Spectral Theory of Orthogonal Polynomials

Periodic and Ergodic Spectral Problems

Issac Newton Institute, January, 2015

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Lecture 7: Chebyshev Polynomials, I

References

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Chebyshev Polynomials

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The minimizer is unique (as we'll see below in the case that $\mathfrak{e} \subset \mathbb{R}$), so it is appropriate to speak of *the* Chebyshev polynomial rather than *a* Chebyshev polynomial.

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Chebyshev invented his explicit polynomials which obey $Q_n(\cos(\theta)) = \cos(n\theta)$ not because of their functional relation but because they are the best approximation on $[-1, 1]$ to x^n by polynomials of degree $n - 1$. In this regard, Sodin and Yuditski unearthed the following quote from a 1926 report by Lebesgue on the work of S. N. Berstein.

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This quote is a little bizarre given that, as we'll see, Borel (who was Lebesgue's thesis advisor) made important contributions to the subject in 1905!

The Alternation Theorem

We will focus for most of this talk on the case $\mathfrak{e} \subset \mathbb{R}$, in which case, T_n is real, since on $\mathbb{R}$, $|\mathrm{Re}(T_n)|$ is smaller than $|T_n|$.

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We say that P_n, a degree n polynomial, has an *alternating set* in $\mathfrak{e} \subset \mathbb{R}$ if there exists $\{x_j\}_{j=0}^n \subset \mathfrak{e}$ with

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While the basic idea of the following theorem goes back to Chebyshev, the result itself is due to Borel and Markov, independently, around 1905.

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Conversely, let P_n be a degree n monic polynomial with an alternating set and suppose that $\|T_n\|_{\mathfrak{e}} < \|P_n\|_{\mathfrak{e}}$. Then at each point, x_j, in the alternating set for P_n, $Q \equiv P_n - T_n$ has the same sign as P_n, so Q has at least n zeros, which is impossible, since it is of degree at most $n-1$. ■

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At the alternating points for Q, we must have $T_n = S_n$, so they must be equal polynomials since there are $n+1$ points and their difference has degree at most $n-1$.

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Here, $\mathrm{cvh}(\mathfrak{e})$ is the convex hull of $\mathfrak{e}$ and that result follows from $x_0, x_n \in \mathfrak{e}$.

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Above the top zero (resp. below the bottom zero) of T_n, $|T_n(x)|$ is monotone increasing (resp. decreasing). It follows that $x_n = \sup_{y \in \mathfrak{e}} y$ (resp $x_0 = \inf_{y \in \mathfrak{e}} y$) so

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$$T_n(z) = z^n; \quad \|T_n\|_{\mathfrak{e}} = 1 = C(\mathfrak{e})^n$$

Example ($[-1,1]$) It is known (and follows from results later) that $C(\mathfrak{e}) = \frac{1}{2}$. By the Alternation Theorem, the polynomials given by $Q_n(\cos(\theta)) = \cos(n\theta)$ (i.e. "*the* Chebyshev polynomials of the first kind") are multiples of Chebyshev polynomials as we've defined them, so

$$T_n(\cos(\theta)) = 2^{-n+1}\cos(n\theta); \quad \|T_n\|_{\mathfrak{e}} = 2^{-n+1} = 2\,C(\mathfrak{e})^n$$

These examples show the two lower bounds have optimal constants.

The Discriminant

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For $z \in \mathbb{C}$ fixed, we are interested in solutions, $\{u_n\}_{n=-\infty}^{\infty}$, of

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We study the p-step transfer (aka update) matrix.

$$M_p(z)\begin{pmatrix} u_1 \\ a_0 u_0 \end{pmatrix} = \begin{pmatrix} u_{p+1} \\ a_p u_p \end{pmatrix}$$

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We put a's in the bottom component so the one step matrix $\frac{1}{a_j}\begin{pmatrix} z-b_j & -1 \\ a_j^2 & 0 \end{pmatrix}$ has determinant 1 and thus $\det(M_p(z)) = 1$.

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$$M_p(z) = \begin{pmatrix} p_p(z) & -q_p(z) \\ a_p p_{p-1}(z) & -a_p q_{p-1}(z) \end{pmatrix}$$

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Recall that we proved that there are real numbers $\alpha_1 < \beta_1 \leq \alpha_2 < \dots \beta_{p-1} \leq \alpha_p < \beta_p$ so that

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Theorem *Let J be a period p periodic Jacobi matrix with Jacobi parameters $\{a_n, b_n\}_{n=-\infty}^{\infty}$ and let $\Delta(z)$ be its discriminant. Then $(a_1 \dots a_p)\Delta(z)$ is the pth Chebyshev polynomial for spec(J).*

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Potential Theory and the Discriminant

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where the branch of the square root is taken which, when $|z|$ is large, is $cz^p + \mathrm{O}(z^{p-1})$ with $c > 0$ and with branch cuts on $\mathfrak{e}$.

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We note that, of course, $\frac{\Delta(z)}{2}$ can be replaced by $\frac{T_p(z)}{\|T_p\|_{\mathfrak{e}}}$.

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Reading the constant at infinity from the coefficient of the highest order term of $\Delta(z)$, we see that

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Theorem *A subset $\mathfrak{e} \subset \mathbb{R}$ is the spectrum of a period p Jacobi matrix if and only if it has no more than p connected components where each such component has harmonic measure $\frac{k}{p}$ with $k \in \mathbb{Z}_+$.*

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Call a set, $\tilde{\mathfrak{e}}$, which is the spectrum of a period n Jacobi matrix, a *period n set*.

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Fact 1 $\mathfrak{e} \subset \mathfrak{e}_n \equiv T_n^{-1}([-Q,Q]) \subset \mathrm{cvh}(\mathfrak{e})$. $\mathfrak{e}_n$ is a period n set with the same Chebyshev polynomial as $\mathfrak{e}$ and

$$\|T_n\|_{\mathfrak{e}} = 2C(\mathfrak{e}_n)^n$$

Since $\mathfrak{e} \subset \mathfrak{e}_n$, we have $C(\mathfrak{e}_n) \geq C(\mathfrak{e})$ and thus

Theorem (Schiefermayr's Theorem) $\|T_n\|_{\mathfrak{e}} \geq 2C(\mathfrak{e})^n$

Suppose that $\tilde{\mathfrak{e}} \supset \mathfrak{e}$ is a period n set and let S_n be its nth Chebyshev polynomial. Since it is monic we must have that $\|S_n\|_{\mathfrak{e}} \geq \|T_n\|_{\mathfrak{e}}$ which means that $C(\tilde{\mathfrak{e}}) \geq C(\mathfrak{e}_n)$. Taking into account the uniqueness of T_n we see that

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Fact 2 For all period n sets $\tilde{\mathfrak{e}} \supset \mathfrak{e}$, we have $C(\tilde{\mathfrak{e}}) \geq C(\mathfrak{e}_n)$ with equality only if $\tilde{\mathfrak{e}} = \mathfrak{e}_n$.

FFS Theorem

Theorem (*Faber–Fekete–Szegő Theorem*) For any compact subset $\mathfrak{e} \subset \mathbb{C}$, we have that

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He also obtained asymptotics for the polynomials themselves. The unbounded component, Ω, of $(\mathbb{C}\cup\{\infty\})\setminus\mathfrak{e}$ is simply connected, so $G_{\mathfrak{e}}(z)$ has a single valued harmonic conjugate and thus, by exponentiating, there is a function, $B_{\mathfrak{e}}(z)$, on Ω with $|B_{\mathfrak{e}}(z)| = \exp(G_{\mathfrak{e}}(z))$ with an overall phase determined by demanding that as $z\to\infty$, we have that $B_{\mathfrak{e}}(z) = \frac{z}{C(\mathfrak{e})} + \mathrm{O}(1)$.

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Fekete's work on transfinite diameters and its connection to capacity for some special cases is from 1923. Szegő had the full theorem in a 1924 paper whose title started "Comments on a paper by Mr. M. Fekete".

Character Automorphic Functions

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To recap, as in the work of Faber, it is natural to look for an analytic function, $B_{\mathfrak{e}}(z)$, with $|B_{\mathfrak{e}}(z)| = \exp(G_{\mathfrak{e}}(z))$ on Ω, the unbounded component of $(\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e}$.

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Put differently, $B_\mathfrak{e}(z)$ can be continued along any curve in Ω and there is a map from the fundamental group of Ω to $\partial\mathbb{D}$, which is a character (i.e. group homomorphism), so that after continuation around a closed curve, $B_\mathfrak{e}(z)$ is multiplied by the character applied to that curve.

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If $T_n(z)B_{\mathfrak{e}}(z)^{-n}C(\mathfrak{e})^{-n}$ had a limit, that limit cannot be n independent since the character is n dependent. Widom had the idea that there should be functions $F_\chi(z)$ defined for each χ in the character group and continuous in χ so the limit is the F_χ, call it F_n, associated to the character of $B_{\mathfrak{e}}(z)^{-n}$. As a function of n, the limit will be almost periodic!

Widom's Minimizers

He even found a candidate for the functions! Let $F_\chi(z)$ be that function among all character automorphic functions, $A(z)$, on Ω with character χ and with $A(\infty) = 1$, that minimizes $\sup_{z\in\Omega}\{|A(z)|\}$.

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Widom proved uniqueness of the minimizer and found a formula for it (in terms of some theta functions and solutions of some implicit equations). He also proved that $\|F_\chi\|_\Omega$ is continuous in χ. Because of the uniqueness, one can prove that the functions, $F_\chi(z)$, defined for $z \in \Omega$, are continuous in χ on the compact set of characters, uniformly locally in z (but as functions on the covering space not uniformly in all z).

Widom's Theorems and Conjecture

Theorem (Widom) *Let $\mathfrak{e}$ be a finite union of disjoint analytic Jordan curves. Let $F_n(z)$ be as above for the character of $B_{\mathfrak{e}}(z)^{-n}$. Then:*

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where the limit is uniform on compact subsets of Ω.

Since $|B_{\mathfrak{e}}(z)| \to 1$ and $\|F_n\|_\Omega$ is taken as $z \to \mathfrak{e}$, the z asymptotics and norm limit fit together.

Theorem (Widom) *Let $\mathfrak{e}$ be a finite gap subset of $\mathbb{R}$. Let $F_n(z)$ be as above. Then*

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The norm, $\|T_n\|_{\mathfrak{e}}$ is twice as large as one might expect! Note: This is Widom's conjecture for $\mathfrak{e} \subset \mathbb{R}$; he made the conjecture for more general cases of $\mathfrak{e} \subset \mathbb{C}$.

Back to $[-1, 1]$

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It was this example that led Widom to his conjecture.

Outer Approximation for General Sets

In order to extend Markov and other polynomial inequalities to general sets, Totik proved that:

Theorem (Totik's Approximation Theorem) *For any compact set* $\mathfrak{e} \subset \mathbb{R}$*, there exist period n sets* $\tilde{\mathfrak{e}}_n \supset \mathfrak{e}$ *so that* $C(\tilde{\mathfrak{e}}_n) \to C(\mathfrak{e})$

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Theorem (Totik's $1/n$ bound) *If $\mathfrak{e}$ is a finite gap set, the period n sets $\tilde{\mathfrak{e}}_n \supset \mathfrak{e}$ can be chosen so that $C(\tilde{\mathfrak{e}}_n) \le C(\mathfrak{e}) \left(1 + \frac{E}{n}\right)$ for some constant E.*

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This complements the $2C(\mathfrak{e})^n$ lower bound. Because of his asymptotic result, Widom already had this bound in 1969 but Totik's proof was much simpler. Neither proof has very explicit estimates for D. Even though they only had the result for finite gap sets, we will say that a general set $\mathfrak{e}$ has *Totik–Widom bounds*, if there is an upper bound of the above form.