 0
    \end{bmatrix}.$

- We solve the following optimization problem:

  \[
  \min \sum_{i,j} l_{ij} \ln\left(\frac{l_{ij}}{x_{ij}}\right)
  \]

subject to $\sum_j l_{ij} = \text{observed interbank asset}$ and $\sum_i l_{ij} = \text{observed interbank liabilities}$
Numerical Example: Three Network Configurations (Con’d)

Three kinds of network configurations for the core banks are examined.

Figure: Complete Network
Figure: Ring-Like Network
Figure: Core-Periphery
Numerical Example: Market Environment

- $Q(s) = \exp(-\gamma s)$.
- Baseline case:
  - $\gamma = 1 \times 10^{-7}$
  - 30% of the total asset of each bank is illiquid

Under this choice, the market price of the illiquid asset will drop to $0.864$ from its face value $1$ when every bank in the system depletes its holdings.
Numerical Example: Liquidity Contagion

We let one of the banks receive shocks on its asset in the following experiment, i.e., $\hat{y}_1 = (1 - \theta)\bar{y}_1$, $\theta \in [0, 1]$. 

![Graphs showing numerical example](image_url)
Numerical Example: Liquidity Contagion (Con’d)

- No significant contagion in the absence of market liquidity.
- Introducing illiquid asset into the banking system amplifies the contagion effect significantly.
- Network structure matters when contagion kicks off.
Extensions: Capital Ratio Requirement and Spillover

- **Capital ratio requirement:**

\[
\frac{\hat{y}_i + \bar{a}_i + \bar{s}_i q + \sum_{j \neq i} l_j \pi_{ji} - x_i}{\hat{y}_i + (\bar{a}_i - a_i) + (\bar{s}_i - s_i) q + \sum_{j \neq i} l_j \pi_{ji}} \geq R.
\]

- **Sensitivity and undesirable spillover:**

Let

\[
\begin{align*}
L_1^* &= \left\{ i \in \mathcal{N}^* : s_i^* > 0, \frac{\beta_i + \tilde{y}_i + \bar{s}_i q^* + \sum_j x_j^* p_{ji} - \ell_i}{\beta_i + \tilde{y}_i - y_i^* + (\bar{s}_i - s_i^*) q^* + \sum_j x_j^* p_{ji}} > R \right\}, \\
L_2^* &= \left\{ i \in \mathcal{N}^* : s_i^* > 0, \frac{\beta_i + \tilde{y}_i + \bar{s}_i q^* + \sum_j x_j^* p_{ji} - \ell_i}{\beta_i + \tilde{y}_i - y_i^* + (\bar{s}_i - s_i^*) q^* + \sum_j x_j^* p_{ji}} = R \right\}. 
\end{align*}
\]

- **Sensitivity analysis:**

\[
\frac{\partial q^*}{\partial \beta_i} = \gamma \left( 1 - \gamma (s_L^* 1 + \bar{s}_D^* (l_D^* - n_D^*))^{-1} (P_{D^*}, L_1^* 1 + \frac{1-R}{R} n_{D^*}, L_2^* 1) + \frac{1-R}{R} \bar{s}_{L_2^* 1} \right).
\]
Data incompleteness and robust optimization:

- Typically, we do not possess detailed observation for all bilateral exposures in the network. What we know is at most that $\Pi$ is a member of an uncertain set $\mathcal{P}$.

$$\min_{\Pi \in \mathcal{P}} \left( \max_{\Pi \in \mathcal{P}} \max_{q} \right)$$

s.t. 

$(p, a, s, q)$ satisfy constraints 1-4.

- Ben-Tal, El Ghaoui, and Nemirovski (2009), Bertsimas, Brown, and Caramanis (2011)
Concluding Remarks

- We present an equilibrium-constrained optimization formulation to model the systemic risk.
- Our research identifies network and liquidity multipliers explicitly and points out different roles played by these two factors in systemic risk development.