Spectral Theory of Orthogonal Polynomials

Periodic and Ergodic Spectral Problems

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Barry Simon
IBM Professor of Mathematics and Theoretical Physics
California Institute of Technology
Pasadena, CA, U.S.A.

Lecture 5: Isospectral Tori
Spectral Theory of Orthogonal Polynomials

- Lecture 1: Introduction and Overview
- Lecture 2: Szegő Theorem for OPUC
- Lecture 3: Three Kinds of Polynomial Asymptotics
- Lecture 4: Potential Theory
- Lecture 5: Isospectral Tori
- Lecture 6: Fuchsian Groups
- Lecture 7: Chebyshev Polynomials, I
- Lecture 8: Chebyshev Polynomials, II


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So \( \{a_n, b_n\}_{n=-\infty}^{\infty} \) are two-sided sequences with some \( p > 0 \) in \( \mathbb{Z} \) so that

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a_{n+p} = a_n \quad b_{n+p} = b_n
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For \( z \in \mathbb{C} \) fixed, we are interested in solutions \( \{u_n\}_{n=0}^{\infty} \) of

\[
a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1} = zu_n
\]
that also obey for some $\lambda \in \mathbb{C}$ ($\lambda = e^{i\theta}, \theta \in \mathbb{C}$)

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The (twisted) periodic boundary condition Jacobi matrix $J_{\text{per}, \lambda}$ is $p \times p$. 

Conversely, if $\tilde{u}$ solves this, the unique $u$ with $u_{n+p} = \lambda u_n$ and $\tilde{u}$ is a Floquet solution.
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If $\lambda \neq \pm 1$, then $\bar{\lambda} \neq \lambda$. If $u$ is a Floquet solution for $\lambda$, since $z$ is real, $\bar{u}$ is a Floquet solution for $\bar{\lambda}$ so there is a unique solution for that $z$. 
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$$T_p(z) \begin{pmatrix} a_1 \\ a_0 \\ u_0 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ a_0 \\ u_0 \end{pmatrix}$$

is equivalent to $\begin{pmatrix} u_1 \\ a_0 \\ u_0 \end{pmatrix}$ generating a Floquet solution!
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In terms of the OP’s for $\{a_n, b_n\}_{n=1}^{\infty}$,

$$T_p(z) = \begin{pmatrix} p_p(z) & -q_p(z) \\ a_p p_{p-1}(z) & -a_p q_{p-1}(z) \end{pmatrix}$$
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\[
\Delta(z) = \text{Tr}(T_p(z)) = p_p(z) - a_pq_{p-1}(z)
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is a (real) polynomial of degree exactly \( p \).
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Floquet solutions correspond to geometric eigenvalues for $T_p(z)$. If $\lambda \neq \pm 1$, it has multiplicity one, so is geometric. $\lambda = \pm 1$ has multiplicity 2, so there can be one or two Floquet solutions.
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An important consequence of the fact that $\Delta(z) \in (-2, 2)$ implies all $z$’s are real is $\Delta^{-1}((-2, 2)) \subset \mathbb{R}$. 
The Discriminant

A basic fact of analytic functions is that if \( f(z) \) is real (i.e., \( f(\bar{z}) = \overline{f(z)} \)), \( x_0 \in \mathbb{R} \) with \( f'(x_0) = 0 \), there are non-real \( z \)'s near \( x_0 \) with \( f(z) \) real and near \( f(x_0) \).
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Could be orientation reversing or not.
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If $\beta_{j-1} = \alpha_j$, there is one less point where $\Delta(x) = (-1)^{p-j-1}2$, but $\Delta'(\alpha_j) = 0$ since $\Delta - (-1)^{p-j-1}2$ has the same sign on both sides of $\alpha_j$. It follows that
Open and Closed Gaps

Theorem. \( \Delta^{-1}([-2, 2]) = \bigcup_{j=1}^{p} [\alpha_j, \beta_j] \) and

\[ \Delta'\left(\beta_j\right) = 0 \iff \beta_j = \alpha_j + 1 \]

and in that case, \( \Delta'' \) is not zero at that point.

The \([\alpha_j, \beta_j] \) are called the bands and \((\beta_j, \alpha_j+1)\) the gaps.

If \( \beta_j < \alpha_j + 1 \), we say that gap \( j \) is open.

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The \([\alpha_j, \beta_j]\) are called the *bands* and \((\beta_j, \alpha_{j+1})\) the *gaps*. If \(\beta_j < \alpha_{j+1}\), we say that gap \(j\) is open.
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Each of the gaps where $\Delta(x) \geq 2$ has two periodic solutions—either two at $\beta_j = \alpha_{j+1}$ or one each at $\beta_j$ and $\alpha_{j+1}$ so there are $p$ periodic Floquet solutions, as there must be from the $J_{\text{per}}$ analysis.
If \( z \) is such that \( \Delta(z) \not\in [-2, 2] \), then the roots of \( \lambda + \lambda^{-1} = \Delta(z) \) have \( |\lambda| > 1 \), \( |\lambda^{-1}| < 1 \).
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$$G_{nm}(z) = u^+_{\max(n,m)}(z)u^-_{\min(m,n)}(z)/W(z)$$

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The continuity implies $d\mu(n)$ is purely a.c., so we have proven Theorem. A periodic two-sided Jacobi matrix has purely absolutely continuous spectrum. One can write out an explicit spectral representation with Floquet solutions with $z \in (\alpha_j, \beta_j)$ as continuum eigenfunctions.
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That constant is determined by $\alpha_p$ when $\Delta$ is $-2$. Thus, $\beta_p, \alpha_{p-1}, \beta_{p-2}$ plus $\alpha_p$ determine the remaining $p - 1$ $\alpha$'s and $\beta$'s.
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The answer will lie in potential theory.
For any \( z \in \mathbb{C} \), there are two Floquet indices, \( \lambda_\pm \), solving
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\gamma(z) = \lim_{n \to \infty} \frac{1}{n} \log ||T_n(\lambda)|| = \frac{1}{p} \log |\lambda_+(z)|
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and $\gamma(z) = \log (|z|) + O(1)$ at $\infty$, since $\Delta(z)$ is a degree $p$ polynomial.
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**Theorem.** $\gamma(z)$ as given above is the potential theorists’ Green’s function and periodic Jacobi parameters are associated to regular measures (in the Stahl–Totik sense).

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By general principles, if $G_\epsilon$ is smooth up to $\epsilon$ on $\epsilon^{\text{int}}$, the equilibrium measure $d\rho_\epsilon(x) = f_\epsilon(x)\,dx$ where

$$f_\epsilon(x) = \frac{1}{\pi} \frac{\partial}{\partial y} G_\epsilon(x + iy) \big|_{y=0}$$
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Thus, the equilibrium measure is

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f_\epsilon(z) = \frac{1}{p\pi} \frac{|\Delta'(x)|}{\sqrt{4 - \Delta^2(x)}} = \frac{1}{p\pi} \left| \frac{d}{dx} \arccos\left(\frac{\Delta(x)}{2}\right) \right|
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This is also a density of zeros way of understanding why the above $f_{\epsilon}$ is the DOS. For the periodic eigenfunctions with a box of size $kp$ are the Floquet solutions with $\lambda = e^{2\pi ij/k}$, $j = 0, 1, 2, \ldots, k - 1$. 
Weyl Solutions

An important property of second kind OPRL, sometimes used as the definition is that for \( n \geq 0 \),

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q_n(x) = \int \frac{p_n(x) - p_n(y)}{x - y} d\mu(y)
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m_\mu(z) = \int \frac{d\mu(x)}{x - z}
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If $\inf a_n > 0$, constancy of the Wronskian shows this is the unique $\ell^2$-solution.
In terms of initial data for \( (u_1, a_0, u_0) \), the Weyl solution has initial data \( (m(z), -1) \).
In terms of initial data for \( (\frac{u_1}{a_0 u_0}) \), the Weyl solution has initial data \( (m(z)) \). Thus if \( m_1 \) is the m-function for \( \{a_{n+1}, b_{n+1}\}_{n=1}^{\infty} \), we have that, for a constant, \( c(z) \),

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\begin{pmatrix}
 m(z) \\
 -1
\end{pmatrix} = c(z)
\begin{pmatrix}
 m_1(z) \\
 -1
\end{pmatrix}
\]

which leads to the recursion relation:

\[
m(z) = \frac{1}{b_1 - z - a_1^2 m_1(z)}
\]
In terms of initial data for \((\frac{u_1}{a_0 u_0})\), the Weyl solution has initial data \((m(z))\). Thus if \(m_1\) is the m-function for \(\{a_{n+1}, b_{n+1}\}_{n=1}^{\infty}\), we have that, for a constant, \(c(z)\),

\[
\begin{pmatrix}
z - b_1 & -1 \\
a_1^2 & 0
\end{pmatrix}
\begin{pmatrix}
m(z) \\ -1
\end{pmatrix}
= c(z)
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\]

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m(z) = \frac{1}{b_1 - z - a_1^2 m_1(z)}
\]

which upon iterations yields the continued fraction of Jacobi, Markov and Stieltjes:

\[
m(z) = \frac{1}{b_1 - z - \frac{a_1^2}{b_2 - z - \frac{a_2^2}{b_3 - z - \ldots}}}
\]
In the period $p$ case, stripping $p$ times leave $J$ invariant.
In the period $p$ case, stripping $p$ times leave $J$ invariant so $m$ must obey:

$$
\begin{pmatrix}
    p_p(z) & -q_p(z) \\
    a_p p_{p-1}(z) & -a_p q_{p-1}(z)
\end{pmatrix}
\begin{pmatrix}
    m(z) \\
    -1
\end{pmatrix}
= 
\begin{pmatrix}
    m(z) \\
    -1
\end{pmatrix}
$$

This is reminiscent of the results of Legendre and Galois on numeric continued fractions.
In the period $p$ case, stripping $p$ times leave $J$ invariant so $m$ must obey:

$$\begin{pmatrix} p_p(z) & -q_p(z) \\ a_pp_{p-1}(z) & -a_pq_{p-1}(z) \end{pmatrix}\begin{pmatrix} m(z) \\ -1 \end{pmatrix} = \begin{pmatrix} m(z) \\ -1 \end{pmatrix}$$

which leads to the quadratic equation

$$\alpha(z)m(z)^2 + \beta(z)m(z) + \gamma(z) = 0$$
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\alpha(z)m(z)^2 + \beta(z)m(z) + \gamma(z) = 0 \\
\alpha(z) = a_p p_{p-1}(z) \\
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This is reminiscent of the results of Legendre and Galois on numeric continued fractions.

By a direct calculation, the two discriminants are related by

$$
\beta^2 - 4\alpha\gamma = \sqrt{\Delta(z)^2} - 4
$$
We have thus proven that the m-function of a periodic Jacobi matrix has a continuation as a meromorphic function on the two sheeted Riemann surface of $\sqrt{\Delta(z)^2 - 4}$.
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We have thus proven that the m-function of a periodic Jacobi matrix has a continuation as a meromorphic function on the two sheeted Riemann surface of $\sqrt{\Delta(z)^2 - 4}$, $S$, including points at infinity. At closed gaps $\Delta(z)^2 - 4$ has double zeros so there are only branch points at the ends of open gaps and the surface is of genus $\ell$ where $\ell$ is the number of open gaps.

Further analysis shows any meromorphic function which is different on the two sheets has degree at least $\ell + 1$. 
We have thus proven that the m-function of a periodic Jacobi matrix has a continuation as a meromorphic function on the two sheeted Riemann surface of $\sqrt{\Delta(z)^2 - 4}$, $S$, including points at infinity. At closed gaps $\Delta(z)^2 - 4$ has double zeros so there are only branch points at the ends of open gaps and the surface is of genus $\ell$ where $\ell$ is the number of open gaps.

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Further analysis shows any meromorphic function which is different on the two sheets has degree at least $\ell + 1$. Moreover the m-function has a pole at $\infty$ on the “top” sheet and in order that Im m not change sign over a gap, on the two sheeted surface, m must have a pole in each of the \( \ell \) gaps.
We have thus proven that the m-function of a periodic Jacobi matrix has a continuation as a meromorphic function on the two sheeted Riemann surface of $\sqrt{\Delta(z)^2 - 4}$, $S$, including points at infinity. At closed gaps $\Delta(z)^2 - 4$ has double zeros so there are only branch points at the ends of open gaps and the surface is of genus $\ell$ where $\ell$ is the number of open gaps.

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Those points on the surface which “project down” to a gap, are a circle
We have thus proven that the m-function of a periodic Jacobi matrix has a continuation as a meromorphic function on the two sheeted Riemann surface of $\sqrt{\Delta(z)^2 - 4}, S$, including points at infinity. At closed gaps $\Delta(z)^2 - 4$ has double zeros so there are only branch points at the ends of open gaps and the surface is of genus $\ell$ where $\ell$ is the number of open gaps.

Further analysis shows any meromorphic function which is different on the two sheets has degree at least $\ell + 1$. Moreover the m-function has a pole at $\infty$ on the “top” sheet and in order that $\operatorname{Im} m$ not change sign over a gap, on the two sheeted surface, $m$ must have a pole in each of the $\ell$ gaps.

Those points on the surface which “project down” to a gap, are a circle - two intervals glued together at the ends.
There is thus a map from the m-function to the set of points on the product over the gaps of the points on the surface that project down to that gap, i.e. onto a $\ell$-dimensional torus.
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There is thus a map from the m-function to the set of points on the product over the gaps of the points on the surface that project down to that gap, i.e. onto a $\ell$-dimensional torus. Moreover, if we consider the map as defined on all periodic Jacobi matrices with a given discriminant, $\Delta$, one can prove with some effort that this map is a bijection. Thus, this set is a torus of dimensions $\ell$, called the \textit{isospectral torus}. 
There is thus a map from the $m$-function to the set of points on the product over the gaps of the points on the surface that project down to that gap, i.e. onto a $\ell$-dimensional torus. Moreover, if we consider the map as defined on all periodic Jacobi matrices with a given discriminant, $\Delta$, one can prove with some effort that this map is a bijection. Thus, this set is a torus of dimensions $\ell$, called the isospectral torus.

The possible $m$-functions are thus precisely the Herglotz functions (i.e. analytic functions in the upper half plane with positive imaginary part) that have a meromorphic continuation to the Riemann surface of $\sqrt{\Delta(z)^2 - 4}$ that have minimal degree and which are normalized to look like $-1/z$ near $\infty$. Christiansen, Simon and Zinchenko call these minimal Herglotz functions.
Now consider a general compact subset $e \subset \mathbb{R}$ which has $\ell + 1$ components so its complement in $\mathbb{R}$ has $\ell$ gaps.
Now consider a general compact subset $e \subset \mathbb{R}$ which has $\ell + 1$ components so its complement in $\mathbb{R}$ has $\ell$ gaps. Thus

$$e = [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2] \cup \ldots \cup [\alpha_{\ell+1}, \beta_{\ell+1}]$$
Now consider a general compact subset $\epsilon \subset \mathbb{R}$ which has $\ell + 1$ components so its complement in $\mathbb{R}$ has $\ell$ gaps. Thus

$$\epsilon = [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2] \cup \ldots \cup [\alpha_{\ell+1}, \beta_{\ell+1}]$$

where the meaning of the $\alpha$’s and $\beta$’s has changed subtly from the prior notation when we have a periodic problem with closed gaps.
Finite Gap Sets

Now consider a general compact subset $e \subset \mathbb{R}$ which has $\ell + 1$ components so its complement in $\mathbb{R}$ has $\ell$ gaps. Thus

$$e = [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2] \cup \ldots \cup [\alpha_{\ell+1}, \beta_{\ell+1}]$$

where the meaning of the $\alpha$’s and $\beta$’s has changed subtly from the prior notation when we have a periodic problem with closed gaps.

The same method which we didn’t describe that constructs minimal Herglotz functions in the periodic case lets us do the same for the Reimann surface of

$$\sqrt{\prod_{j=1}^{\ell+1} (z - \alpha_j)(z - \beta_j)}$$
Almost Periodic Isospectral Torus

There is again an $\ell$ dimensional torus of half line Jacobi matrices, each of them almost periodic with essential spectrum exactly equal to $\epsilon$. 
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There is again an $\ell$ dimensional torus of half line Jacobi matrices, each of them almost periodic with essential spectrum exactly equal to $\epsilon$. The frequency spectrum of the almost periodic function is generated by the harmonic measures of the intervals, i.e. $\rho_\epsilon([\alpha_j, \beta_j])$. The result is thus

**Theorem** Every finite gap set, $\epsilon$ has an isospectral torus of almost periodic Jacobi matrices associated to it.
Almost Periodic Isospectral Torus

There is again an \( \ell \) dimensional torus of half line Jacobi matrices, each of them almost periodic with essential spectrum exactly equal to \( \varepsilon \). The frequency spectrum of the almost periodic function is generated by the harmonic measures of the intervals, i.e. \( \rho_\varepsilon([\alpha_j, \beta_j]) \). The result is thus

**Theorem** Every finite gap set, \( \varepsilon \) has an isospectral torus of almost periodic Jacobi matrices associated to it. These are all periodic with period \( p \) if and only if each band has harmonic measure \( \frac{j}{p} \) for \( j \in \{1, \ldots, p - 1\} \).