Spectral Theory of Orthogonal Polynomials

Periodic and Ergodic Spectral Problems

Issac Newton Institute, January, 2015

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Lecture 2: Szegő Theorem for OPUC

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Szegő's Theorem as a Variational Principle

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In these papers, Szegő assumed $d\mu$ was purely a.c. The addition of a singular continuous part is a discovery of Verblunsky in 1934–35 but his work was largely ignored and he didn't get credit until about fifteen years ago when, in a different context, Killip and Simon rediscovered his proof and then his paper.

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Φ_n has a variational form. Since $\Phi_n =$ Projection of z^n onto the orthogonal complement of $\{1, \ldots, z^{n-1}\}$,

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This implies $\|\Phi_{n+1}\| \leq \|\Phi_n\|$ which is obvious from $\|\Phi_n\| = \rho_0\, \rho_1 \dots \rho_{n-1}$ and $\rho_j \leq 1$.

Thus, clearly, $\lim_{n\to\infty}\|\Phi_n\|$ exists and

$$\lim_{n\to\infty}\|\Phi_n\| = \inf\{\|P\| \mid P(0)=1, P \text{ is a polynomial } \}$$

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$$d\mu = f(\theta)\,\frac{d\theta}{2\pi} + d\mu_s$$

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Before turning to the proof, we consider some equivalent forms and consequences.

Szegő's Theorem as a Sum Rule

As we've seen, $\|\Phi_n\| = \rho_1 \dots \rho_{n-1}$ so

$$\lim \|\Phi_n\|^2 = \prod_{j=0}^{\infty} \left(1 - |\alpha_j|^2\right)$$

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Szegő Theorem (Sum Rule Version). *If* $d\mu = f(\theta)\frac{d\theta}{2\pi} + d\mu_s$, *then*

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This is a precursor of KdV sum rules. It is clearly equivalent to the variational form.

Szegő's Theorem as a Sum Rule

Corollary. $\sum_{j=0}^{\infty} |\alpha_j|^2 < \infty \Leftrightarrow \int \log\bigl(f(\theta)\bigr) \frac{d\theta}{2\pi} > -\infty.$

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A consequence of this is that $d\mu_s$ can be more or less arbitrary while one still has $\sum_{j=0}^{\infty} |\alpha_j|^2 < \infty$; for example, if $\int d\mu_s = \eta < 1$, $(1-\eta)\frac{d\theta}{2\pi} + d\mu_s = d\mu$ has $\sum_{j=0}^{\infty} |\alpha_j(\mu)|^2 < \infty$.

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This is remarkable because one can show that $\sum_{j=0}^{\infty}|\alpha_j| < \infty \Rightarrow d\mu$ is purely a.c. and $\varepsilon < |f(\theta)| < \varepsilon^{-1}$ for some $\varepsilon > 0$ and all θ.

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It is also remarkable because it is not easy to construct operators with mixed spectrum and potential decay.

Given $\{c_n\}_{n=-\infty}^{\infty}$, the corresponding $N \times N$ Toeplitz matrix $T_N(c)$ has the form

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{N-1} \\ c_{-1} & c_0 & \cdots & c_N \\ \vdots & & \ddots & \vdots \\ c_{-N+1} & c_{-N+2} & \cdots & c_0 \end{pmatrix}$$

i.e., $\big(T_N(c)\big)_{ij} = c_{j-i}$.

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i.e., $\big(T_N(c)\big)_{ij} = c_{j-i}$. If μ is a measure, we set $c_j = \int e^{-ij\theta} d\mu(\theta)$ and write (μ is called the *symbol*)

$$D_N(\mu) = \det\big(T^{N+1}(\mu)\big)$$

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$$D_N(\mu) = \det\big(\langle \Phi_j, \Phi_k \rangle\big)_{0 \le j,\, k \le N}$$

$$= \|\Phi_0\|^2 \cdots \|\Phi_N\|^2$$

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Thus,

Toeplitz Determinant Form of Szegő's Theorem. *For any μ,*

$$\lim_{N\to\infty} \frac{1}{N+1} \log D_N(\mu) = \int \log f(\theta) \frac{d\theta}{2\pi}$$

Aside: It is known that if $d\mu_s = 0$ and

$$\log\bigl(f(\theta)\bigr) \equiv \sum_{n=-\infty}^{\infty} \widehat{L}_n \, e^{in\theta}$$

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This is the Strong Szegő Theorem. [OPUC1], Chap. 6 has many proofs of this.

When are Polynomials Dense in $L^2(\partial\mathbb{D}, d\mu)$?

By Weierstrass' Theorem, for any μ of compact support on $\mathbb{R}$, the polynomials in x are dense in $L^2(\mathbb{R}, d\mu)$.

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Theorem (Kolmogorov-Krein). *If* $d\mu = f\,\frac{d\theta}{2\pi} + d\mu_s$*, then the polynomials in* z *are dense in* $L^2(\partial\mathbb{D}, d\mu)$ *if and only if* $\int \log f(e^{i\theta})\,\frac{d\theta}{2\pi} = -\infty$.

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$$= \inf_{Q|Q(0)=1} \|Q\|^2_{L^2} = \exp\Big(\int \log f \frac{d\theta}{2\pi}\Big)$$

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On the other hand, if $z^{-1} = \lim P_n$, then $z^{-2} = \lim_{n\to\infty} P_n \big[\lim_{m\uparrow\infty} P_m\big]$ so all polynomials in z and z^{-1} are in closure of polys and they are dense (by Weierstrass' other density theory).

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Krein used this to show (see [SzThm], p. 319) that on $\mathbb{R}$, if $d\rho = F\,dx + d\rho_\nu$, then $\{e^{i\alpha x}\}_{\alpha\geq 0}$ are dense in $L^2 \Leftrightarrow \int \frac{\log F(x)}{1+x^2}\,dx = -\infty$.

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$$d\rho(x) = e^{-|x|^\alpha}\,dx, \quad \alpha < 1$$

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The proof that $\lim_{n\to\infty} \|\Phi_n^*\|^2$ is an upper bound will be variational. We'll show that for any polynomial with $P(0) = 1$, we have $\|P\|^2 \geq$ entropy term.

Strategy of the Proof

The lower bound on the entropy term will come from the fact that $\mu \mapsto$ entropy term is weakly upper-semicontinuous (usc), i.e., $\mu_n \to \mu \Rightarrow S(\mu) \geq \limsup S(\mu_n)$.

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We'll prove that $S(\mu) = \prod_{j=0}^{\infty} \bigl(1 - |\alpha_j|^2\bigr)^{1/2}$ for Bernstein–Szego measures by direct calculation and then use this and usc to get the other inequality.

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Proof. If $\{z_j\}_{j=1}^m$ are zeros in $\mathbb{D}$, let

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Then $\log Q(z)$ is analytic in $\mathbb{D}$, so

Upper Bound

$$\log|Q(0)| = \lim_{r\uparrow 1} \int \log|Q(re^{i\theta})| \frac{d\theta}{2\pi} = \int \log|Q(e^{i\theta})| \frac{d\theta}{2\pi}$$

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$$= \int \log\left|P(e^{i\theta})\right|\frac{d\theta}{2\pi}$$

But, $|Q(0)| = \prod_{j=1}^{m}|z_j|^{-1}\,|P(0)| \geq |P(0)|$.

Upper Bound

For any polynomial, P, with $P(0) \neq 0$, $d\mu = f\,\frac{d\theta}{2\pi} + d\mu_s$, we have

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$$\geq |P(0)|^2 \exp\biggl(\int \log|f(\theta)|\frac{d\theta}{2\pi}\biggr)$$

by the Lemma. Thus

$$\inf_{P|P(0)=1} \int |P(e^{i\theta})|^2\,d\mu \geq \exp\bigl(\int \log\bigl(f(\theta)\bigr)\bigr)\,\frac{d\theta}{2\pi}$$

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If $d\mu_s = 0$, we have $P(z) = D(0)/D(z)$ has $P(0) = 0$ and

$$\int |P(z)|^2\, d\mu = D(0)^2 \int f(\theta)^{-2}\left[f(\theta)\frac{d\theta}{2\pi}\right] = D(0)^2$$
$$= \exp\left(\int \log(f(0))\frac{d\theta}{2\pi}\right)$$

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Suppose $\alpha_j = 0$ for $j \geq N$. Then, we've seen that

$$d\mu = f(\theta)\frac{d\theta}{2\pi}, \quad f(\theta) = |\varphi_N^*(e^{i\theta})|^{-2}$$

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proving Szegő's Theorem in this case.

The Szegő Integral as an Entropy

Given two prob. measures on $\partial\mathbb{D}$, we define their relative entropy by

$$S(\mu \mid \nu) = \begin{cases} -\infty & \text{if } \mu \text{ is not } \nu\text{-a.c.} \\ -\int \log\bigl(\frac{d\mu}{d\nu}\bigr) d\mu & \text{if } \mu \text{ is } \nu\text{-a.c.} \end{cases}$$

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Usually ν is fixed and we vary μ.

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We claim that

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For μ is ν-a.c. iff $f(\theta)\neq 0$ for $\frac{d\theta}{2\pi}$-a.e. θ. If $f(\theta)=0$ on a positive Lebesgue measure set, the integral is $-\infty$, so both sides are $-\infty$.

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If $f(\theta)\neq 0$ for a.e. θ, $\frac{d\mu}{d\nu}=f^{-1}\chi_S$ where χ_S is a set with $d\mu_s(S)=0$ and $|S|=1$.

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If $f(\theta) \neq 0$ for a.e. θ, $\frac{d\mu}{d\nu} = f^{-1}\chi_S$ where χ_S is a set with $d\mu_s(S) = 0$ and $|S| = 1$. Clearly

$$-\int \log\left(\frac{d\mu}{d\nu}\right) = \int \log\big(f(\theta)\big)\frac{d\theta}{2\pi}$$

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Sketch. If $d\mu = gd\nu$ with $g \in \mathcal{E}(\partial\mathbb{D})$, then

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By an approximation argument (and control of $d\mu_s$) one obtains

$$S(\mu \mid \nu) \geq \inf \mathcal{S}$$

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Thus

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Theorem. *$S(\mu \mid \nu)$ is jointly converse and jointly weakly usc in μ and ν.*

Corollary. *Define $Sz(\mu) = \int \log f \frac{d\theta}{2\pi}$ if $d\mu = f \frac{d\theta}{2\pi} + d\mu_s$. Then $\mu \mapsto Sz(\mu)$ is weakly usc.*

The end of the proof

Let μ have Verblunsky coefficients, $\{\alpha_n\}_{n=0}^{\infty}$. Let μ_n be the Bernstein–Szegő approximation.

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By weak usc

$$Sz(\mu) \geq \overline{\lim}\, Sz(\mu_n) = \prod_{j=0}^{\infty} \rho_j^2$$

which is the other inequality that we needed to prove.