Spectral Theory of Orthogonal Polynomials

Periodic and Ergodic Spectral Problems

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Lecture 1: Introduction and Overview
Spectral Theory of Orthogonal Polynomials

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- Lecture 2: Szegő Theorem for OPUC
- Lecture 3: Three Kinds of Polynomial Asymptotics
- Lecture 4: Potential Theory
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References


What is spectral theory?

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The *direct problem* goes from the object to spectra. The *inverse problem* goes backwards. The direct problem is typically easy while the inverse problem is typically hard. For example, the domain of definition of the harmonic oscillator isospectral “manifold” is unknown. It is not even known if it is connected!
OPs

Orthogonal polynomials on the real line (OPRL) and on the unit circle (OPUC) are particularly useful because the inverse problems are easy—indeed the inverse problem is the OP definition as we’ll see.
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OPs also enter in many application—both specific polynomials and the general theory.

Indeed, my own interest came from studying discrete Schrödinger operators on $\ell^2(\mathbb{Z})$

$$ (Hu)_n = u_{n+1} + u_{n-1} + Vu_n $$

and the realization that when restricted to $\mathbb{Z}_+$, one had a special case of OPRL.
$\mu$ will be a probability measure on $\mathbb{R}$. We’ll always suppose that $\mu$ has bounded support $[a, b]$ which is not a finite set of points. (We then say that $\mu$ is non-trivial.) This implies that $1, x, x^2, \ldots$ are independent since $\int |P(x)|^2 \ d\mu = 0 \Rightarrow \mu$ is supported on the zeroes of $P$. 
μ will be a probability measure on \( \mathbb{R} \). We’ll always suppose that \( \mu \) has bounded support \([a, b]\) which is not a finite set of points. (We then say that \( \mu \) is non-trivial.) This implies that \( 1, x, x^2, \ldots \) are independent since
\[
\int |P(x)|^2 \, d\mu = 0 \Rightarrow \mu \text{ is supported on the zeroes of } P.
\]
Apply Gram Schmidt to \( 1, x, \ldots \) and get monic polynomials
\[
P_j(x) = x^j + \alpha_{j,1}x^{j-1} + \ldots
\]
and orthonormal (ON) polynomials
\[
p_j = P_j/\|P_j\|
\]
More generally we can do the same for any probability measure of bounded support on $\mathbb{C}$. 
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One difference from the case of $\mathbb{R}$, the linear combination of $\{x^j\}_{j=0}^{\infty}$ are dense in $L^2(\mathbb{R}, d\mu)$ by Weierstrass. This may or may not be true if $\text{supp}(d\mu) \not\subset \mathbb{R}$. 
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If \( d\mu = d\theta/2\pi \) on \( \partial \mathbb{D} \), the span of \( \{z^j\}_{j=0}^{\infty} \) is not dense in \( L^2 \) (but is only \( H^2 \)). Perhaps, surprisingly, we’ll see later that there are measures \( d\mu \) on \( \partial \mathbb{D} \) for which they are dense (e.g., \( \mu \) purely singular).
More generally we can do the same for any probability measure of bounded support on $\mathbb{C}$.

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If $d\mu = d\theta/2\pi$ on $\partial \mathbb{D}$, the span of $\{z^j\}_{j=0}^{\infty}$ is not dense in $L^2$ (but is only $H^2$). Perhaps, surprisingly, we’ll see later that there are measures $d\mu$ on $\partial \mathbb{D}$ for which they are dense (e.g., $\mu$ purely singular).

More significantly, the argument we’ll give for our recursion relation fails if $\text{supp}(d\mu) \not\subset \mathbb{R}$.
Since $P_k$ is monic and $\{P_j\}_{j=0}^{k+1}$ span polynomials of degree at most $k + 1$, we have

$$x P_k = P_{k+1} + \sum_{j=0}^{k} B_{k,j} P_j$$

Clearly

$$B_{k,j} = \langle P_j, x P_k \rangle / \| P_j \|^2$$
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Now we use

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If $j < k - 1$, this is zero.

If $j = k - 1$, $\langle P_{k-1}, xP_k \rangle = \langle xP_{k-1}, P_k \rangle = \|P_k\|^2$. 
Thus \((P_{-1} \equiv 0)\); \(\{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty\) : Jacobi recursion

\[ xP_N = P_{N+1} + b_{N+1}P_N + a_N^2 P_{N-1} \]
Thus \( (P_{-1} \equiv 0) \); \( \{a_j\}_{j=1}^{\infty}, \{b_j\}_{j=1}^{\infty} : \text{Jacobi recursion} \)

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Thus \( (P_{-1} \equiv 0); \{a_j\}_j^\infty, \{b_j\}_j^\infty : \text{Jacobi recursion} \)

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These are called \textit{Jacobi parameters}. This implies \( \|P_N\| = a_N a_{N-1} \ldots a_1 \) (since \( \|P_0\| = 1 \)).
OPRL basics

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These are called \textit{Jacobi parameters}. This implies

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\|P_N\| = a_N a_{N-1} \ldots a_1 \quad \text{(since} \quad \|P_0\| = 1).
\]

This, in turn, implies \(p_n = P_n/a_1 \ldots a_n\) obeys

\[
xp_n = a_{n+1}p_{n+1} + b_n p_n + a_n p_{n-1}
\]
We have thus solved the inverse problem, i.e., $\mu$ is the spectral data and $\{a_n, b_n\}_{n=1}^{\infty}$ are the descriptors of the object.
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In the orthonormal basis $\{p_n\}_{n=0}^{\infty}$, multiplication by $x$ has the matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \ldots \\ a_1 & b_2 & a_2 & 0 & \ldots \\ 0 & a_2 & b_3 & a_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

called a \textit{Jacobi matrix}.
Since

\[ b_n = \int xp_{n-1}^2(x) \, d\mu, \quad a_n = \int xp_{n-1}(x)p_n(x) \, d\mu \]

\[ \text{supp}(\mu) \subset [-R, R] \Rightarrow |b_n| \leq R, \, |a_n| \leq R. \]
Favard’s Theorem

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Conversely, if \( \sup_n (|a_n| + |b_n|) = \alpha < \infty \), \( J \) is a bounded matrix of norm at most \( 3\alpha \). In that case, the spectral theorem implies there is a measure \( d\mu \) so that

\[ \langle (1, 0, \ldots)^t, J^\ell (1, 0, \ldots)^t \rangle = \int x^\ell \, d\mu(x) \]
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If one uses Gram-Schmidt to orthonormalize \( \{ J^\ell (1, 0, \ldots)^t \}_{\ell=0}^\infty \), one finds \( \mu \) has Jacobi matrix exactly given by \( J \).
Favard’s Theorem

We have thus proven Favard’s Theorem (his paper was in 1935; really due to Stieltjes in 1894 or to Stone in 1932).
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**Favard’s Theorem.** *There is a one–one correspondence between bounded Jacobi parameters*  

\[
\{a_n, b_n\}_{n=1}^\infty \in \left((0, \infty) \times \mathbb{R}\right)^\infty
\]

*and non-trivial probability measures, \(\mu\), of bounded support via:*

\[
\mu \Rightarrow \{a_n, b_n\} \quad (OP \ recursion)
\]

\[
\{a_n, b_n\} \Rightarrow \mu \quad (Spectral \ Theorem)
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\[ \{a_n, b_n\} \Rightarrow \mu \quad (Spectral \ Theorem) \]

There are also results for \( \mu \)'s with unbounded support so long as \( \int x^n \, d\mu < \infty \). In this case, \( \{a_n, b_n\} \Rightarrow \mu \) may not be unique because \( J \) may not be essentially self-adjoint on vectors of finite support.
Let $d\mu$ be a non-trivial probability measure on $\partial \mathbb{D}$. As in the OPRL case, we use Gram-Schmidt to define monic OPs, $\Phi_n(z)$ and ON OP’s $\varphi_n(z)$. 
OPUC basics

Let $d\mu$ be a non-trivial probability measure on $\partial \mathbb{D}$. As in the OPRL case, we use Gram-Schmidt to define monic OPs, $\Phi_n(z)$ and ON OP’s $\varphi_n(z)$.

In the OPRL case, if $z$ is multiplication by the underlying variable, $z^* = z$. This is replaced by $z^*z = 1$. 
Let \( d\mu \) be a non-trivial probability measure on \( \partial \mathbb{D} \). As in the OPRL case, we use Gram-Schmidt to define monic OPs, \( \Phi_n(z) \) and ON OP’s \( \varphi_n(z) \).

In the OPRL case, if \( z \) is multiplication by the underlying variable, \( z^* = z \). This is replaced by \( z^*z = 1 \).

In the OPRL case, \( P_{n+1} - xP_n \perp \{1, x_1, \ldots, x_{n-2}\} \).
In the OPUC case, \( \Phi_{n+1} - z\Phi_n \perp \{z, \ldots, z^n\} \), since

\[
\langle z\Phi, z^j \rangle = \langle \Phi, z^{j-1} \rangle
\]

if \( j \geq 1 \).
In the OPUC case, $\Phi_{n+1} - z\Phi_n \perp \{z, \ldots, z^n\}$, since

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In the OPRL case, we used $\deg P = n$ and $P \perp \{1, x, \ldots, x^{n-2}\} \Rightarrow P = c_1 P_n + c_2 P_{n-1}$.

In the OPUC case, we want to characterize $\deg P = n$, $P \perp \{z, z^2, \ldots, z^n\}$. 
Define * on degree $n$ polynomials to themselves by

$$Q^*(z) = z^n Q\left(\frac{1}{\bar{z}}\right)$$

(bad but standard notation!) or

$$Q(z) = \sum_{j=0}^{n} c_j z^j \Rightarrow Q^*(z) = \sum_{j=0}^{n} \bar{c}_{n-j} z^j$$
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Then, * is unitary and so for deg $Q = n$

$$Q \perp \{1, \ldots, z^{n-1}\} \iff Q = c \Phi_n$$

is equivalent to

$$Q \perp \{z, \ldots, z^n\} \iff Q = c \Phi_n^*$$
Thus, we see, there are parameters $\{\alpha_n\}_{n=0}^{\infty}$ (called Verblunsky coefficients) so that

$$\Phi_{n+1}(z) = z\Phi_n - \alpha_n \Phi_n^*(z)$$

This is the Szegő Recursion (History: Szegő and Geronimus in 1939; Verblunsky in 1935–36)
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Applying * for deg \( n + 1 \) polynomials to this yields

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\Phi^*_{n+1}(z) = \Phi^*_n(z) - \alpha_n z \Phi_n
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\]

The strange looking \(-\bar{\alpha}_n\) rather than say \(+\alpha_n\) is to have the \(\alpha_n\) be the Schur parameter of the Schur function of \(\mu\) (Geronimus); also the Verblunsky parameterization then agrees with \(\alpha_n\). These are discussed in [OPUC1].
\( \Phi_n \) monic \( \Rightarrow \) constant term in \( \Phi^*_n \) is 1 \( \Rightarrow \) \( \Phi^*_n(0) = 1 \).
Szegő recursion and Verblunsky coefficients

\[ \Phi_n \text{ monic } \Rightarrow \text{ constant term in } \Phi_n^* \text{ is } 1 \Rightarrow \Phi_n^*(0) = 1. \]

This plus \( \Phi_{n+1} = z\Phi_n - \bar{\alpha}_n \Phi_n^*(z) \) implies

\[ -\Phi_{n+1}(0) = \alpha_n \]

i.e., \( \Phi_n \) determines \( \alpha_{n-1} \).
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$$\Phi_{n+1} \perp \Phi^*_n \Rightarrow \|\Phi_{n+1}\|^2 + |\alpha_n|^2 \|\Phi_n\|^2 = \|z\Phi_n\|^2$$
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Multiplication by $z$ unitary plus $^*$ antiunitary $\Rightarrow$

$$\|\Phi_{n+1}\|^2 = \rho_n^2 \|\Phi_n\|^2; \quad \rho_n^2 = 1 - |\alpha_n|^2$$
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Multiplication by $z$ unitary plus $^*$ antiunitary $\Rightarrow$

$$\|\Phi_{n+1}\|^2 = \rho_n^2 \|\Phi_n\|^2; \quad \rho_n^2 = 1 - |\alpha_n|^2$$

which implies $|\alpha_n| < 1$ (i.e., $\alpha_n \in \mathbb{D}$) and

$$\|\Phi_n\| = \rho_{n-1} \cdots \rho_0$$
Szegő recursion and Verblunsky coefficients

\[
\begin{pmatrix}
\varphi_{n+1} \\
\varphi^*_{n+1}
\end{pmatrix}
= A_n(z)
\begin{pmatrix}
\varphi_n \\
\varphi^*_n
\end{pmatrix}
x; \quad A_n = \rho_n^{-1}
\begin{pmatrix}
z & -\bar{\alpha}_n \\
-\alpha_n z & 1
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\[
\det A_n \neq 0 \text{ if } z \neq 0, \text{ so we can get } \varphi_n (\Phi_n) \text{ from } \varphi_{n+1} (\Phi_{n+1}) \text{ by}
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\[z\Phi_n = \rho_n^{-2} [\Phi_{n+1} + \bar{\alpha}_n \Phi_{n+1}^*]\]
\[\Phi_n^* = \rho_n^{-2} [\Phi_{n+1} + \alpha_n \Phi_{n+1}]\]
We see that \( \Phi_{n+1} \) determines \( \alpha_n \), so by induction and inverse recursion,
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**Theorem (Geronimus-Wendroff Theorem)** *If two measures have the same $\Phi_n$, they have the same $\{\Phi_j\}_{j=0}^{n-1}$ and $\{\alpha_j\}_{j=0}^{n-1}$.***
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A similar argument to the one that led to $|\alpha_n| < 1$ yields

**Theorem** All zeros of $\Phi_n$ lie in $\mathbb{D}$. 
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**Theorem** *All zeros of $\Phi_n$ lie in $\mathbb{D}$.*

**Proof** $\Phi_n(z_0) = 0 \Rightarrow \Phi_n = (z - z_0)p$, $\deg p = n - 1$
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$zp = \Phi_n + z_0p$ and $p \perp \Phi_n \Rightarrow \|p\|^2 = \|\Phi_n\|^2 + |z_0|^2\|p\|^2$
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**Proof** $\Phi_n(z_0) = 0 \Rightarrow \Phi_n = (z - z_0)p$, $\deg p = n - 1$

$zp = \Phi_n + z_0p$ *and* $p \perp \Phi_n \Rightarrow \|p\|^2 = \|\Phi_n\|^2 + |z_0|^2\|p\|^2$

$\Rightarrow |z_0| < 1$
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A similar argument to the one that led to $|\alpha_n| < 1$ yields

**Theorem** *All zeros of $\Phi_n$ lie in $\mathbb{D}$.*

**Proof** $\Phi_n(z_0) = 0 \Rightarrow \Phi_n = (z - z_0)p$, deg $p = n - 1$

$zp = \Phi_n + z_0 p$ and $p \perp \Phi_n \Rightarrow \|p\|^2 = \|\Phi_n\|^2 + |z_0|^2\|p\|^2$

$\Rightarrow |z_0| < 1$

**Corollary.** *All zeros of $\Phi_n^*(z)$ lie in $\mathbb{C} \setminus \overline{\mathbb{D}}$.***
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If $\Phi_{n+1}^{(\beta)} = z\Phi_n + \beta\Phi_n^*$, then at $\beta = 0$, all zeros of $\Phi_{n+1}^{(\beta)}$ are in $\mathbb{D}$. 
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If $\Phi_n^{(\beta)} = z\Phi_n + \beta \Phi_n^*$, then at $\beta = 0$, all zeros of $\Phi_n^{(\beta)}$ are in $\mathbb{D}$.

As $\beta$ varies in $\mathbb{D}$, all zeros of $\Phi_n^{(\beta)}$ are trapped in $\mathbb{D}$. QED.
Bernstein–Szegő Approximation

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**Theorem** (Bernstein–Szegő measures) Let \( \{\alpha_j^{(0)}\}_{j=0}^{n-1} \in \mathbb{D}^n \). Let \( \varphi_n(z) \) be the normalized degree \( n \) polynomial obtained by Szegő recursion. Let

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\frac{d\mu_n(\theta)}{d\theta} = \frac{d\theta}{2\pi |\varphi_n(e^{i\theta})|^2}
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Then \( d\mu_n \) has Verblunsky coefficients

\[
\alpha_j(d\mu_n) = \begin{cases} 
\alpha_j^{(0)} & j = 0, \ldots, n - 1 \\
0 & j \geq n
\end{cases}
\]
The first step of the proof is to show that

\[ k, \ell, n \text{ with } k < n + \ell \Rightarrow \int_{z=e^{i\theta}} \bar{z}^k z^\ell \varphi_n(z) d\mu_n(\theta) = 0 \]
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\[ \oint \frac{\overline{z}^k z^\ell \phi_n(z)}{z^{-n} \phi_n(z) \phi_n^*(z)} \frac{dz}{2\pi i z} = \frac{1}{2\pi} \oint z^{\ell+n-k-1} \frac{dz}{\phi_n^*(z)} \]
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is zero since \([\varphi_n^*(z)]^{-1}\) is analytic on a neighborhood of \(\overline{\mathbb{D}}\) and \(\ell + n - k - 1 \geq 0\).
Thus, \( z^l \varphi_n \) is a multiple of the OP’s for \( d\mu_n \).
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$$\varphi_j(z; d\mu) = \begin{cases} 
\varphi_j(x) & j = 0, \ldots, n \\
z^{j-n} \varphi_n(z) & j = n, n + 1, \ldots
\end{cases}$$

implying the claimed result.
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Given \( \{\alpha_j\}_{j=0}^\infty \subset \mathbb{D}^\infty \), we can form \( d\mu_n \) as above. Via
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\int \Phi_j(e^{i\theta})d\mu(e^{i\theta}) = 0,
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\( \{\Phi_j\}_{j=0}^n \) determines \( \{\int z^j d\mu\}_{j=0}^n \) inductively (actually they determine more moments).
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Thus, \( d\mu_n \) has a weak limit \( d\mu_\infty \). Clearly, \( \alpha_j(d\mu_\infty) = \alpha_j \).
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We have thus proven

**Verblunsky’s Theorem.** \( \mu \rightarrow \{\alpha_j(\mu)\}_{j=0}^{\infty} \) sets up a 1–1 correspondence between non-trivial probability measures on \( \partial \mathbb{D} \) and \( \mathbb{D}^\infty \).
While Verblunsky’s Theorem is an analog of Favard’s theorem, the proofs we’ve given are very different.
CMV Matrices

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In 2001, Cantero, Moral and Velázquez had the lovely idea of orthonormalizing \( \{ 1, z, z^{-1}, z^2, z^{-2}, \ldots \} \) which always produces a complete set. Remarkably, they found that this ON basis can be expressed in terms of suitable \( z^\ell \varphi_k \) and \( z^\ell \varphi_k^* \).
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One can use the CMV matrix for a different proof of Verblunsky’s theorem. Given $\{\alpha_j\}_{j=1}^{\infty}$, one can form the CMV matrix which is unitary and apply the spectral theorem to get a spectral measure which one shows has the right Verblunsky coefficients. Details can be found in [OPUC1].
Simon [CRM Proc. and Lecture Notes 42 (2007), 453–463] has proven an analog of the Bernstein–Szegő approximation for OPRL (the analog for Schrödinger operators is due to Carmona; hence the name):
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Let \( d\rho \) be a probability measure on \( \mathbb{R} \) with \( \int |x|^n \, d\rho < \infty \) for all \( n \). Let \( \{p_n\}_{n=0}^{\infty} \) be its orthonormal polynomials and \( \{a_n, b_n\}_{n=1}^{\infty} \) its Jacobi parameters. Let

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One strange aspect of this formula is that for $\ell > 2n - 1$, the moments of $d\nu_n$ are infinite!
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**Theorem.** If $I \subset \mathbb{R}$ is an interval and for all $x \in I$ and some $c > 0$, we have that

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**Remark.** A much stronger result is known (see e.g., Simon [Proc AMS 124 (1996), 3361-3369]); $I$ can be any set and $c$ can be $x$-dependent.