When Micro Prudence increases Macro Risk: The Destabilizing Effects of Financial Innovation, Leverage, and Diversification

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Credit growth is driven by the dynamics of balance sheet size. What drives the balance sheet dynamics of FIs and what are the systemic risk implications?

- What is the effect of financial innovations allowing a more efficient diversification of risk?

- What are the systemic risk consequences of a change in capital requirements and in the maximum leverage allowed to banks?

- How does contagion propagate when financial players change their level of diversification? Is it always beneficial for the system to have more diversified portfolios?

- Can actions to reduce risk of a single financial institution increase the risk of a systemic event i.e. are micro-prudential policies always coherent with macro-prudential objectives?


Related literature

Our paper tries to combine several strands of literature:

• on the impact of capital requirements on the behavior of FIs (Danielsson et al., 2004, 2009; Adrian & Shin, 2009; Adrian et al., 2011);

• on the effects of diversification and overlapping portfolios on systemic risk (Tasca & Battiston, 2012; Caccioli et al., 2012)

• on the risks of financial innovation (Brock et al. 2009, Haldane & May, 2011)

• on distressed selling and its impact on the market price dynamics (Kyle & Xiong, 2001; Cont & Wagalath, 2011, Thurner et al., 2012; Caccioli et al., 2012)

• on the determinants of balance sheet dynamics of FIs and credit supply (Stein 1998; Bernanke & Gertler 1989; Bernanke, Gertler, & Gilchrist, 1996, 1999; Kiyotaki & Moore, 1997)

Contribution: propose a solvable model that, by combining these different streams of literature, provides full analytical quantification of the links between micro prudential rules and macro prudential outcomes
The portfolio choice

- $N$ financial institutions (FIs or banks), $i = 1, \ldots, N$
- For simplicity, we assume that FIs adopt a simple investment strategy: 
etually weighted portfolio of $m$ randomly selected investments (out of $M$)
- We consider the existence of "costs of diversification" $c$ reflecting the presence of transaction costs, firms specialization and other types of frictions.
- With $r_L$ the avg interest rate on liabilities the portfolio expected return is $\mu - r_L$,
- FI maximizes portfolio returns under VaR constraints.
  \[ \text{VaR} = \alpha \sigma_p A \leq E. \]
  with $\sigma_p$ the holding period volatility, $A$ asset of bank $i$, and $\alpha$ a constant,
The investment set:

- collection of risky investment \( j = 1, \ldots, M \)

- FIs, correctly perceive that each risky investment entails both an idiosyncratic (diversifiable) risk component and a systematic (undiversifiable) risk component,

\[
\sigma_i^2 = \sigma_s^2 + \sigma_d^2
\]

where

- \( \sigma_s^2 \) is the systematic risk
- \( \sigma_d^2 \) is the diversifiable risk component.

- Hence, the expected mean and volatility per dollar invested in the portfolio chosen by a given institution are \( \mu \) and

\[
\sigma_p = \sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}}
\]
Then, facing cost of diversification and VaR constraints, FI chooses the total asset $A$ ($E$ is sticky) and diversification $m$ which max their portfolio returns.

$$\max_{A,m} A(\mu - r_L) - \tilde{c}m \quad \text{s.t.} \quad \alpha A \sqrt{\sigma^2_s + \frac{\sigma^2_d}{m}} \leq E.$$ 

Dividing by $E$ and with $c = \frac{\tilde{c}}{E}$, the max can be written in terms of the leverage $\lambda = \frac{A}{E}$,

$$\max_{\lambda,m} \lambda(\mu - r_L) - cm \quad \text{s.t.} \quad \alpha \lambda \sqrt{\sigma^2_s + \frac{\sigma^2_d}{m}} \leq 1.$$ 

→ chooses the optimal leverage $\lambda^* = \frac{A^*}{E}$ and $m^*$ which max ROE under the VaR

Squaring the constraint the Lagrangian can be written as

$$L = \lambda(\mu - r_L) - cm - \frac{1}{2} \gamma \left( \alpha^2 \lambda^2 \left( \sigma^2_s + \frac{\sigma^2_d}{m} \right) - 1 \right).$$ 

where $\gamma$ is the Lagrange multiplier for the VaR constraint.
Optimal leverage and diversification

F.O.C. $\Rightarrow \lambda^* = \frac{1}{\gamma} \frac{1}{\alpha^2} \frac{\mu - r_L}{\sigma_p^2}$ with Lagrange multiplier $\gamma = \frac{1}{\alpha} \frac{\mu - r_L}{\sigma_p}$

- The optimal leverage is

$$\lambda^* = \frac{1}{\alpha \sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}}} = \frac{1}{\alpha \sigma_p}$$

- The optimal level of diversification is

$$m^* = \frac{\sqrt{\gamma} \alpha \lambda \sigma_d}{\sqrt{2c}} = \lambda \sigma_d \sqrt{\frac{\alpha}{2c} \frac{\mu - r_L}{\sigma_p}}$$

Bottom line:

- leverage $\lambda$ is an inverse function of the portfolio volatility $\sigma_p$
- portfolio size $m$ is an inverse function of diversification costs $c$
parameters are: $M = 20$, $\alpha = 0.05$, $\mu - r_L = 0.8$, $\sigma_d = 1$. We then choose $\sigma_s$ equal to 0 (solid line), 0.3 (dashed line), and 0.6 (dotted line).
Portfolio overlap between two financial institutions

m = 3

N Financial Institutions

M Investments

portfolio overlap between red and green FI = 2/3
parameters are: $M = 20$, $\alpha = 0.05$, $\mu - r_L = 0.8$, $\sigma_d = 1$.
We then choose $\sigma_s$ equal to 0 (solid line), 0.3 (dashed line), and 0.6 (dotted line).
### Leverage targeting and balance sheet adjustments

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*Increase in value of securities*
*Increase in equity*

---

*New purchase of securities*
*New borrowing*

(from Adrian and Shin (2010))
Balance sheet adjustments: empirical evidence

Investment Banks (1994Q1 - 2011Q2)

Change in Equity & Changes in Debt (Billions)

Change in Assets (Billions)

$y = 0.9948x - 0.6713$

$y = 0.0052x + 0.6713$

from Adrian, Colla, and Shin (2010)
Dynamics of asset with portfolio rebalancing

At the beginning of each investment period FIs rebalance their portfolio by the difference between the desired amount of asset $A_{j,t}^* = \lambda E_{j,t}$ and the actual one $A_{j,t}$

$$\Delta R_{j,t} \equiv A_{j,t}^* - A_{j,t} = \lambda E_{j,t} - A_{j,t},$$

By defining realized portfolio return $r_{j,t}^P$, can be rewritten as

$$\Delta R_{j,t} = (\lambda - 1) r_{j,t}^P A_{j,t-1}^*$$

⇒ any P&L from the investments portfolio $r_{j,t}^P A_{j,t-1}^*$ results in a change of FI asset value amplified by the target leverage (for $\lambda > 1$).

⇒ VaR induces a perverse demand function: buy if $r_{j,t}^P > 0$, sell if $r_{j,t}^P < 0$

⇒ positive feedback
Dynamics of investments demand

The aggregate demand of asset $i$ will be simply the sum of the individual demands of the FIs who picked asset $i$ in their portfolio.

$$D_{i,t} = \sum_{j=1}^{N} I_{\{i \in j\}} \frac{1}{m} \Delta R_{j,t} \approx \sum_{j=1}^{N} I_{\{i \in j\}} (\lambda - 1) r_{j,t}^{p} \frac{A_{j,t-1}^{*}}{m}$$

where $I_{\{i \in j\}}$ is 1 if asset $i$ is in the portfolio of institution $j$ and zero otherwise.

Considering total assets approximately the same across FIs, $A_{j,t-1}^{*} \approx A_{t-1}^{*}$, demand of investment $i$ can be approximated as

$$D_{i,t} \approx (\lambda - 1) \frac{A_{t-1}^{*}}{m} \frac{N}{M} \left( r_{i,t} + \frac{m - 1}{M - 1} \sum_{k \neq i} r_{k,t} \right)$$

Note: it can be shown that demand correlation between two assets $\rho(D_{i}, D_{k}) \xrightarrow{m \to M} 1$
Parameters are $M = 20$, $N = 100$, and $\sigma_d = 1$. We then choose $\sigma_s$ equal to 0 (solid line), 0.3 (dashed line), and 0.6 (dotted line).
Risky asset dynamics with endogenous feedbacks

With rebalancing feedbacks, the return process is now made of 2 components

\[ r_{i,t} = e_{i,t-1} + \varepsilon_{i,t} \]

endogenous exogenous

We assume that the exogenous component has a multivariate factor structure

\[ \varepsilon_{i,t} = f_t + \epsilon_{i,t} \]

factor idiosyncratic

uncorrelated and distributed with mean 0 and constant volatility, \( \sigma_f \) and \( \sigma_\epsilon \) (the same for all investments).

Thus, the variance of the exogenous component of the risky investment \( i \) is

\[ V(\varepsilon_i) = \sigma_f^2 + \sigma_\epsilon^2 \]
Risky asset dynamics with endogenous feedbacks II

Assuming a linear price impact function the endogenous component becomes

\[ e_{i,t} = \frac{D_{i,t}}{\gamma_i C_{i,t}} \]

where

- \( \gamma_i \) is the market liquidity of asset \( i \)
- \( C_{i,t} = \sum_{j=1}^{N} I_{\{i \in j\}} \frac{A_{j,t-1}^*}{m} \approx \frac{N}{M} A_{t-1}^* \) is a proxy for market cap

Substituting \( D, r, \) and \( C, \) we obtain the following VAR(1) for \( e_t \)

\[ e_t = \Phi (e_{t-1} + \varepsilon_t) \]

where \( \Phi \equiv (\lambda - 1) \Gamma^{-1} \Psi \) with

\[
\Gamma_{M \times M} = \begin{bmatrix}
\gamma_1 & 0 & \ldots & 0 \\
0 & \gamma_2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \gamma_M \\
\end{bmatrix}, \quad \Psi_{M \times M} = \begin{bmatrix}
\frac{1}{m} & \frac{1}{m} & \ldots & \frac{1}{m} \\
\frac{1}{m} & \frac{1}{m} & \ldots & \frac{1}{m} \\
\frac{1}{m} & \frac{1}{m} & \ldots & \frac{1}{m} \\
\frac{1}{m} & \frac{1}{m} & \ldots & \frac{1}{m} \\
\end{bmatrix}.
\]
Multivariate return dynamics

The VAR(1) dynamics

\[ e_t = \Phi (e_{t-1} + \varepsilon_t) \]

is dictated by the eigenvalues of the matrix

\[ \Phi \equiv (\lambda - 1) \Gamma^{-1} \Psi. \]

Being the max eigenvalue of \( \Psi \) equal 1 \( \forall m \), we have:

\[ \Lambda_{\text{max}} \approx (\lambda - 1) \overline{\gamma^{-1}} \]

where \( \overline{\gamma^{-1}} \) is the average of all the \( \gamma_i^{-1} \).

\[ \Rightarrow \text{the max eig depends on leverage and on the average illiquidity of the assets.} \]

When \( \Lambda_{\text{max}} > 1 \), the return processes become non-stationary and explosive.
parameters are: $M = 20$, $\alpha = 0.05$, $\mu - r_L = 0.8$, $\gamma = 40$, and $\sigma_d = 1$. We then choose $\sigma_s$ equal to 0 (solid line), 0.3 (dashed line), and 0.6 (dotted line). The horizontal solid line shows the condition $\Lambda_{\text{max}} = 1$, \[
\Lambda_{\text{max}} \quad \text{diversification cost} \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5
\]
\[
\text{stationary} \quad \text{non-stationary}
\]
\[
\Lambda_{\text{max}} \quad \text{portfolio overlap} \quad 0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1
\]
\[
\text{stationary} \quad \text{non-stationary}
\]
Alternative representation of the endogenous dynamics

We can write,

\[ m\Psi = \begin{bmatrix}
1 & \frac{m-1}{M-1} & \cdots & \frac{m-1}{M-1} \\
\frac{m-1}{M-1} & 1 & \cdots & \frac{m-1}{M-1} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{m-1}{M-1} & \frac{m-1}{M-1} & \cdots & 1 \\
\end{bmatrix} = (1 - b)I + bu' \]

with \( b = \frac{m-1}{M-1} \).

Thus, the endogenous component of an individual investment \( i \) becomes

\[ e_{i,t} = (1 - b) a_i (e_{i,t-1} + \varepsilon_{i,t}) + b M a_i (\bar{e}_{t-1} + \bar{\varepsilon}_t) \]

with \( a_i = \frac{\lambda - 1}{m \gamma_i} \).

Moreover, assuming all investments have the same liquidity, we can show:

- the average process \( \bar{e}_t \) is an AR(1) (systemic component)
- the distance from the avg \( \Delta e_{i,t} \equiv e_{i,t} - \bar{e}_t \) is an AR(1) (idiosyncratic)

\[ \Rightarrow \] the endogenous return dynamics can be seen as a multivariate “ARs around AR"
Thanks to this representation we can explicitly compute the variance and covariances of endogenous components $e_{i,t}$

$$V(e_{i,t}) = \frac{(\lambda - 1)^2 \left( m^2 \left( \sigma_e^2 \left( (\lambda - 1)^2 - \gamma^2 (M-1)^2 \right) \right) + \sigma_f^2 (\gamma^2 - (\lambda - 1)^2) \left( \lambda - 1 \right)^2 \sigma_f^2 - \gamma^2 \sigma_f^2 + M \left( \sigma^2 \left( (\lambda - 1)^2 - \gamma^2 \right) + (\lambda - 1)^2 \sigma_f^2 + \gamma^2 \sigma_f^2 \right) \right)}{(\gamma^2 - (\lambda - 1)^2) \left( m^2 (\sigma_e^2 (M-1)^2 - (\lambda - 1)^2) + 2(\lambda - 1)^2 mM - (\lambda - 1)^2 M^2 \right)}$$

$$Cov(e_{i,t}, e_{j,t}) = -\frac{(\lambda - 1)^2 \left( m^2 \left( \sigma_e^2 \left( (\lambda - 1)^2 - \gamma^2 (M-1)^2 \right) - \gamma^2 (M-2) \sigma_f^2 \right) - 2m \left( (\lambda - 1)^2 M \sigma_f^2 + \gamma^2 \sigma_f^2 \right) + M \left( (\lambda - 1)^2 M \sigma_f^2 + \gamma^2 \sigma_f^2 \right) \right)}{(\gamma^2 - (\lambda - 1)^2) \left( m^2 (\sigma_e^2 (M-1)^2 - (\lambda - 1)^2) + 2(\lambda - 1)^2 mM - (\lambda - 1)^2 M^2 \right)}$$

and show that:

- ↑ leverage → ↑ both var and cov of $e_{i,t}$
- ↑ diversification → ↓ var and ↑ cov
- Both → ↑ correlations
- $\text{Corr}(e_{i,t}, e_{j,t}) \xrightarrow{m \to M} 1$
Return Var-Cov with rebalancing feedbacks

The feedback induced by portfolio rebalancing, introduces a new endogenous component in the variances and covariances of individual and portfolio returns

- **individual returns:**
  \[ V(r_{i,t}) = V(e_{i,t}) + V(\varepsilon_{i,t}) \]
  with \( V(\varepsilon_{i,t}) = \sigma_f^2 + \sigma_{\varepsilon_i}^2 \)

  \[ \text{endogenous} + \text{exogenous} \]

  \[ \text{Cov}(r_{i,t}, r_{j,t}) = \text{Cov}(e_{i,t}, e_{j,t}) + \sigma_f^2 \]
  \[ \text{endogenous} + \text{exogenous} \]

- **portfolio returns:**
  \[ V(r_t^p) = \frac{V(e_{i,t})}{m} + \frac{m - 1}{m} \text{Cov}(e_{i,t}, e_{j,t}) + \sigma_f^2 + \frac{\sigma_{\varepsilon_i}^2}{m} \]
  \[ \text{endogenous} + \text{exogenous} \]

  \[ \text{Cov}(r_t^p, r_k^p) = V(\bar{\varepsilon}) + V(\varepsilon_{M,t}) \]
  \[ \text{endogenous} + \text{exogenous} \]

  \[ V(r_t^M) = \frac{1}{1 - \Lambda_{\text{max}}^2} V(\varepsilon_{M,t}) \]
  \[ \text{"variance multiplier"} \]
Parameters are: $M = 20$, $\alpha = 0.05$, $\mu - r_L = 0.8$, $\gamma = 40$, and $\sigma_d = 1$. We then choose $\sigma_s$ equal to 0 (solid line), 0.3 (dashed line), and 0.6 (dotted line).
Systemic risk

- Correlation between assets and between institutions increases with overlaps. Diversification tends to increase the probability of a systemwide failures.

- A negative realization of the factor $f_t$, now triggers a sequence of portfolio rebalances causing the price of all risky assets to decay for several periods.

Being

$$r_t = e_{t-1} + \nu f_t + \epsilon_t = \Phi r_{t-1} + \nu f_t + \epsilon_t,$$

the $h$-period cumulative mean return conditioned on systematic shock $f_t^{shock}$ is

$$E \left[ r_{t:t+h} | f_t = f_t^{shock} \right] \approx (I - \Phi)^{-1} \nu f_t^{shock}.$$
Including systemic risk in the haircuts on repo

\[ \lambda = 50, \ b_2 = 50\% \]

Introduction of financial innovation: summary

High costs of diversification $c \Rightarrow$ small diversification $m \Rightarrow$ heterog. portfolios and P&L
$\Rightarrow$ individual feedbacks weak and uncoordinated.

Introduction of financial innovation makes: $\downarrow c \uparrow m \downarrow \sigma_p \uparrow \lambda$

Hence we have:

1) Increase in leverage $\lambda \Rightarrow$ increases risk exposure

2) Increase in diversification $m \Rightarrow$ increases correlations

3) Increase in $\lambda$ and $m \Rightarrow$ increases endogenous feedback $\Rightarrow \uparrow$ var, cov & corr

So, individual reaction more aggressive (due to higher leverage) and more coordinated (due to higher correlation) $\Rightarrow$ aggregate feedback between prices and total asset

$\Rightarrow$ makes bank total asset $A$ more erratic $\Rightarrow$ liquidity and funding booms and bursts
Simulation results: simulated structural break

Structural Break at 1000:
1) low diversification and leverage
2) high diversification and leverage
In the model $\downarrow c \uparrow m \downarrow \sigma_p \uparrow \lambda \uparrow A$.

The model presented so far assumes that financial institutions do not consider their effect on prices and do not use past information to create future expectations of assets’ volatilities and correlation.

This leads to a dynamics described by a VAR(1) model and predicts the existence of a transition between a stationary and a nonstationary phase when the control parameter (the cost of diversification or the parameter of the Value at Risk) crosses a given threshold.

What happens if one uses other (more realistic) expectation formation schemes?
The portfolio dynamics between \( t - 1 \) and \( t \) is:

- at time \( t - 1 \), financial institutions choose \( m_{t-1} \) and \( \lambda_{t-1} \)
- between \( t - 1 \) and \( t \), financial institutions adopt a rebalancing strategy in order to maintain the target leverage \( \lambda_{t-1} \)
- at time \( t \), financial institutions estimate their actual risk position \((\tilde{\sigma}_{p,t})^2\)
- at time \( t \), financial institutions deduce \((\tilde{\sigma}_{d,t})^2\) and they form the expectation \((\tilde{\sigma}_{d,t}^e)^2\) about the future \((t+1)\) risk
- at time \( t \), according to their expectation \((\tilde{\sigma}_{d,t}^e)^2\), financial institutions solve the problem of portfolio optimization choosing \( m_t \) and \( \lambda_t \)
Model setting

- We assume that the exogenous systematic risk component $\sigma_s$ is equal to zero. Hence the risk of the portfolio, realized at time $t$, is

$$ (\tilde{\sigma}_{p,t})^2 = \left(\frac{\tilde{\sigma}_{d,t}}{m_{t-1}}\right)^2 $$

where $(\tilde{\sigma}_{d,t})^2$ is the diversifiable risk of an asset as a consequence of the impact of rebalancing feedback.

- When banks maintain the optimal target leverage $\lambda_{t-1}$ rebalancing their portfolio in the time period between $t - 1$ and $t$, the portfolio risk realized at time $t$ is

$$ (\tilde{\sigma}_{p,t})^2 = \frac{\sigma^2_d}{m_{t-1}} + \frac{V(e_{j,t-1})}{m_{t-1}} + \frac{m_{t-1} - 1}{m_{t-1}} \text{Cov}(e_{j,t-1}, e_{k,t-1}) $$

- In the previous slides we have derived the equations for the variance and covariance of the different assets when investors have a certain leverage $\lambda_t$ and a degree of diversification $m_t$
At each time step banks have an expectation $\tilde{\sigma}^e_{d,t}$ about the future value of the volatility. The optimization of the Lagrangian leads to the choice of leverage $\lambda_t$ and a degree of diversification $m_t$ given by

$$m_t = \lambda_t \tilde{\sigma}^e_{d,t} \sqrt{\frac{\alpha}{2c} \frac{\mu - r_L}{\sqrt{(\tilde{\sigma}^e_{d,t})^2/m_t}}}$$ \hspace{1cm} (3)$$

$$\lambda_t = \frac{1}{\alpha \sqrt{(\tilde{\sigma}^e_{d,t})^2/m_t}}$$ \hspace{1cm} (4)$$

$$\tilde{\sigma}^e_{d,t} \equiv \tilde{\sigma}^e_d (m_{t-1}, \lambda_{t-1})$$ \hspace{1cm} (5)$$

The system is closed when we set the expectation scheme. We assume that banks use past observations of volatility to create expectations.

This is a deterministic dynamical system for the variables $\{m, \lambda, \tilde{\sigma}^e_d\}$ in the domain

$$\{m, \lambda, \tilde{\sigma}^e_d\} \in [1, M] \otimes [1, \gamma + 1) \otimes \mathbb{R}^+$$ \hspace{1cm} (6)$$
Under naive expectations, last period volatility of the risky investments will coincide to the future one

\[(\tilde{\sigma}_{d,t}^e)^2 = (\tilde{\sigma}_{d,t})^2 \quad (7)\]

When the systematic component \(\sigma_s = 0\), \(m_t\) is proportional to \(\lambda_t\). We will show the dynamics of the latter.

The dynamical system has as control parameters the diversification cost \(c\) and the Value at Risk parameter \(\alpha\).
Dynamics of the leverage

Figure: Dynamics of the leverage $\lambda_t$ changing the value of diversification cost, $c = 0.4$ (left) and $c = 0.0175$ (right). The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous volatility $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

- Similar dynamics for the diversification $m_t$; when leverage is high (low), diversification is high (low)
- For large diversification costs (or conservative VaR), the dynamics converges quickly to a stable state
- For small diversification costs (or aggressive VaR), a 2-period orbit appears. High values of leverage are alternated by low leverage periods
Figure: Orbit plots as function of the cost parameter $c$. Top left shows the orbit plot for the leverage $\lambda_t$, top right for the endogenous risk, and bottom left for the Pearson Correlation Coefficient between two endogenous components, $e_j$ and $e_k$. The bottom right panel is the orbit plot for the leverage, using the VaR parameter $\alpha$ as controlling variable. The used parameters are: $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$, $\gamma = 100$. 
There is a threshold $c_2$ for the cost (or $\alpha_2$ for the VaR) under which financial cycles of period 2 appear. These are due to the feedback effect induced by the target leverage strategy adopted by financial institutions.

There is also a second threshold $c^* < c_2$ corresponding to the transition from stationarity to non-stationarity return dynamics.

Technically, at $c_2$ we have a period-doubling (flip) bifurcation.

The value of $c_2$ grows with the illiquidity. For a more illiquid market (higher $\gamma^{-1}$) the transition occurs for larger transaction costs.

Near criticality ($c \gtrsim c_2$) the system converges to the stable state very slowly.

By using the Poincaré-Birkhoff normal form it is possible to estimate the flip bifurcation’s amplitude for $c \lesssim c_2$. 
Figure: The linear autocorrelation function of the errors time series for different values of diversification costs for $c = 0.24$ (top left), 0.22 (top right), 0.18 (bottom left), 0.16 (bottom right). The amplitude of the gaussian noise is $\sigma_{\text{noise}} = \frac{\sigma_d}{10}$.

- Below the bifurcation the autocorrelation function of the forecasting errors

$$E_t = \tilde{\sigma}_p^e, t - \tilde{\sigma}_p, t+1$$

converges quickly to zero

- Above the bifurcation the autocorrelation oscillates remaining different from zero and financial institutions might recognize this structure in the autocorrelation function and they might improve their forecast.

→ Adaptive expectations
Adaptive expectations

- The adaptive expectation about the volatility of a typical asset is
  \[
  (\tilde{\sigma}_d^e, t)^2 = (\tilde{\sigma}_d^e, t-1)^2 + \omega \left[ (\tilde{\sigma}_d, t)^2 - (\tilde{\sigma}_d^e, t-1)^2 \right]
  \]
  \[(9)\]
  where \(0 \leq \omega \leq 1\)
- Naive expectations corresponds to \(\omega = 1\).
- An equivalent form for adaptive expectations is
  \[
  (\tilde{\sigma}_d^e, t)^2 = (1 - \omega) (\tilde{\sigma}_d^e, t-1)^2 + \omega (\tilde{\sigma}_d, t)^2
  \]
  \[(10)\]
  that is, expected risk at time \(t\) is a weighted average of the most recently observed risk and the most recently expected one.
- Using recursion
  \[
  (\tilde{\sigma}_d^e, t)^2 = \sum_{i=0}^{\infty} \omega (1 - \omega)^i (\tilde{\sigma}_d, t-i)^2
  \]
  \[(11)\]
  i.e. adaptive expectation is an Exponential Weighted Moving Average of all past observed values of the realized volatility. The weights with which the past realizations of risk are considered, depend on the memory parameter, \(\omega\).
- In financial markets volatility is strongly correlated and moving averages are frequently used to forecast future volatility (e.g. RiskMetrics by JP Morgan)
**Dynamics of leverage**

**Figure:** Dynamics of the diversification $m_t$ and the financial leverage $\lambda_t$ for a fixed value of $\omega$ equal to 0.95 and for two "small" values of the diversification cost, $c = 0.15$ (top) and $c = 0.14$ (bottom). The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$. 
Figure: Bifurcation diagrams of the leverage $\lambda_t$ as function of the parameter $c$ (left) or of the parameter $\alpha$ (right). We have fixed the value of $\omega$ equal to 0.95. The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$. 

Bifurcation diagram: chaos appears!
The role of memory and of noise

Figure: Bifurcation diagrams of the leverage $\lambda_t$ for a fixed value of $c$ equal to 0.14, varying the value of the expectations weight factor $\omega$. The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

A longer memory stabilizes the market
Figure: Largest Lyapunov Exponent (LLE, black circle), the Kolmogorov-Sinai entropy (KS, red circle), the order-two Renyi entropy ($K_2$, blue circle) as a function of the parameter $\omega$. The parameters are fixed at $c = 0.14$, $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

The dynamical behavior of the system is truly chaotic (deterministic chaos)
Conclusions

An analytical model of systemic risk where the bipartite network of investments, illiquidity, and leverage management play a crucial role

- In the baseline model
  - the feedback between investment prices and bank asset induced by portfolio rebalancing leads to a multivariate VAR process whose max eigenvalue depends on the degree of leverage and average illiquidity of the assets;
  - both the variance and correlation of individual investments monotonically increase with reduction in diversification costs;
  - reduction in diversification costs, by increasing the strength and coordination of individual feedbacks, increases the variability of bank total asset.
  - A transition between a stationary and a non stationary VAR is predicted

- In the model with naive expectations
  - A new transition corresponding to a flip bifurcation appears, when the costs or the VaR parameter are small enough
  - 2 period cycles of high and low leverage (diversification) is predicted in the unstable phase

- In the model with adaptive expectations
  - A full bifurcation cascade is observed, as well as a transition to a chaotic phase of the dynamics of leverage or diversification
  - Small changes in the control parameters ($c$ and $\alpha$) can trigger complex dynamics