Spectral Theory of Orthogonal Polynomials

Periodic and Ergodic Spectral Problems

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Lecture 8: Chebyshev Polynomials, II

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Theorem *For any compact set* $\mathfrak{e} \subset \mathbb{C}$, $|T_n(z)|^{1/n} \to \exp(-\Phi_{\mathfrak{e}}(z))$ *uniformly on compact subsets of* $\mathbb{C} \setminus \mathrm{cvh}(\mathfrak{e})$.

Totik Widom Bounds

Recall that we say that $\mathfrak{e} \subset \mathbb{R}$ obeys a Totik-Widom bound if there is a D with $\|T_n\|_{\mathfrak{e}} \leq DC(\mathfrak{e})^n$ and that this was only known for finite gap sets.

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Theorem *Every homogeneous set obeys a Totik Widom bound.*

Recall that a compact set $\mathfrak{e} \subset \mathbb{C}$ is called regular if $G_{\mathfrak{e}}$ is continuous on all of $\mathbb{C}$ and vanishes everywhere (rather than only q.e.) on $\mathfrak{e}$.

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Interesting Open Question Does potential theory regularity + Parreau-Widom $\Rightarrow$ Totik-Widom bound for general $\mathfrak{e} \subset \mathbb{C}$ (our proof is only for $\mathfrak{e} \subset \mathbb{R}$).

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Our Proof of Root Asymptotics for $T_n(z)$

We begin with the Bernstein Walsh Lemma which implies that $|T_n(z)| \leq \|T_n\|_{\mathfrak{e}} \exp(nG_{\mathfrak{e}}(z))$.

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Note that this theorem implies convergence of the density of zeros when $\mathfrak{e} \subset \mathbb{R}$, another result of Saff-Totik.

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since regularity of $\mathfrak{e}$ implies $G_{\mathfrak{e}}$ vanishes at the ends of each gap so the maximum is taken a critical point w_j. Exponentiating and using $\|T_n\|_{\mathfrak{e}} \leq 2C(\mathfrak{e}_n)^n$ we get the result. ∎

Proof of the Lemma

Because the integrated equilibrium measure of $\mathfrak{e}_n$ is $\frac{1}{\pi n}\arccos(\frac{T_n(x)}{\|T_n\|_{\mathfrak{e}}})$, each band of $\mathfrak{e}_n$ has $\rho_{\mathfrak{e}_n}$ measure $\frac{1}{n}$ and the part of a band from a zero of T_n to a nearby band edge has $\rho_{\mathfrak{e}_n}$ measure $\frac{1}{2n}$.

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Case 2 (T_n *has a zero in* K) By the Alternation Theorem, one of the two extreme points immediately below the zero must lie in $\mathfrak{e}$, so there is at most a half band below the zero. Similarly, at most a half band above, so no more than a full band. ■

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In fact, one can prove if there is a zero not too close to a gap edge and n is large, then there is exactly a full exponentially small (in Lebesgue measure) band of $\mathfrak{e}_n$ totally inside K.

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Similarly if, for j large T_{n_j} has a zero, x_j, in K and $x_j \to x_\infty \in K$, then only x_∞ is asymptotically in $\mathfrak{e}_{n_j} \cap K$ in the sense that $\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} (K \cap \mathfrak{e}_{n_j}) = \{x_\infty\}$.

Pointwise Asymptotics in Terms of B_n

We fix $\mathfrak{e} \subset \mathbb{R}$. We will use $B(z)$ for what we previously called $B_{\mathfrak{e}}(z)$, that is the multivalued function on $\Omega \equiv (\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e}$,

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$$B_n(z)^n = \frac{T_n(z)}{\|T_n\|} + \sqrt{\left(\frac{T_n(z)}{\|T_n\|}\right)^2 - 1}$$

Pointwise Asymptotics in Terms of B_n

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$$2\frac{T_n(z)}{\|T_n\|} = B_n(z)^n + B_n(z)^{-n}$$

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As we've seen, as sets in $\mathbb{R}$ increase, the Green's functions decrease, so for all n, we have that $G_n(z) \geq \widetilde{G}(z)$, which implies that uniformly on compact subsets of $\widetilde{\Omega}$, $B_n(z)^{-2n} \to 0$. Thus on such compacts

$$\lim_{n\to\infty} \frac{2T_n(z)}{\|T_n\| B_n(z)^n} = 1$$

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We want to prove that uniformly on compact subsets of Ω,

$$\lim_{n\to\infty}\left[\frac{T_n(z)}{C(\mathfrak{e})^n B(z)^n} - F_n(z)\right] = 0$$

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Pointwise Asymptotics on $\widetilde{\Omega}$

Once, we have things in this form, we can essentially follow one of Widom's arguments. Define

$$H_n(z) = \left[\frac{C(\mathfrak{e}_n)^n B_n(z)^n}{C(\mathfrak{e})^n B(z)^n} \right]$$

which is character automorphic on Ω_n with essentially the same character as $B(z)^{-n}$ since $B_n(z)^n$ is automorphic.

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$$|H_n(z)| \leq \left[\frac{C(\mathfrak{e}_n)}{C(\mathfrak{e})}\right]^n$$

which the Totik-Widom bound says is uniformly bounded in z and n.

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- On $\widetilde{\Omega}$, the $H_{n_j}(z)$ have a limit $H_\infty(z)$ (by Montel's Theorem)

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Let $\mathcal{J}$ be the set of gaps, K_ℓ, with a limit point of zeros.

Pointwise Asymptotics on $\widetilde{\Omega}$

The H_{n_j} are defined and character automorphic on sets which converge to the universal cover of Ω with the points that map into the $\{x_\ell\}_{\{K_\ell \in \mathcal{J}\}}$ removed,

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Since near ∞, $B_{\mathfrak{g}}(z) = \frac{z}{C(\mathfrak{g})} + \mathrm{O}(1)$, we see that for each n, $H_n(\infty) = 1 \Rightarrow H_\infty(\infty) = 1$.

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Pointwise Asymptotics on $\widetilde{\Omega}$

We have thus proven that

$$\lim_{n\to\infty}\left[\frac{T_n(z)}{C(\mathfrak{e})^n B(z)^n} - F_n(z)\right] = 0$$

for $z \in \widetilde{\Omega}$.

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We have thus proven that

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