Spectral Theory of Orthogonal Polynomials

Periodic and Ergodic Spectral Problems

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Lecture 6: Fuchsian Groups

- Lecture 1: Introduction and Overview
- Lecture 2: Szegő Theorem for OPUC
- Lecture 3: Three Kinds of Polynomial Asymptotics
- Lecture 4: Potential Theory
- Lecture 5: Isospectral Tori
- Lecture 6: Fuchsian Groups
- Lecture 7: Chebyshev Polynomials, I
- Lecture 8: Chebyshev Polynomials, II

References

[OPUC] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series **54.1**, American Mathematical Society, Providence, RI, 2005.

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

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Here, the Joukowski map $z \mapsto z + z^{-1}$ enters as the Riemann map of $(\mathbb{C} \cup \{\infty\}) \setminus \mathbb{D}$ to $(\mathbb{C} \cup \{\infty\}) \setminus [-2,2]$ that takes ∞ to ∞ and $z \mapsto \frac{z}{2} + \sqrt{\frac{z^2}{4} - 1}$ is its inverse.

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We also note that for $\mathfrak{e} = [-2, 2]$ we have that $|B(z)| = \exp(G_{\mathfrak{e}}(z))$ making Szegő asymptotics consistent with regularity.

Multivalued Functions

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What determines the phase of $B(z)$? Since $G_{\mathfrak{e}}(z)$ is harmonic, it has a local harmonic conjugate so, *locally*, we can define the phase to make B analytic. Thus we can analytically continue along any curve. But there is no reason to expect the local phase is preserved after continuation along a closed curve. Indeed, it is not hard to see that if one loops around a subset $\mathfrak{e}_1 \subset \mathfrak{e}$, the phase changes by $2\pi i \rho_{\mathfrak{e}}(\mathfrak{e}_1)$.

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A function like $B(z)$ in the general case is called *character automorphic*. When $\mathfrak{e}$ is an ℓ-gap set, the family of possible characters is an ℓ dimensional torus. Typically, the characters of $\{B(z)^n\}_{n=1}^{\infty}$ are dense in this torus.

Widom's Surmise

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It follows that $p_n(z)B(z)^{-n}$ is asymptotically almost periodic.

Some History

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This Fuchsian group approach to finite gap problems is due to Sodin–Yuditskii [J. Geom. Anal. **7** (1997), 387–435] and developed to get Szegő asymptotics by Peherstorfer–Yuditskii [J. Anal. Math. **89** (2003), 113-154].

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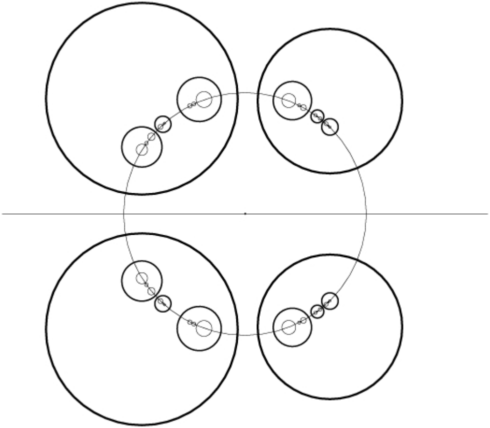
A Pretty Graphic

One goal will be explain how the following picture is associated to a finite gap $\ell = 2$ problem:



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As we saw, if $\ell \geq 1$, $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ isn't simply connected and we want to lift functions to its universal cover.

Any Riemann surface has a universal cover which is also a Riemann surface since the local analytic structure "below" lifts. The uniformization theorem says this universal cover is $\mathbb{D}$ except for a few special cases of the underlying surface: $\mathbb{C} \cup \{\infty\}$, $\mathbb{C}$, a torus, $\mathbb{C} \setminus \{0\}$.

So there is a covering map, $\mathbf{x}(z)$, from $\mathbb{D}$ to $(\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e}$ which is many to one. As with any covering map, there is a discrete group of transformations which in this case preserve the complex structure so are Möbius transformations of $\mathbb{D}$ to $\mathbb{D}$.Thus, there is a discrete group of Möbius transformations (aka Fuchsian group), Γ, so that $\mathbf{x}(\gamma(z)) = \mathbf{x}(z)$. Indeed, $\mathbf{x}(z) = \mathbf{x}(w) \Leftrightarrow \exists\, \gamma \in \Gamma$ with $\gamma(z) = w$.

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We normalize $\mathbf{x}$ by demanding $\mathbf{x}(0) = \infty$ and $\lim_{z \to 0, z \neq 0} z\,\mathbf{x}(z) > 0$. This implies $\mathbf{x}$ maps the region in $\mathbb{D} \cap \mathbb{C}_+$ just above $\mathbb{R}$ to $\mathbb{C}_-$.

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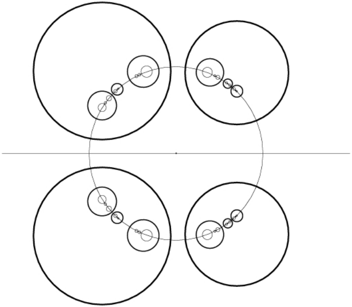

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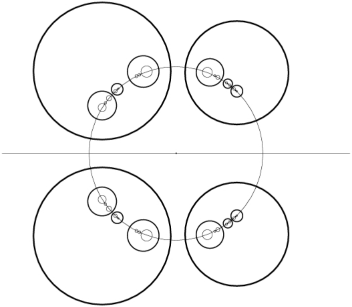

We have 2 gaps and 3 bands and we can understand the fundamental domain.

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which is a conjugate Möbius transform with $R_j \infty = z_j$; R_j leaves the orthocircle pointwise fixed.

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It is not hard to show that Γ is generated by the γ_j's.

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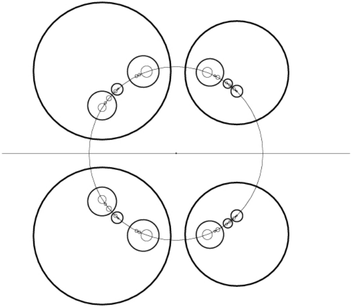

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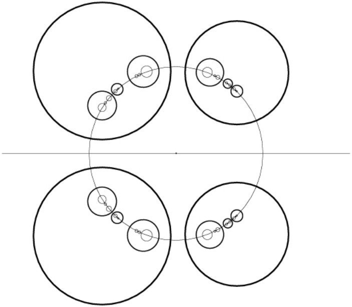

The second generation circles inside one of the first generation circles are exactly the image of the three other first generation circles, etc.

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Indeed, $\mathbf{x}$ has a meromorphic continuation to $\mathbb{C} \cup \{\infty\} \setminus \Lambda$. By mapping $\mathbb{C} \setminus \overline{\mathbb{D}}$ to $\mathcal{S}_-$, the lower sheet of the Riemann surface of $\sqrt{\prod_{j=1}^{\ell+1}(z - \alpha_j)(z - \beta_j)}$, one sees this extended $\mathbf{x}$ is essentially a covering map of $\mathcal{S}$.

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- For some $s < 1$, we have

$$\sum_{\gamma\in\Gamma} (1 - |\gamma(z)|)^s < \infty \quad \text{for all } z \in \mathbb{D}$$

Beardon's Theorem

A. F. Beardon (Acta Math. **127** (1971), 221–258) proved an important result about certain finitely generated Fuchsian groups that include the ones associated to finite gap sets. It has all of the following consequences:

- The set of limit points of the orbit $\{\gamma(0) \mid \gamma \in \Gamma\}$ (which is the same as the limit points of $\{\gamma(z) \mid \gamma \in \Gamma\}$ for any $z \in \mathbb{D}$) has Hausdorff dimension strictly less than 1.
- If $\mathcal{R}_k$ is the union of the interiors of all $2\ell(2\ell-1)^{k-1}$ orthocircles at generation k, and $\partial\mathcal{R}_k = \partial\mathbb{D} \cap \mathcal{R}_k$ and $|\cdot|$ is $d\theta/2\pi$ measure, then $|\partial\mathcal{R}_k| \leq C_0 e^{-C_1 k}$.
- For some $s < 1$, we have

$$\sum_{\gamma\in\Gamma} (1 - |\gamma(z)|)^s < \infty \quad \text{for all } z \in \mathbb{D}$$

so, in particular, this holds for $s = 1$.

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$b_{\gamma_0(z_j)}(z)$ and $b_{z_0}(\gamma_0^{-1}(z))$ have the same zeros and poles and so the ratio is a constant,

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