Planar Lattice Model

CHAPTER 1

Introduction

In this class:

- The graphs will be mainly subsets of the square and honeycomb lattices; most results can be generalized to other planar lattices.
- We will focus on models which are
  - Simple enough to be "exactly solvable": many quantities can be computed exactly and explicitly
  - Sophisticated enough to exhibit an interesting behavior: phase transition, connections with geometry, complex analysis, representation theory, stochastic calculus
  - Quite realistic: applications in physics, chemistry, biology, operation research, economics, statistics
- We will discuss general ideas after treating the central examples.
- Today: we discuss a number of models and elementary (but nontrivial) examples of results

Example 1: simple random walk.

- Let $(S_n)_{n\geq 0}$ be a random walk on $\mathbb{Z}^2$ starting from $(0,0)$ and let $\left(\tilde{S}_n\right)_{n\geq 0}$ be an independent r.w. on $\mathbb{Z}^2$ starting from $(0,1)$.
- What is the probability that their paths are disjoint until time n, for n large?
  - Answer: $\mathbb{P}\left\{S\left([0,n]\right)\cap\tilde{S}\left([0,n]\right)=\emptyset\right\}\approx n^{-5/8}$ as $n\to\infty$ [LSW01].

Example 2: self-avoiding walk. (model of polymer)

- Let C_n be the number of honeycomb lattice paths of n steps that are *self-avoiding.*
- What is the asymptotic behavior of C_n?
  - Answer $C_n \approx \sqrt{2+\sqrt{2}}^n$ as $n\to\infty$ [DcSm10].
  - Unknown for square lattice.
- Conjecture: $C_n \sim \mathrm{Cst}\cdot\sqrt{2+\sqrt{2}}^n n^{11/32}$ [Nie82].

Example 3: uniform spanning tree. (aka random mazes)

- A spanning tree of a graph G is a tree that contains all the vertices of G.
- Pick, uniformly at random, a spanning tree $\mathcal{T}$ of an $n\times n$ square grid (in other words: a random maze). There is a unique path in $\mathcal{T}$ that goes from the top-left corner to the bottom-right corner. What is, typically, the length of that path?
  - Answer: with high probability, it is of size $\approx n^{5/4}$ as $n\to\infty$. [Ken00]

Example 4: percolation. The simplest statistical mechanics model.

- Fix $p \in [0,1]$. Color the faces of the honeycomb lattice in white with probability p and in black with probability $1-p$, independently of each other.
- If we think of black as matter and white as void, we can model a porous medium and ask if a fluid will flow in the medium (i.e. will percolate).
- Is there an infinite white cluster (i.e. an infinite connected component of white hexagons)? Yes, there is a unique one if $p > \frac{1}{2}$, and there is no if $p \leq \frac{1}{2}$ (with probability 1).
- What happens at the critical point $p = \frac{1}{2}$? Many interesting things. For instance: take two hexagons x and y that are far apart. What is the probability that x and y are connected by a path of white hexagons?
  - Answer: $\mathbb{P}\left\{x \leftrightsquigarrow_{\text{white}} y\right\} \approx \operatorname{dist}(x,y)^{-5/24}$ as $\operatorname{dist}(x,y) \to \infty$.
- What happens for $p = \frac{1}{2} + \epsilon$, for small $\epsilon > 0$? There is an infinite white cluster. Fix a hexagon x. What is the chance that x is in the infinite cluster?
  - Answer: $\mathbb{P}\left\{x \leftrightsquigarrow_{\text{white}} \infty\right\} \approx \epsilon^{5/36}$ as $\epsilon \to 0$ [SmWe00].
- Where this comes from: percolation at large scales is intimately connected with complex analysis and Brownian motion, and we can use this to get exact computations.

Example 5: dimer model. The word 'dimer' stands for '2-mer' (like in 'polymer')

- A *dimer tiling* (or *perfect matching*) of a graph G is a collection $\mathcal{D}$ of edges of G such that each vertex v of G is covered exactly once by an edge of $\mathcal{D}$.
- What is the number of dimer coverings of an $m \times n$ square grid (how many ways to pave an $m \times n$ checkerboard by dominos) ? If mn is odd, 0. And otherwise? Answer:

$$\sqrt{\prod_{j=1}^{m} \prod_{k=1}^{n} \left(2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right)\right)}$$

- If G is a finite subgraph of the honeycomb lattice, we can represent $\mathcal{D}$ by a surface living in $\mathbb{R}^3$.
- When G becomes large, how does a typical, uniformly chosen at random, dimer cover of G look like? How does the random surface look like?
  - Answer: depending on the boundary of G, we see a limit shape appear, with frozen regions (dimers are ordered) and liquid regions (dimers are disordered).

Example 6: Ising model. Model for ferromagnetism and many other phenomena.

- The Ising model is a random assignment of ± 1 spins to the vertices V of a graph G. There are $2^{\#V}$ *spin configurations* $(\sigma_x)_{x \in V}$ with $\sigma_x \in \{\pm 1\}$ for all $x \in V$.
- The probability of a spin configuration (σ_x) is proportional to $e^{-\beta H(\sigma)}$, where $\beta > 0$ is a fixed parameter (for applications, $\beta \propto \frac{1}{T}$, where T is the

temperature) and the energy H is given by

$$H(\sigma) = -\sum_{x\sim y} \sigma_x \sigma_y,$$

where the sum is over all pairs of adjacent vertices. The spins interact via the edges: if the spins at the ends of an edge are aligned, they add -1 to the energy, otherwise $+1$.

- What does the model do? The model favors local alignment of spins: it favors lower energy spins configurations and H quantifies the local dis-alignment of spins.
- If we consider the Ising on $\mathbb{Z}^2$, what happens? There is a phase transition at $\beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right)$: for $\beta < \beta_c$ disordered system, for $\beta > \beta_c$ long-range alignment phenomenon.
- More precisely, we have [Ons44]

$$\lim_{|x-y|\to\infty} \mathbb{E}\left[\sigma_x\sigma_y\right] \quad = \quad \begin{cases} 0 & \text{for } \beta \leq \beta_c \\ \left(1-\sinh^{-2}(2\beta)\right)^{\frac{1}{4}} & \text{for } \beta \geq \beta_c \end{cases}$$

- What happens at β_c? We have that

$$\mathbb{E}\left[\sigma_x\sigma_y\right] \sim \frac{2^{\frac{1}{6}} e^{-3\zeta'(-1)}}{|x-y|^{\frac{1}{4}}} \quad |x-y| \to \infty.$$

CHAPTER 2

Percolation

Critical percolation.

- We consider critical percolation on the honeycomb lattice: each hexagon is colored in black/white, with probability $\frac{1}{2}/\frac{1}{2}$, independently of the others.
- Percolation is originally a model of a porous medium: we can think that black hexagons are filled with matter, while white ones are empty.
- It is natural to ask about connectivity questions for that medium: is there a path made of white hexagons joining two sets?
- The probability $\frac{1}{2}/\frac{1}{2}$ is critical [Kes80]: if one color is slightly favored over the other, then we will have an infinite connected component of that color.

Cardy's formula.

- Consider a quad (Ω, a, b, c, d) (i.e. domain with $\partial\Omega$ a simple curve and four points $a, b, c, d \in \partial\Omega$ in ccw order), a discretization $(\Omega_\delta)_{\delta>0}$ by a honeycomb domain of mesh size δ, and identify a, b, c, d with the closest boundary vertice.
- Crossing probabilities: what is the chance $\mathbb{P}_{\Omega_\delta}\{[ab] \leftrightsquigarrow_w [cd]\}$ that there is a path of white hexagons linking $[ab]$ to $[cd]$?
- Conformal invariance of crossing probabilities [Smi01]: if (Ω, a, b, c, d) and $\left(\tilde{\Omega}, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\right)$ are conformally equivalent (i.e. there exists a conformal map $\Omega \to \tilde{\Omega}$ with $a, b, c, d \mapsto \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$), then
$$\lim_{\delta\to 0} \mathbb{P}_{\Omega_\delta}\{[ab] \leftrightsquigarrow_w [cd]\} = \lim_{\delta\to 0} \mathbb{P}_{\tilde{\Omega}_\delta}\left\{\left[\tilde{a}\tilde{b}\right] \leftrightsquigarrow_w \left[\tilde{c}\tilde{d}\right]\right\}.$$
- There exists an explicit formula (Cardy's formula) given in terms of a conformal mapping to an equilateral triangle.

Riemann's mapping theorem.

- Let Ω and $\tilde{\Omega}$ be two Jordan domains (i.e. $\partial\Omega$ and $\partial\tilde{\Omega}$), and let $a, b, c \in \partial\Omega$ and $\tilde{a}, \tilde{b}, \tilde{c} \in \partial\tilde{\Omega}$ be boundary points, in ccw order.
- Then there exists a unique conformal mapping $\varphi : \Omega \to \tilde{\Omega}$ with $a, b, c \mapsto \tilde{a}, \tilde{b}, \tilde{c}$.
- This statement can be generalized to arbitrary simply-connected domains, provided the boundary points are replaced by prime ends.
- How to see this?
  - We have proven (at least for Ω smooth) that for any $d \in [ca]$, we can map (Ω, a, b, c, d) to a rectangle $(0, L) \times (0, i)$
  - Using CR equations, it is fairly easy to see that L moves continuously from 0 to ∞ as d moves from c to a (see HW).

- So, there exists a $d \in [ca]$, we can map (Ω, a, b, c, d) to the square $(0,1) \times (0,i)$, with $a, b, c \mapsto 0, 1, 1+i$.
- So we can compose the maps $\Omega \to (0,1) \times (0,i) \to \tilde{\Omega}$ and map $a, b, c \mapsto 0, 1, 1+i \mapsto \tilde{a}, \tilde{b}, \tilde{c}$.

Cardy's formula and main statement.

- For a domain (Ω, a_1, a_2, a_3), there exists a unique conformal mapping φ from Ω to the equilateral triangle $\triangle$ with vertices $1, \pm\frac{\sqrt{3}}{2}i$, with $a, b, c \mapsto 1, \frac{\sqrt{3}}{3}i, -\frac{\sqrt{3}}{3}i$.
- Carleson's formulation of Cardy's formula [Smi01]: $\lim_{\delta \to 0} \mathbb{P}_{\Omega_\delta} \{[a_1 a_2] \leftrightsquigarrow_w [a_3 a_4]\} = \mathfrak{Re}\left(\varphi\left(a_4\right)\right)$.
- By uniqueness of the conformal mapping $\varphi : \Omega \to \Delta$, this proves the conformal invariance.
- To prove Cardy's formula, we prove the following statement:
  - Set $A_1 := [a_2 a_3]$, $A_2 := [a_3 a_1]$ and $A_3 := [a_1 a_2]$. For a vertex $z \in \Omega_\delta$ define $H_\delta^1(z)$, $H_\delta^2(z)$ and $H_\delta^3(z)$ by

$$H_\delta^\mu(z) := \mathbb{P}_{\Omega_\delta}\{a_\mu \text{ and } z \text{ are separated from } A_\mu \text{ by a white path}\}.$$

  - We set $\varphi_\delta := H_\delta^1 + \frac{\sqrt{3}}{2} i H_\delta^2 - \frac{\sqrt{3}}{2} i H_\delta^3$. The main statement reads

$$\varphi_\delta \xrightarrow[\delta \to 0]{} \varphi.$$

  - Evaluating with $a_4 \in [a_3 a_1]$, we get Cardy's formula.

Strategy for the proof.

- Precompactness: we want to show that $(\varphi_\delta)_{\delta>0}$ is uniformly equicontinuous on $\overline{\Omega}$. Let φ be a subsequential scaling limit $\varphi_{\delta_n} \to \varphi$ (uniqueness of the limit will yield full convergence).
- Boundary conditions: we show that if $\varphi_{\delta_n} \to \varphi$, then φ is a homeomorphism $\partial\Omega \to \partial\triangle$ with $a_1, a_2, a_3 \mapsto 1, \frac{\sqrt{3}}{3}i, -\frac{\sqrt{3}}{3}i$.
- Analyticity: we show that if $\varphi_{\delta_n} \to \varphi$, then φ is analytic. This will follow from approximate discrete Cauchy-Riemann relations.
- To conclude, i.e. show that $\varphi : \Omega \to \triangle$ is a bijection inside Ω. We use the argument principle, take $w \in \mathbb{C}$ and want to show that $\varphi(z) - w$ has a single zero in Ω iff $w \in \triangle$:

$$\#\{\text{zeros of } \varphi(z) - w\} = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{\varphi'(z)}{\varphi(z) - w} \mathrm{d}z = \frac{1}{2\pi i} \oint_{\partial\triangle} \frac{1}{\zeta - w} \mathrm{d}\zeta = \mathbf{1}_\triangle(w).$$

Symmetry and self-duality. Like for the random walks, we first need some simple a priori estimates for the precompactness part.

- Suppose that the honeycomb lattice has edges parallel to $1, e^{\pm 2\pi i/3}$ and consider the square $S = [0,1] \times [0,i]$, and discretize S by a symmetric honeycomb domain S_δ, with $\delta > 0$ small.
- We have $\mathbb{P}_{S_\delta}\{[0,i] \leftrightsquigarrow_w [1, 1+i]\} = \frac{1}{2}$: either we have a horizontal white path or a vertical black path crossing, both events have the same probability. This works on any symmetric domain.

RSW estimate.

- Consider the rectangle $R = [0,2] \times [0,i]$ and a symmetric discretization R_δ. We would like to show the Russo-Seymour-Welsh (RSW) estimate: we have$\mathbb{P}_{R_\delta}\left\{[0,i] \leftrightsquigarrow_w [2,2+i]\right\} \geq \frac{1}{16}$.
- To prove RSW, we will want to paste white paths together. We need the Fortuin-Kasteleyn-Ginibre (FKG) inequality:
  - For $\mathcal{A}$ and $\mathcal{B}$ events of the type 'there is a white path from here to there', then $\mathbb{P}(\mathcal{A}\cap\mathcal{B}) \geq \mathbb{P}(\mathcal{A})\,\mathbb{P}(\mathcal{B})$, or equivalently $\mathbb{P}(\mathcal{A}|\mathcal{B}) \geq \mathbb{P}(\mathcal{A})$.
  - The latter is intuitive, because the existence of a white path somewhere can only increase the chances of seeing a white path elsewhere.
- With FKG, we can prove RSW:
  - With probability $\frac{1}{2}$, there is a path $\gamma : [0,i] \leftrightsquigarrow [1,1+i]$, made only of white hexagons, take the lowest such path possible. Whatever is above it is independent percolation.
  - Let $\tilde{\gamma}$ be the reflection of γ with respect to the line $1+i\mathbb{R}$ and let D_δ be the connected component $R_\delta \setminus (\gamma \cup \tilde{\gamma})$ lying above $\gamma \cup \tilde{\gamma}$ and intersecting the line $1+i\mathbb{R}$.
  - With probability $\frac{1}{2}$, there is a path $\lambda \subset D_\delta$ linking the bottom-left part of ∂D_δ to the top-right one, again by symmetry. Joining γ and λ yields a white path $[0,i] \leftrightsquigarrow [1+i,2+i]$.
  - So, we have $\mathbb{P}_{R_\delta}(\mathcal{A}) \geq \frac{1}{4}$, where $\mathcal{A} := \{[0,i] \leftrightsquigarrow_w [1+i,2+i]\}$. By symmetry, we have $\mathbb{P}_{R_\delta}(\mathcal{B}) \geq \frac{1}{4}$, where $\mathcal{B} := \{[2,2+i] \leftrightsquigarrow_w [i,1+i]\}$.
  - If both $\mathcal{A}$ and $\mathcal{B}$ occur, there is a white path $[0,i] \leftrightsquigarrow [2,2+i]$. By FKG, $\mathbb{P}(\mathcal{A}\cap\mathcal{B}) \geq \frac{1}{16}$, which is the desired result.
- With FKG and RSW, the probability of a crossing in the discretizations of rectangles $[0,L] \times [0,i]$ are uniformly bounded from below with respect to $\delta > 0$ (and from above by duality):
  - We can paste several crossings which exist with positive probability from RSW.

Annulus crossing estimate.

- By FKG and RSW, we get that the probability of a white loop in an annulus of inner and outer radii 1 and 2 is uniformly bounded from below with respect to δ (we paste again).
- For $r, R > 0$ consider the discretization A_δ of an annulus of inner radius r and outer radius R. The probability that a black path links the inner circle to the outer circle is bounded by $C\left(\frac{r}{R}\right)^\alpha$, for universal $\alpha, C > 0$.
  - To prove that, we decompose A_δ into $k := \lfloor \log_2\left(\frac{R}{r}\right) \rfloor$ concentric annuli of inner and outer radii $2^{j-1}r$ and $2^j r$ for $j = 1, \ldots, k$
  - For each annulus, there is a uniformly positive chance of a white crossing in it.

Precompactness.

- To show precompactness, we will show that $\left(H_\delta^1\right)_\delta$ is uniformly Hölder continuous (the same reasoning applies to H_δ^2 and H_δ^3)
  - There exists $C > 0$ and $\alpha > 0$ such that for any $x, y \in \Omega_\delta$, $\left|H_\delta^1(x) - H_\delta^1(y)\right| \leq C \mathrm{d}_\Omega(x,y)^\alpha$, where $\mathrm{d}_\Omega(x,y)$ is the length of the shortest path from x to y in $\overline{\Omega}$

- How to prove this? Let us assume x and y close (otherwise, there is nothing to prove).
  - We have that (writing $\|_w$ for 'there is a white path separating')

$$\begin{aligned} H_\delta^1(x) - H_\delta^1(y) &= \mathbb{P}_{\Omega_\delta}\{a_1, x\|_w A_1\} - \mathbb{P}_{\Omega_\delta}\{a_1, y\|_w A_1\} \\ &= \mathbb{P}_{\Omega_\delta}(\{a_1, x\|_w A_1\} \setminus \{a_1, y\|_w A_1\}) - \mathbb{P}(\{a_1, y\|_w A_1\} \setminus \{a_1, x\|_w A_1\}). \end{aligned}$$

  - Let us study the probability of the event $E_1(x,y) = \{a_1, x\|_w A_1\} \setminus \{a_1, y\|_w A_1\}$.
    - We see that the occurence of E_1 implies the existence of a white path from A_2 to A_3 passing between x and y, and a black path from A_1 to A_1 separating x and y.
    - In turn, each such path, implies that a white or a black path goes from a 'microscopic' circle (i.e. of radius $\mathrm{d}_\Omega(x,y)$) to a macroscopic circle (i.e. of radius $\mathrm{dist}(\{x,y\}, A_j)$ for $j = 1,2,3$).
    - By topology argument, at least one of the macroscopic circles is 'big' (i.e. greater than a uniform ϵ), and we can bound the probability this path by the application of RSW above.
  - Hence, we get $\mathbb{P}(E_1(x,y)) \leq C\mathrm{d}(x,y)^\alpha$, and by symmetry, we deduce the Hölder-continuity.
- To extract converging subsequences, we extend φ_δ into a continuous function (by piecewise affine interpolation for instance), and use Arzelà-Ascoli: obviously $(\varphi_\delta)_\delta$ is bounded and equicontinuous.
- From now on, we will assume that φ is a subsequential scaling limit $\lim_{\delta_n \to 0} \varphi_{\delta_n}$, with $\varphi = H^1 + i\frac{\sqrt{3}}{3}H^2 - i\frac{\sqrt{3}}{3}H^3$.

Boundary conditions.

- We want to prove that φ is a homeomorphism $\partial\Omega \to \partial\Delta$, so we want to prove that it is a homeomorphism $A_1 \to \left[\frac{i\sqrt{3}}{3}, -\frac{i\sqrt{3}}{3}\right]$, $A_2 \to \left[-\frac{i\sqrt{3}}{3}, 1\right]$, $A_3 \to \left[1, \frac{i\sqrt{3}}{3}\right]$.
- To prove that $\varphi : A_1 \to \left[\frac{i\sqrt{3}}{3}, -\frac{i\sqrt{3}}{3}\right]$ is a homeomorphism, we should prove that
  - For any $z \in A_1$ $H^1(z) = 0$. This follows from RSW:
    - If $z \in \Omega_\delta$ is at microscopic distance from A_1, then if $Q_\delta^1(z)$ happens, there is a white path from z to A_2 and to A_3
    - At least one of A_2 and A_3 is at macroscopic distance from z. Hence $\mathbb{P}_{\Omega_\delta}\left(Q_\delta^1(z)\right) \to 0$.
  - For any $z \in A_1$ $H^2(z) + H^3(z) = 1$. This follows essentially from self-duality:
    - For $z \in \Omega_\delta$, $\mathbb{P}_{\Omega_\delta}\left(Q_\delta^2(z)\right) = \mathbb{P}_{\Omega_\delta}\left(\tilde{Q}_\delta^2(z)\right)$, where $\tilde{Q}_\delta^2 = \{a_2, z \|_\mathrm{b} A_2\}$.
    - At least one of $Q_\delta^3(z)$ and $\tilde{Q}_\delta^2(z)$ happens by self-duality and $Q_\delta^3(z) \cap \tilde{Q}_\delta^2(z) = \emptyset$.
    - This is 'regular' by RSW (i.e. we can exchange limit $z \to A_1$ and $\delta \to 0$).
  - As $z \in A_1$ moves from a_2 to a_3, $H^3(z)$ increases from 0 to 1.

  - Let $z, \tilde{z} \in A_1$ with z closer to a_2 than $\tilde{z}$. We have that $Q_\delta^3(z) \subset Q_\delta^3(\tilde{z})$, so let $R_\delta := Q_\delta^3(\tilde{z}) \setminus Q_\delta^3(z) = \{[z\tilde{z}] \leftrightsquigarrow_{\mathrm{b}} A_2\}$.
  - By RSW, the probability of the latter even is strictly positive, so $H_\delta^3(\tilde{z}) - H_\delta^3(z) = Q_\delta^3(\tilde{z}) \setminus Q_\delta^3(z)$ is uniformly positive as $\delta \to 0$.
  - By RSW, making concetric annuli, we see that $H_\delta^3(a_3) \to 1$ as $\delta \to 0$ (we can make concetric annuli around a_3).

Discrete Cauchy-Riemann equations.

- This is the key identity to prove analyticity (which is itself the key property).
- For an oriented edge $\vec{e} \in \Omega_\delta$ from vertex $x \in \Omega_\delta$ to vertex $y \in \Omega_\delta$ and a function $f : \Omega_\delta \to \mathbb{C}$ we define the discrete derivative $\partial_{\vec{e}} f$ by $f(y) - f(x)$.
- For the functions H_δ^μ, we can write $\partial_{\vec{e}} H_\delta^\mu = \partial_{\vec{e}}^+ H_\delta^\mu - \partial_{\vec{e}}^- H_\delta^\mu$, where $\partial_{\vec{e}}^+ H_\delta^\mu = \mathbb{P}_{\Omega_\delta}(Q_\delta^\mu(y) \setminus Q_\delta^\mu(x))$ and $\partial_{\vec{e}}^- H_\delta^\mu = \mathbb{P}_{\Omega_\delta}(Q_\delta^\mu(x) \setminus Q_\delta^\mu(y))$.
- Set $\tau = e^{2\pi i/3}$, write $\tau\vec{e}$ for the rotation of $\vec{e}$ around its origin x by $2\pi/3$ and set $H_\delta^1, H_\delta^\tau, H_\delta^{\tau^2} := H_\delta^1, H_\delta^2, H_\delta^3$.
- The discrete Cauchy-Riemann equation: for $\mu \in \{1, \tau, \tau^2\}$ and an oriented edge $\vec{e}$, we have $\partial_{\vec{e}}^+ H_\delta^\mu = \partial_{\tau\vec{e}}^+ H_\delta^{\tau\mu} = \partial_{\tau^2\vec{e}}^+ H_\delta^{\tau^2\mu}$.
- Proof: suppose $\mu = 1$, $\vec{e}$ an horizontal edge from left to right, let $z, w \in \Omega_\delta$ be the destinations of $\tau\vec{e}$ and $\tau^2\vec{e}$, and let us prove the first identity (everything is symmetric)
  - We have that $\partial_{\vec{e}}^+ H_\delta^\mu$ is the probability of $Q_\delta^\mu(y) \setminus Q_\delta^\mu(x)$: this event means that there is white path $\gamma : A_2 \leftrightsquigarrow A_3$ passing between x and y and that there is a black path $\lambda : A_1 \leftrightsquigarrow \{x, z, w\}$.
  - We have that $\partial_{\tau\vec{e}}^+ H_\delta^{\tau\mu}$ is the probability of $Q_\delta^\mu(z) \setminus Q_\delta^\mu(x)$: this event means that there is white path $\tilde{\gamma} : A_3 \leftrightsquigarrow A_2$ passing between x and z a black path $\tilde{\lambda} : A_2 \leftrightsquigarrow \{x, w, y\}$.
  - We construct a bijection between the $\omega \in Q_\delta^\mu(y) \setminus Q_\delta^\mu(x)$ and the $\tilde{\omega} \in Q_\delta^\mu(z) \setminus Q_\delta^\mu(x)$: because each configuration has the same probability, this will prove the identiy:
    - Let γ_2 be the cw-most white path from $\vec{e}$ to A_2, let λ be the ccw-most black path from $\vec{e}$ to A_1.
    - Flip the color of all the hexagons that on the ccw side of γ_2 and on the cw side of λ: the black path γ_3 becomes black, and this map is clearly invertible.
    - Flip the color of all the hexagons: λ and γ_3 become black, γ_2 becomes white and we get a configuration $\tilde{\omega} \in Q_\delta^\mu(z) \setminus Q_\delta^\mu(x)$.
  - Hence, we have constructed a bijection, and $\partial_{\vec{e}}^+ H_\delta^\mu = \partial_{\tau\vec{e}}^+ H_\delta^{\tau\mu}$.

Analyticity.

- To show analyticity, we use Morera's criterion: a continuous function $f : \Omega \to \mathbb{C}$ is holomorphic if $\oint_\gamma f(z)\, \mathrm{d}z = 0$ for any smooth closed contour γ (of course, the converse is true as well).
- Why Morera's criterion holds: we can define the antiderivative $F(z) := \int_w^z f(\zeta)\, \mathrm{d}\zeta$ (i.e. the definition is independent of the contour) and see that $F(z)$ is holomorphic.

- We will show that $\psi := H_1 + \tau H_\tau + \tau^2 H_{\tau^2}$ and $\sigma := H_1 + H_\tau + H_{\tau^2}$ (notice $\varphi = \frac{1}{3}\sigma + \frac{2}{3}\psi$) are analytic using Morera's criterion, by computing Riemann sums on lattice-level and seeing that they tend to 0.
- Let $\gamma \subset \Omega$ be smooth and let γ_δ be a honeycomb discretization of γ that uses $\mathcal{O}\left(\delta^{-1}\right)$ edges, oriented in ccw direction.
- Discretize $\oint_\gamma \psi(z)\,\mathrm{d}z$ by $I_\delta(\gamma,\psi) := \sum_{\vec{e}\in\gamma_\delta} \psi_\delta(\vec{e})\cdot\vec{e}$, where we write $\psi_\delta(\vec{e}) := \frac{1}{2}\left(\psi_\delta(y) + \psi_\delta(x)\right)$ and $\vec{e} := (y-x)$ (we will identify oriented edges with complex numbers).
- We can write $\sum_{\vec{e}\in\gamma_\delta} \psi_\delta(\vec{e})\cdot\vec{e} = \sum_{f\in\mathcal{F}_{\Omega_\delta}} \sum_{\vec{e}\in\partial f} \psi_\delta(\vec{e})\cdot\vec{e}$, where $\mathcal{F}_{\Omega_\delta}$ is the set of hexagonal faces of Ω_δ and ∂f is the boundary of f, oriented ccw.
- We can rewrite $\sum_{\vec{e}\in\partial f} \psi_\delta(\vec{e})\cdot\vec{e} = \sum_{\vec{e}\in\partial f} \partial_{\vec{e}}\psi_\delta m(\vec{e})$ by discrete resummation, where $m(\vec{e})$ is the midpoint of $\vec{e}$.
- Writing $\vec{e}^*$ for the oriented edge of the dual of Ω_δ that crosses $\vec{e}$ oriented such that $\vec{e}^*/(i\vec{e}) > 0$, and introducing $c(f)$, the center of f, we get $I_\delta(\gamma,\varphi) = \frac{1}{2}\sum_{\vec{e}\in\partial f} \partial_{\vec{e}}\psi\cdot\vec{e}^*$.
- Resumming over all the edges $\vec{e}\in\vec{\mathcal{E}}_{\Omega_\delta}$ (taking each edge in its two possible orientations), we get $I_\delta(\gamma,\psi) = \sum_{\vec{e}\in\vec{\mathcal{E}}_{\Omega_\delta}} \vec{e}^*\partial_{\vec{e}}\psi_\delta +$ boundary terms.
- The boundary terms tend to 0 as $\delta\to 0$: there are $\mathcal{O}\left(\frac{1}{\delta}\right)$ of them, and they are of order $o(\delta)$ (the δ comes from the edge length, the $o(1)$ from $\partial_{\vec{e}}\psi$, by RSW).
- We get

$$\frac{1}{2}\left(\sum_{\vec{e}\in\vec{\mathcal{E}}_{\Omega_\delta}} \vec{e}^*\left(\partial_{\vec{e}}^+ H_\delta^1 + \tau\partial_{\vec{e}}^+ H_\delta^\tau + \tau^2\partial_{\vec{e}}^+ H_\delta^{\tau^2}\right) - \sum_{\vec{e}\in\vec{\mathcal{E}}_{\Omega_\delta}} \vec{e}^*\left(\partial_{\vec{e}}^- H_\delta^1 + \tau\partial_{\vec{e}}^- H_\delta^\tau + \tau^2\partial_{\vec{e}}^- H_\delta^{\tau^2}\right)\right),$$

- We can resum over all the edges, to get $\sum_{\vec{e}\in\vec{\mathcal{E}}_{\Omega_\delta}} \vec{e}^*\left(\partial_{\vec{e}}^+ H_\delta^1 + \tau\partial_{\vec{e}}^+ H_\delta^\tau + \tau^2\partial_{\vec{e}}^+ H_\delta^{\tau^2}\right)$
- We can resum once more to get

$$\sum_{\vec{e}\in\vec{\mathcal{E}}_{\Omega_\delta}} \vec{e}^*\left(\partial_{\vec{e}}^+ H_\delta^1 + \partial_{\tau^2\vec{e}}^+ H_\delta^\tau + \partial_{\tau\vec{e}}^+ H_\delta^{\tau^2}\right)$$

  and resum one last time

$$\sum_{\vec{e}\in\vec{\mathcal{E}}_{\Omega_\delta}} \left(\vec{e}^* + \tau(\tau\vec{e})^* + \tau^2\left(\tau^2\vec{e}\right)^*\right)\partial_{\vec{e}} H_\delta^1$$

  .
- This last term equals 0, as $\vec{e}^* + \tau(\tau\vec{e})^* + \tau^2\left(\tau^2\vec{e}\right)^* = 0$, so $\lim_{\delta\to 0} I_\delta(\gamma,\psi) = 0$, and hence ψ is analytic.
- Similarly for $\sigma := H_1 + H_\tau + H_{\tau^2}$, we get a cancellation because $\vec{e}^* + (\tau\vec{e})^* + \left(\tau^2\vec{e}\right)^* = 0$ and σ is analytic as well.

CHAPTER 3

Ising Model

- We want to (start to) prove a conformal invariance result [CHI12]:

$$\frac{1}{\delta^{\frac{1}{8}}}\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\right] \underset{\delta\to 0}{\longrightarrow} \mathcal{C}\left\langle\sigma_a\right\rangle^{+}_{\Omega},$$

  where $\left\langle\sigma_a\right\rangle_{\Omega} = \left|\phi'\left(a\right)\right|^{\frac{1}{8}}$

$$\frac{1}{\delta^{\frac{1}{4}}}\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_b\right] \underset{\delta\to 0}{\longrightarrow} \mathcal{C}^2\left\langle\sigma_a\sigma_b\right\rangle^{+}_{\Omega},$$

  where $\left\langle\sigma_a\sigma_b\right\rangle_{\Omega} = \left\langle\sigma_a\right\rangle_{\Omega}\left\langle\sigma_b\right\rangle_{\Omega}\left(1-e^{-2d_{\mathcal{H}}(a,b)}\right)$.
- The result extends to other boundary conditions.

Setup.

- We rotate the square grid by 45 degrees and call δ the size of a half-diagonal (the sidelength of a square is now $\sqrt{2}\delta$).
- We consider the Ising model on the faces $\mathcal{F}_\delta$, with + boundary conditions on $\partial\mathcal{F}_\delta$, at critical temperature $\beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right)$.

What we want to compute.

- The key quantities to compute the ratios is

$$\frac{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a+2\delta}\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\right]} \quad\text{and}\quad \frac{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a+2i\delta}\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\right]}$$

- This allows one to compute ratios of values of spins on black squares (if we color $\mathcal{F}_\delta$ in a chessboard fashion).
- More precisely, what allows one to pass to the scaling limit is [uniformly with respect to the location of a]

$$\frac{1}{\delta}\left(\frac{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a+2\delta}\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\right]}-1\right) \underset{\delta\to 0}{\longrightarrow} \partial_{\mathfrak{Re}(a)}\log\left\langle\sigma_a\right\rangle^{+}_{\Omega}.$$

- After that (the same holds true in the y direction), we can integrate the discrete logarithmic derivative on paths parallel to x/y-axes.
- Similarly, we need to compute the logarithmic derivatives

$$\frac{1}{\delta}\left(\frac{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a+2\delta}\sigma_b\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_b\right]}-1\right) \underset{\delta\to 0}{\longrightarrow} \partial_{\mathfrak{Re}(a)}\log\left\langle\sigma_a\sigma_b\right\rangle^{+}_{\Omega},$$

  etc.

Steps of the proof. There are mainly two steps

- Convergence of ratios at different locations $\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_b]} \underset{\delta\to 0}{\longrightarrow} \left|\frac{\phi'(a)}{\phi'(b)}\right|^{\frac{1}{8}}$, and ratios with different boundary conditions.
  - The proof relies on lattice spinors and discrete complex analysis.
- Calibrate the convergence, to compare different domains and different number of points.
  - The proof relies mostly on inequalities and the convergence of ratios.

Ratio of spin correlations. Now, the goal is to compute $\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_{a+2\delta}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_a]}$ and other correlation ratios.

- First, we represent the expectation ratios as special values of lattice spinors
- Then we prove that these functions are discrete holomorphic, with discrete singularities and boundary conditions.
- Then we prove that the lattice spinors converge to continuous spinors, which are holomorphic functions
- Finally, we extract the scaling limit of the expectation ratios as coefficients in the expansion of the continuous spinors.
- We will use the low-temperature expansion: represent the spin configurations by their collections of contours.
- As we have seen before: the probability of a contour configuration $\omega \in \mathcal{L}_\delta$ is proportional to $\alpha^{\#\omega}$, where $\alpha = e^{-2\beta_c} = \sqrt{2} - 1$.
- For a face $a \in \mathcal{F}_\delta$ denote by $[\Omega_\delta, a]$ the double cover of Ω_δ around a, with vertices $[\mathcal{V}_\delta, a]$, edges $[\mathcal{E}_\delta, a]$, etc.

Notation and loop statistics.

- We will denote by $\mathcal{V}_\delta$ the vertices, by $\mathcal{E}_\delta$ the edges, by $\mathcal{F}_\delta$ the faces, by $\mathcal{C}_\delta$ the corners.
- We denote by $\partial\mathcal{V}_\delta$ the boundary vertices, by $\partial\mathcal{E}_\delta$ for the boundary edges, by $\partial\mathcal{F}_\delta$ the boundary faces.
- We consider the low-temperature representation of the model, i.e. trace edges between pairs of different spins.
- The spin configurations are in bijection with the collection of loops $\mathcal{L}_\delta$ and the probability of $\omega \in \mathcal{L}_\delta$ is proportional to $\exp(-2\beta\#\omega) = \alpha^{\#\omega}$.
- In our case (i.e. the critical temperature), we have $\alpha = e^{-2\beta} = \sqrt{2} - 1$.
- The value of the spin a corresponds to $(-1)^{\#\text{loops}(\omega,a)}$, where $\#\text{loops}(\omega, a)$ is the number of loops surrounding a.
- So, we have that $\mathbb{E}[\sigma_{a+2\delta}]/\mathbb{E}[\sigma_a] = \sum_{\omega\in\mathcal{L}} (-1)^{\#\text{loops}(\omega,a+\delta)} \alpha^{\#\omega} / \sum_{\omega\in\mathcal{L}} (-1)^{\#\text{loops}(\omega,a)} \alpha^{\#\omega}$.
- To see how the technique works, let us first compute $\mathbb{E}[\sigma_{a+2\delta}\sigma_a]$, which gives $\sum_{\omega\in\mathcal{L}} (-1)^{\#\text{loops}(\omega,a,a+2\delta)} \alpha^{\#\omega}/\mathcal{Z}_\delta$, where $\mathcal{Z}_\delta := \sum_{\omega\in\mathcal{L}} \alpha^{\#\omega}$ and $\#\text{loops}(\omega, a, a+2\delta)$ is the number of loops separating a and $a + 2\delta$.

Fermion. We now introduce the fermion correlators (easier to work with). The spinors will be a generalization

- Let $\mathcal{M}_\delta$ be the set of midpoints of edges of $\mathcal{E}_\delta$, $\partial\mathcal{M}_\delta$ the midpoints of boundary edges $\partial\mathcal{E}_\delta$, $\mathcal{C}_\delta$ the set of corners of Ω_δ (see Figure).
- We denote by b with the corner $a + \frac{\delta}{2}$.

- For $z \in \mathcal{C}_\delta$, let $\mathcal{L}_b(z)$ denote the set ω, consisting of edges of $\mathcal{E}_\delta$ and two 'corner edges', one from b to $b + \frac{\delta}{2}$, and the other from z to the closest vertex of $\mathcal{V}_\delta$ such that each vertex $v \in \mathcal{V}_\delta$ is incident to a even number of edges or corner edges (see Figure)
- In other words: each $\omega \in \mathcal{L}_b(z)$ contains loops, and a path from a to z (suppose $z \neq a + \frac{\delta}{2}$)
- We define $f_{\Omega_\delta, b}(z)$ as $\frac{1}{\mathcal{Z}} \sum_{\gamma \in \mathcal{L}_b(z)} \alpha^{\#\gamma} e^{-\frac{i}{2}\mathbf{W}(\gamma)}$, where $\#\gamma$ counts the number of edges (not the corners) and $\mathbf{W}(\gamma)$ is the total winding (=rotation) of γ from b to z.
- We count the winding by the rule that whenever there is an ambiguity, we turn left or right, but don't go straight. $e^{-\frac{i}{2}\mathbf{W}(\gamma)}$ is independent of the choice.
- We have that $f_{\Omega_\delta, b}(b+\delta) = \mathbb{E}[\sigma_a \sigma_{a+2\delta}]$: the winding $\mathbf{W}(\gamma)$ is $0, \pm 2\pi, \pm 4\pi$. If the number of loops separating a and $a + 2\delta$ is even, then $e^{-\frac{i}{2}\mathbf{W}(\gamma)}$ gives 1, otherwise, it gives -1.
- We extend $f_{\Omega_\delta, b}$ to $\mathcal{M}_\delta$: for $z \in \mathcal{M}_\delta$, we define $\mathcal{L}_a(z)$ is the collection of edges, a corner edge from b to $b + \frac{\delta}{2}$, and a half-edge to z, such that each vertex in $\mathcal{V}_\delta$ belongs to an even number of edges, corner edges or half-edges.
- We define $f_{\Omega_\delta, b}$ as before (not counting the half-edge as before), but with an extra factor $\frac{1}{\cos\left(\frac{\pi}{8}\right)}$.

Discrete analysis.

- What is the interest of introducing $f_{\Omega_\delta, b}$? Its values at various mid-edges, corners satisfy relations: hence moving z in the domain will allow one to propagate the information to $z = b + \delta$, hopefully.
- More precisely, we will prove that $f_{\Omega_\delta, b}$ is discrete holomorphic, has a discrete singularity and satisfies boundary conditions.
- Discrete holomorphicity implies discrete Cauchy-Riemann equations

S-holomorphicity. This notion of discrete holomorphicity is taylored for the fermionic correlators [Smi06]

- What is the discrete holomorphicity? First notice that the value on each corner have a certain phase, because of $e^{-\frac{i}{2}\mathbf{W}(\gamma)}$: setting $\lambda = e^{\frac{\pi i}{4}}$, we have that the value of f are in $\mu\mathbb{R}$, where $\mu \in \left\{1, \overline{\lambda}, i, \lambda\right\}$ depending on whether a corner is east, north, west or south of the attached vertex.
- The s-holomorphicity relation reads as follows: if $c \in \mathcal{C}_\delta$ is a corner and $m \in \mathcal{M}_\delta$ an adjacent mid-edge (distance $\delta/2\sqrt{2}$ from each other), we have that
$$f_{\Omega_\delta, b}(c) = \mathsf{P}_{\mu\mathbb{R}}\left[f_{\Omega_\delta, b}(m)\right],$$
where $\mathsf{P}_{\mu\mathbb{R}}[X] := \frac{1}{2}\left(X + \mu^2 \overline{X}\right)$ is the orthogonal projection in $\mathbb{C}$ on the line $\mu\mathbb{R}$.
- The s-holomorphicity relation implies a discrete Cauchy-Riemann equation:

$$f_{\Omega, b}\left(z + \delta\frac{1+i}{2}\right) - f_{\Omega_\delta, b}\left(z - \delta\frac{1+i}{2}\right) = i\left(f_{\Omega, b}\left(z + \delta\frac{1-i}{2}\right) - f_{\Omega, b}\left(z - \delta\frac{1-i}{2}\right)\right).$$

- It follows by taking a linear combination of the s-holomorphicity relations around a square face.
  - [Details]
- Proof of s-holomorphicity: let $c \in \mathcal{C}_\delta$ be a corner and $m \in \mathcal{M}_\delta$ be an adjacent mid-edge.
  - We construct an XOR bijection between the configurations of $\mathcal{L}(b,m)$ and $\mathcal{L}(b,c)$ defined by $\gamma \mapsto \tilde{\gamma}$, with $\tilde{\gamma} := \gamma \oplus \langle c,v\rangle \oplus \langle v,m\rangle$ where v is the closest vertex to c.
  - We check that the weight of a configuration $\alpha^{\#\text{edges}(\gamma)} e^{-\frac{i}{2}\mathbf{W}(\gamma)}$ equals the projection on $\mu\mathbb{R}$ of $\frac{1}{\cos\left(\frac{\pi}{8}\right)}\alpha^{\#\text{edges}(\tilde{\gamma})} e^{-\frac{i}{2}\mathbf{W}(\tilde{\gamma})}$.
  - For instance, if $m - v = \frac{1+i}{2}\delta$ and $c - v = \frac{\delta}{2}$, we have that a configuration $\gamma \in \mathcal{L}(b,m)$ arrives to m either from the south-west or the north-east directions.
  - There are two cases to check either $\langle v,m\rangle \in \omega$ or $\langle v,m\rangle \notin \omega$. In both cases, we can see that the projections are the same.
    - In the first case, $e^{-\frac{i}{2}\mathbf{W}(\gamma)} \parallel e^{-\frac{\pi i}{8}}$ and $\tilde{\gamma}$ has the same number of edges: projecting $\alpha^{\#\gamma} e^{-\frac{i}{2}\mathbf{W}(\gamma)}$on the real line will just multiply the modulus by $\cos\left(\frac{\pi}{8}\right)$, and we get $\alpha^{\#\tilde{\gamma}} e^{-i\mathbf{W}(\tilde{\gamma})}$.
    - In the second case, $e^{-\frac{i}{2}\mathbf{W}(\gamma)} \parallel e^{\frac{3\pi i}{8}}$ and $\#\tilde{\gamma} = \#\gamma + 1$, and we have that $\alpha^{\#\tilde{\gamma}} e^{-\frac{i}{2}\mathbf{W}(\tilde{\gamma})} = \alpha e^{-\frac{3\pi i}{8}} \cdot \alpha^{\#\gamma} e^{-\frac{i}{2}\mathbf{W}(\gamma)}$. We can check that $\mathsf{P}_\mathbb{R}\left[e^{\frac{3\pi i}{8}}\right] = \alpha\cos\left(\frac{\pi}{8}\right)$ and hence we get the result.

Discrete singularity.

- The problem is: how to define $f_{\Omega,b}(b)$?
  - What does a configuration in $\mathcal{L}_b(b)$ mean? We get the same thing as $\mathcal{L}_\delta$.
  - What is the winding number? it should be $\pm\pi$, to get the correct phase at b.
  - It gives $f_{\Omega_\delta,b}(b) = \mp - i$, and we get that $\mathsf{P}_{i\mathbb{R}}\left[f_{\Omega_\delta,b}\left(b + \frac{i\delta}{2}\right)\right] = -i$ and $\mathsf{P}_{i\mathbb{R}}\left[f_{\Omega_\delta,b}\left(b - \frac{i\delta}{2}\right)\right] = i$.

Boundary conditions.

- For any $z \in \partial\mathcal{M}_\delta$, we have that the phase of $f_{\Omega_\delta,b}(z)$ is completely determined: we have $f_{\Omega,a}(z) \parallel \nu^{-\frac{1}{2}}$
  - where $\nu(z) = z - v$, with $v \in \mathcal{V}_\delta$ the closest vertex, is the outward-pointing normal vector, viewed as a complex number
- This follows by observing that the winding $\mathbf{W}(\omega)$ of any $\omega \in \mathcal{L}(b,z)$ is determined modulo 2π, hence $e^{-\frac{i}{2}\mathbf{W}(\omega)} \parallel \nu^{-\frac{1}{2}}$

Energy density. It can be shown [HoSm10]:

- That the function $f_{\Omega_\delta,b}$ defined above converges $f_{\Omega_\delta,b} \underset{\delta\to 0}{\longrightarrow} f_{\Omega,a}$
  - Where $f_{\Omega,a}$ is the unique function such that $f_{\Omega,a} - \frac{1}{z-a}$ is holomorphic near a and such that $f_{\Omega,a} \parallel \nu^{-\frac{1}{2}}$ on $\partial\Omega$.
- And that $f_{\Omega,a}(z) = \frac{1}{z-a} + |\phi'(a)| + \mathcal{O}(z-a)$ [b and a collapse to the same point]
- And we can say something interesting about the product of two adjacent spins $\frac{1}{\delta}\mathbb{E}^+_{\Omega_\delta}\left[\sigma_a\sigma_{a+2\delta} - \frac{2}{\pi}\right] \to \mathcal{C}\,|\phi'(a)|$, for some $\mathcal{C} > 0$.

Spinor.

- The idea is that we want to compute $\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a+2\delta}\right]/\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\right]$. We would like to take $\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a+2\delta}\sigma_a\right]=\frac{\sum\alpha^{\text{edges}}(-1)^{\#\text{loops}(\omega,a.a+2\delta)}}{\sum\alpha^{\#\text{edges}}}$ a reweigh the numerator an the denominator by $(-1)^{\#\text{loops}(\omega,a)}$
- So, we would like to reweigh the observable

$$\frac{\sum_{\gamma\in\mathcal{L}(b,z)}\alpha^{\#\gamma}e^{-\frac{i}{2}\mathbf{W}(\gamma)}}{\sum_{\omega\in\mathcal{L}}\alpha^{\#\omega}}\rightsquigarrow\frac{\sum_{\gamma\in\mathcal{L}(b,z)}(-1)^{\#\text{loops}(\omega,a)}\alpha^{\#\gamma}e^{-\frac{i}{2}\mathbf{W}(\gamma)}}{\sum_{\omega\in\mathcal{L}}\alpha^{\#\omega}(-1)^{\#\text{loops}(\omega,a)}}.$$

- The reweighed version gives $\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a+2\delta}\right]/\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\right]$ at $z=b+\delta$.
- But this spoils the holomorphicity: if the bijection destroys a loop around a, then we get the opposite sign to what we should get.
- If we destroy a loop, what does this loop become? A piece of path around a.
- So, we can save the s-holomorphicity if we are ready to go to the double cover.
- Let $[\Omega_\delta,a]$ denote the double cover of Ω_δ around a. It is an analogue of the square root surface:
  - Above a point of Ω_δ, there are two points of $[\Omega_\delta,a]$ (living on two sheets).
  - If we make a loop around a, we end up on the other sheet.
  - From now on, we will identify b with one of the points of $[\Omega_\delta,a]$ living on one of the sheets.
- For $z\in[\Omega_\delta,a]$, we define the spinor $f_{[\Omega_\delta,a]}(b,z)$, which we will abbreviate $f_{[\Omega_\delta,a]}(z)$ by

$$f_{[\Omega_\delta,a]}(b,z):=\frac{\sum_{\gamma\in\mathcal{L}(b,z)}(-1)^{\#\text{loops}(\omega,a)}\alpha^{\#\gamma}e^{-\frac{i}{2}\mathbf{W}(\gamma)}\text{sheet}(\gamma,\text{z})}{\sum_{\omega\in\mathcal{L}}\alpha^{\#\omega}(-1)^{\#\text{loops}(\omega,a)}}$$

  - where $\text{sheet}(\gamma,z)$ is equal to 1 if the lift of γ to the double cover $[\Omega_\delta,a]$, starting from b (i.e. its identification on $[\Omega_\delta,a]$) ends on the same sheet at z
  - where $\text{sheet}(\gamma,z)$ is equal to -1 otherwise.
- Notice that $\mathcal{L}(b,z)$ is the same as before, there is no double cover in the loop configurations we consider, the double cover is involved in how we count them.
- Now, we have restored the s-holomorphicity: if we destroy a loop, there is a -1 sign that we lose, but we also change sheet.
- The boundary conditions are the same, the singularity much the same (besides the monodromy).

Convergence: outline. Now, we have constructed $f_{[\Omega_\delta,a]}$ such that $f_{[\Omega_\delta,a]}(b+2\delta)=\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a+2\delta}\right]/\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\right]$ ($b+2\delta$ same sheet as b).

- We would like to show

$$f_{[\Omega_\delta,a]}(b+2\delta)=1+2\delta\partial_{\mathfrak{Re}}\log\langle\sigma_a\rangle^{+}_{\Omega}+o(\delta)$$

- We will show that there exists $\vartheta(\delta) \asymp \delta^{\frac{1}{2}}$ such that $\frac{1}{\vartheta(\delta)} f_{[\Omega_\delta,a]} \to f_{[\Omega,a]}$ where $f_{[\Omega,a]}$ is holomorphic (explicit) and

$$f_{[\Omega,a]}(z) = \frac{1}{\sqrt{z-a}} + 2\mathcal{A}_{[\Omega,a]} \log \langle \sigma_a \rangle_\Omega^+ \sqrt{z-a} + o\left(\sqrt{z-a}\right)$$

  where $\mathcal{A}_{[\Omega,a]} = \partial_{\Re\mathfrak{e}(a)} \log \langle \sigma_a \rangle_\Omega^+ - i\partial_{\Im\mathfrak{m}(a)} \log \langle \sigma_a \rangle_\Omega^+$
- After, we will have to explain why $f_{[\Omega_\delta,a]}(b+2\delta) = 1+\delta\Re\mathfrak{e}\left(\mathcal{A}_{[\Omega,a]}\right)+o(\delta)$, which is not quite trivial.

Preliminary remarks.

- The values of an s-holomorphic function on the corners of type 1 (pointing from left to right from the closest vertex) play the role of real part of that function.
- The values on the corners of type i (pointing from right to left from the closest vertex) play the role i times the imaginary part.
- The values on the mid-edges are the 'plain' values of an s-holomoprhic function (i.e. real + i imaginary).
- From the restriction of an s-holomorphic function to the horizontal corners (i.e. corners of type 1 and i), we can recover the s-holomorphic function everywhere.
- The restriction of an s-holomorphic function to the horizontal corners satisfy Cauchy-Riemann equations.
- The restriction of an s-holomorphic function to the corners of type 1 (or to the corners of type i) is discrete harmonic and determines uniquely the function up to an additive constant.

Definition of the full plane spinor. If there were a full-plane version $f_{[\mathbb{C}_\delta,a]}$ of the spinor (where $\mathbb{C}_\delta := \delta(1+i)\mathbb{Z}^2$) , what would it look like?

- If we look at the restriction f_δ of $f_{[\mathbb{C}_\delta,a]}$ to the corners of type 1, what do we see?
  - It is discrete harmonic
  - By symmetry (seeing what the winding does), we have that f_δ vanishes on the line $\{a-x, x>0\}$
  - We have $f_\delta(a_\to + \delta) = 1$ (ratios of expected values of spin should be 1 in the full plane)
  - And $f_\delta \to 0$ as $z \to \infty$ (natural to expect by probabilistic interpretation, using high temperature expansion of Ising correlations)
- Let Σ_+ denote the sheet over $[\mathbb{C}_\delta, a] \setminus \{a-x : x>0\}$ that contains $a_\to$.
- We have that $f_\delta(z)$, restricted to Σ_+, should equal the probability that a simple random walk on $\mathbb{C}_\delta+\frac{\delta}{2}$ started from z hits $a_\to$before $\{a-x, x>0\}$.
- We can take this as a definition of f_δ restricted to corners of type 1 of Σ_+. We can extend on the other sheet Σ_- by taking minus this and to $\{a-x : x>0\}$ by the value 0.
- Using the vanishing condition at ∞, we can reconstruct $f_{[\mathbb{C}_\delta,a]}$ from there.

Convergence of the full plane spinor.

- What kind of convergence result can we prove? Let $\vartheta(\delta)$ be the value $f_{[\mathbb{C}_\delta,a]}(a+1)$ (take the closest corner of type 1 to $a+1$).

- We can prove that
$$\frac{1}{\vartheta(\delta)} f_{[\mathbb{C}_\delta, a]}(z) \to \frac{1}{\sqrt{z-a}},$$
for any z away from a.
- How can we prove that? Look at the sheet Σ_+ the function is positive there:
  - By compactness arguments, we can say that there is a subsequential scaling limit (if things would blow up, they would blow up at $z = a+1$, but the function equals to 1 there).
  - We can get that subsequential limits are continuous harmonic, non-negative, equal to 1 at 1, equal to 0 on $\{a - x : x > 0\}$, tending to 0 at ∞.
  - There is no choice of this subsequential scaling limit to be $\mathfrak{Re}\left(\frac{1}{\sqrt{z-a}}\right)$.
- We take the harmonic conjugate, etc, and get the convergence of the spinor.

Convergence of the domain spinor.

- Now we have the normalization: we want to prove that
$$\frac{1}{\vartheta(\delta)} f_{[\Omega_\delta, a]}(z) \to f_{[\Omega, a]}(z),$$
where
$$f_{[\Omega,a]}(z) = \frac{1}{\sqrt{z-a}} + \mathcal{O}\left(\sqrt{z-a}\right)$$
and
$$f_{[\Omega,a]} \parallel \nu^{-\frac{1}{2}} \quad \text{on } \partial\Omega.$$
- How do we prove that? It is tricky, because of the boundary conditions.
- We have that $f_{[\Omega,a]}^2 \parallel \nu^{-1}$ and $f_{[\Omega,a]}^2(z)\,\mathrm{d}z \parallel i$ on $\partial\Omega$ and hence $\mathfrak{Re}\left(\int f_{[\Omega,a]}^2\right)$ is constant on $\partial\Omega$.
- We will construct a lattice version of $\mathfrak{Re}\left(\int f_{[\Omega,a]}^2\right)$, which will enable to pass to the scaling limit.

Discrete integral of the square.

- First observation: if we want to integrate $f_{[\Omega_\delta,a]}^2$, there is no need to look at the double cover (because of the -1 monodromy).
- We define the discrete integral $I_\delta : \mathcal{V}_\delta \cup \mathcal{C}_\delta \to \mathbb{R}$, by local integration, as follows:
  - If $v \in \mathcal{V}_\delta$ and $x \in \mathcal{F}_\delta$ are adjacent (i.e. at distance $\frac{\delta}{2}$) and $c \in \mathcal{C}_\delta$ is the corner between them, we set $I_\delta(v) - I_\delta(x) = \delta \left| f_{[\Omega_\delta,a]}^2(c) \right|^2$.
  - To check that I_δ is well-defined by this, let's see what happen around a midpoint of edge $m \in \mathcal{M}_\delta$, if we sum the increments counterclockwise.
  - We end up with $|\mathsf{P}_{\mathbb{R}}[f(m)]|^2 - \left|\mathsf{P}_{\overline{\lambda}\mathbb{R}}[f(m)]\right|^2 + |\mathsf{P}_{i\mathbb{R}}[f(m)]|^2 - |\mathsf{P}_{\lambda\mathbb{R}}[f(m)]|^2$. This is zero, by Pythagoras theorem (make a picture).

- Why is it a discrete analogue of $\mathfrak{Re}\left(\int f_{[\Omega,a]}^2\right)$? Let $v_1, v_2 \in \mathcal{V}_\delta$ be two adjacent vertices, such that $v_2 = v_1 + (1+i)\,\delta$, say. We have that

$$I_\delta\left(v_2\right) - I_\delta\left(v_1\right) = \mathfrak{Re}\left(\left(f_{[\Omega_\delta,a]}\left(v_1 + \frac{1+i}{2}\delta\right)\right)^2 (1+i)\right)$$

- What about the boundary values? They are exactly constant on $\partial\mathcal{F}_\delta$.

Convergence.

- One can prove that $\frac{1}{\vartheta^2(\delta)} I_\delta \to I$, where $I = \mathfrak{Re}\left(\int f_{[\Omega,a]}^2\right)$ as $\delta \to 0$, using that the boundary condition is Dirichlet and that $f_{[\Omega,a]} - \frac{1}{\sqrt{z-a}}$ is regular near a (true on discrete level, too).
- The proof is tricky and involves defining I_δ and $I_\delta^{(a)} := \mathfrak{Re}\left(\int f_{[\Omega_\delta,a]} - f_{[\mathbb{C}_\delta,a]}^2\right)$.

Discrete square root.

- We want to show that $\left(f_{[\Omega_\delta,a]} - f_{[\mathbb{C}_\delta,a]}\right)\left(a_\to + \delta\right) = \mathfrak{Re}\left(\mathcal{A}_{[\Omega_\delta,a]}\right)\delta$, where $f_{[\Omega,a]} = \frac{1}{\sqrt{z-a}} + \mathcal{A}_{[\Omega,a]}\sqrt{z-a}$.
- We introduce a discrete square root r_δ, such that $r_\delta\left(a_\to + \delta\right) = \delta$ and such that $\frac{1}{\vartheta(\delta)} r_\delta \to \sqrt{z-a}$ as $\delta \to 0$.
- We define r_δ as the antiderivative of $f_{[\mathbb{C}_\delta,a]}$ on lattice level, normalized to have -1 monodromy around a.
- The convergence follows from that of $\frac{1}{\vartheta(\delta)} f_{[\mathbb{C}_\delta,a]}$
- Now the problem is to show that $\left(f_{[\Omega_\delta,a]} - f_{[\mathbb{C}_\delta,a]} - \mathfrak{Re}\left(\mathcal{A}_{[\Omega_\delta,a]}\right) r_\delta\right)\left(a_\to + \delta\right) = o\left(\delta\right)$.

Reflection trick.

- Let $\mathcal{R}_a$ denote the reflection with respect to the axis $\{a + x : x \in \mathbb{R}\}$. Consider $f_\delta := f_{[\Omega_\delta,a]}$ and $g_\delta := f_{[\mathcal{R}_a[\Omega_\delta],a]}$. Consider $f_\delta^+ := f_\delta + g_\delta$ and $f_\delta^- := f_\delta - g_\delta$ and notice that $f_\delta = \frac{1}{2}\left(f_\delta^+ + f_\delta^-\right)$.
- Now, we should show that $\left(f_\delta^+ - 2f_{[\mathbb{C}_\delta,a]} - 2\mathcal{A}_{[\Omega,a]} r_\delta\right)\left(a_\to + \delta\right) = o\left(\delta\right)$ and that $f_\delta^-\left(a_\to + \delta\right) = 0$.
- The second identity follows from symmetry considerations: the logarithmic derivative of spin correlation in x-direction in Ω_δ and $\mathcal{R}_a\left(\Omega_\delta\right)$ are the same
- For the first identity, notice that f_δ^+ is 0 on the corners of type 1 of $\{a - x : x > 0\}$, by symmetry: half of the time, the curve comes in one direction, half of the time in the other direction.
- Now $\mathfrak{Re}\left(\frac{1}{\vartheta(\delta)}\left(f_\delta^+ - 2f_{[\mathbb{C}_\delta,a]} - 2\mathcal{A}_{[\Omega,a]} r_\delta\right)(z)\right)$ is $o\left(\sqrt{z-a}\right)$ near $z = a$, by definition of $\mathcal{A}_{[\Omega,a]}$.
- Looking at the real part, and using the discrete Beurling estimate, we see a function that vanishes on a branch cut, and is really small around, and we evaluate the function at one lattice step away from the cut: we get that it is $\leq C\delta^{\frac{1}{2}}$ for any C, i.e. $o\left(\delta^{\frac{1}{2}}\right)$
- Since we know that $\vartheta\left(\delta\right) \asymp \delta^{\frac{1}{2}}$, we get the desired result.

Output of discrete complex analysis. Using methods seen in the previous classes, we can compute:

- Ratios of spin correlations at different locations

$$\frac{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a_1}\cdots\sigma_{a_n}\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{b_1}\cdots\sigma_{b_n}\right]} \underset{\delta\to 0}{\longrightarrow} \frac{\langle\sigma_{a_1}\cdots\sigma_{a_n}\rangle^{+}_{\Omega}}{\langle\sigma_{b_1}\cdots\sigma_{b_n}\rangle^{+}_{\Omega}}$$

- Ratios of 2-point functions + and free boundary condittions

$$\frac{\mathbb{E}^{\text{free}}_{\Omega_\delta}\left[\sigma_a\sigma_b\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_b\right]} \underset{\delta\to 0}{\longrightarrow} \frac{\langle\sigma_a\sigma_b\rangle^{\text{free}}_{\Omega}}{\langle\sigma_a\sigma_b\rangle^{+}_{\Omega}}$$

- Exact computation in the full plane [Wu]: for $x > 0$

$$\mathbb{E}_{\mathbb{C}_\delta}\left[\sigma_a\sigma_{a+x}\right] = \frac{1}{(2\pi)^N}\prod_{\ell=1}^{N-1}\left(1-\frac{1}{4\ell^2}\right)^{\ell-N},$$

  where $N \sim \frac{x}{2\delta}$ is the number of diagonal steps between σ_a and σ_{a+x}. Asymptotic formula for the right hand side is $\mathcal{A}N^{-\frac{1}{4}}$, where $\mathcal{A} = e^{-3\zeta'(-1)-\frac{1}{6}\log 2}$ and hence

$$\frac{1}{\delta^{\frac{1}{4}}}\mathbb{E}_{\mathbb{C}_\delta}\left[\sigma_a\sigma_{a+x}\right] \to \mathcal{C}^2\left\langle\sigma_a\sigma_{a+x}\right\rangle_{\mathbb{C}}$$

Calibration of the two-point function. The idea is that when we take two points very close to each other, they don't feel the domain anymore, and we can compare to the full plane

- For any domain Ω_δ, we have, by FKG inequality

$$\mathbb{E}^{\text{free}}_{\Omega_\delta}\left[\sigma_a\sigma_{a+x}\right] \quad\leq\quad \mathbb{E}_{\mathbb{C}_\delta}\left[\sigma_a\sigma_{a+x}\right] \leq \mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_{a+x}\right].$$

  In the continuum, we have the formulae

$$\langle\sigma_a\sigma_{a+x}\rangle^{\text{free}}_{\Omega} \leq \langle\sigma_a\sigma_{a+x}\rangle_{\mathbb{C}} \leq \langle\sigma_a\sigma_{a+x}\rangle^{+}_{\Omega}.$$

- We get that for any a, b

$$\frac{1}{\delta^{\frac{1}{4}}}\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_b\right] = \frac{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_b\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_{a+x}\right]}\frac{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_{a+x}\right]}{\mathbb{E}_{\mathbb{C}_\delta}\left[\sigma_a\sigma_{a+x}\right]}\frac{\mathbb{E}_{\mathbb{C}_\delta}\left[\sigma_a\sigma_{a+x}\right]}{\delta^{\frac{1}{4}}} \leq \frac{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_b\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_{a+x}\right]}\frac{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_{a+x}\right]}{\mathbb{E}^{\text{free}}_{\Omega_\delta}\left[\sigma_a\sigma_{a+x}\right]}\frac{\mathbb{E}_{\mathbb{C}_\delta}\left[\sigma_a\sigma_{a+x}\right]}{\delta^{\frac{1}{4}}}$$

  Passing this to the limit, we get

$$\begin{aligned}
\mathcal{C}^2\frac{\langle\sigma_a\sigma_b\rangle^{+}_{\Omega}}{\langle\sigma_a\sigma_{a+x}\rangle^{+}_{\Omega}}\langle\sigma_{a+x}\rangle_{\mathbb{C}} \quad&\leq\quad \liminf_{\delta\to 0}\frac{1}{\delta^{\frac{1}{4}}}\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_b\right] \\
&\leq\quad \limsup\frac{1}{\delta^{\frac{1}{4}}}\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_b\right] \\
&\leq\quad \mathcal{C}^2\frac{\langle\sigma_a\sigma_b\rangle^{+}_{\Omega}}{\langle\sigma_a\sigma_{a+x}\rangle^{+}_{\Omega}}\frac{\langle\sigma_a\sigma_{a+x}\rangle^{+}_{\Omega}}{\langle\sigma_a\sigma_{a+x}\rangle^{\text{free}}_{\Omega}}\langle\sigma_{a+x}\rangle_{\mathbb{C}}
\end{aligned}$$

  We can now let $x \to 0$ and make explicit computations to get $\langle\sigma_{a+x}\rangle_{\mathbb{C}} / \langle\sigma_a\sigma_{a+x}\rangle^{+}_{\Omega} \to 1$ and $\langle\sigma_{a+x}\rangle_{\mathbb{C}} / \langle\sigma_a\sigma_{a+x}\rangle^{\text{free}}_{\Omega} \to 1$.

Calibration of the one-point function. The idea is that the two-point function factorizes into a product of one-point functions as one of the point goes to the boundary.

- FKG inequality (first) and GHS inequality (second) imply
$$0 \leq \mathbb{E}^{+}_{\Omega_\delta}[\sigma_a\sigma_b] - \mathbb{E}^{+}_{\Omega_\delta}[\sigma_a]\,\mathbb{E}^{+}_{\Omega_\delta}[\sigma_b] \leq \mathbb{E}^{\text{free}}_{\Omega_\delta}[\sigma_a\sigma_b]$$
In the continuum, we have
$$0 \leq \langle\sigma_a\sigma_b\rangle^{+}_{\Omega} - \langle\sigma_a\rangle^{+}_{\Omega}\langle\sigma_b\rangle^{+}_{\Omega} \leq \langle\sigma_a\sigma_b\rangle^{\text{free}}_{\Omega}$$
- Hence
$$1 - \frac{\mathbb{E}^{\text{free}}_{\Omega_\delta}[\sigma_a\sigma_b]}{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a\sigma_b]} \leq \frac{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a]\,\mathbb{E}^{+}_{\Omega_\delta}[\sigma_b]}{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a\sigma_b]} \leq 1$$
- We deduce that
$$1 - \frac{\langle\sigma_a\sigma_b\rangle^{\text{free}}}{\langle\sigma_a\sigma_b\rangle^{+}_{\Omega}} \leq \liminf_{\delta\to 0} \frac{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a]\,\mathbb{E}^{+}_{\Omega_\delta}[\sigma_b]}{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a\sigma_b]} \leq \limsup_{\delta\to 0} \frac{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a]\,\mathbb{E}^{+}_{\Omega_\delta}[\sigma_b]}{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a\sigma_b]} \leq 1$$
- We write
$$\left(\frac{1}{\delta^{\frac{1}{8}}}\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a]\right)^2 = \frac{1}{\delta^{\frac{1}{4}}}\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a\sigma_b]\,\frac{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a]\,\mathbb{E}^{+}_{\Omega_\delta}[\sigma_b]}{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a\sigma_b]}\,\frac{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a]}{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_b]}$$
- Using the previous result, we get that
$$\mathcal{C}^2\langle\sigma_a\sigma_b\rangle^{+}_{\Omega}\frac{\langle\sigma_a\rangle^{+}_{\Omega}}{\langle\sigma_b\rangle^{+}_{\Omega}}\left(1 - \frac{\langle\sigma_a\sigma_b\rangle^{\text{free}}}{\langle\sigma_a\sigma_b\rangle^{+}_{\Omega}}\right) \leq \liminf\left(\frac{1}{\delta^{\frac{1}{8}}}\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a]\right)^2 \leq \limsup\left(\frac{1}{\delta^{\frac{1}{8}}}\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a]\right)^2 \leq \mathcal{C}^2\langle\sigma_a\sigma_b\rangle^{+}_{\Omega}\frac{\langle\sigma_a\rangle^{+}_{\Omega}}{\langle\sigma_b\rangle^{+}_{\Omega}}$$
- Now, direct computations in the continuum imply that, as $b \to \partial\Omega$
$$\langle\sigma_a\sigma_b\rangle^{+}_{\Omega}\frac{\langle\sigma_a\rangle^{+}_{\Omega}}{\langle\sigma_b\rangle^{+}_{\Omega}} \to \left(\langle\sigma_a\rangle^{+}_{\Omega}\right)^2$$
$$\left(1 - \frac{\langle\sigma_a\sigma_b\rangle^{\text{free}}}{\langle\sigma_a\sigma_b\rangle^{+}_{\Omega}}\right) \to 1$$
- We get
$$\left(\frac{1}{\delta^{\frac{1}{8}}}\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a]\right)^2 \to \mathcal{C}^2\langle\sigma_a\rangle^{+}_{\Omega}$$

N-point calibration. This part relies on an induction argument: we factorize the $n+1$-point function into a product of 1-point and n-point functions as the 1-point moves to the boundary.

- On lattice level, we can show that $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a\sigma_{a_1}\cdots\sigma_{a_n}] \sim \mathbb{E}^{+}_{\Omega_\delta}[\sigma_a]\,\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}]$ in the double limit $\delta \to 0$ and then $a \to \partial\Omega$.
  - The $\geq$ is always true.
  - The $\leq$ is asymptotically (in the double limit) true: we cut the domain into two domains, one containing a and the other the other points, bound the left hand side from above and use the induction assumption to that cutting the domain has not changed much when a is close to $\partial\Omega$ and the cut is close (but not too close) to a.
- We have that the $(n+1)$-point function $\langle\sigma_a\sigma_{a_1}\cdots\sigma_{a_n}\rangle^{+}_{\Omega} \sim \langle\sigma_a\rangle^{+}_{\Omega}\langle\sigma_{a_1}\cdots\sigma_{a_n}\rangle^{+}_{\Omega}$ as $a \to \partial\Omega$. This part is tricky, because exact formulae are hard to get.

CHAPTER 4

Convergence of Interfaces to SLE

A simple convergence statement. There is a simple way to say: 'the critical percolation interface converges to SLE(6)'. This is also the simplest way to prove it.

- Consider a percolation interface γ_δ, let γ be a subsequential scaling limit as $\delta \to 0$ (guaranteed to exist by general arguments [AiBu99]).
- Let $\varphi : \Omega \to \mathbb{H}$ be a conformal map with $a \mapsto 0$ and $b \mapsto \infty$ and let $\lambda := \varphi(\gamma)$.
- Encode λ by a Löwner chain with driving force $(U_t)_{t\geq 0}$. Then we have that $U_t = \sqrt{6}B_t$ (in law).

Martingales.

- The Brownian motion is a martingale: $\mathbb{E}\left[B_{t+s}\middle|B_{[0,t]}\right] = B_t$ for any $t, s \geq 0$ (more precisely, it is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(B_u : u \in [0,t])$).
- A martingale plays informally the role of a (stochastic) conservation law: it is a quantity which is conserved 'on average'.
- Lévy's characterization theorem: if $(M_t)_{t\geq 0}$ and $\left(M_t^2 - \kappa t\right)_{t\geq 0}$ are martingales, then $M_t = \sqrt{\kappa}B_t$.

From percolation to SLE.

- The idea is to show that for any $\ell < 0 < r$, the evaluation of Cardy's formula $\mathfrak{C}(H_t, \ell, \gamma_t, r, \infty)$ is a martingale (with respect to the filtration generated by $(U_t)_t$), up to the time when γ_t hits $(-\infty, \ell] \cup [r, \infty)$.
- We have that Cardy's formula is conformally invariant (by definition) and hence

$$\begin{aligned}\mathfrak{C}(H_t, \ell, \gamma_t, r, \infty) &= \mathfrak{C}(\mathbb{H}, g_t(\ell), U_t, g_t(r), \infty) \\ &= \mathfrak{C}(\mathbb{H}, g_t(\ell) - U_t, 0, g_t(r) - U_t, \infty) \\ &= \mathfrak{C}\left(\mathbb{H}, \frac{g_t(\ell) - U_t}{g_t(r) - U_t}, 0, 1, \infty\right)\end{aligned}$$

  is a martingale.
- We have that $\mathfrak{C}(\mathbb{H}, \cdot)$ is given in terms of the conformal mapping to the equilateral triangle, and has an explicit formula $F\left(\frac{g_t(\ell)-U_t}{g_t(r)-U_t}\right)$.
- Once we have that $\left(F\left(\frac{g_t(\ell)-U_t}{g_t(r)-U_t}\right)\right)_t$ is a martingale, we get that $(U_t)_t$ and $\left(U_t^2 - 6t\right)_t$ are martingales:
  - Take $r = z$ and $\ell = -2z$
  - We have that $g_t(z) = z + \frac{2t}{z} + \ldots$ as $z \to \infty$.

– We get (plugging the values of the explicit formula F)

$$
\begin{aligned}
& F\left(\frac{z-U_t+2t/z+\mathcal{O}\left(1/z^2\right)}{\left(z+2t/z+\mathcal{O}\left(1/z^2\right)\right)-\left(-2z+2t/\left(-2z\right)+\mathcal{O}\left(1/z^2\right)\right)}\right) \\
= \; & F\left(\frac{1}{3}-\frac{U_t}{3}\frac{1}{z}+\frac{t}{3}\frac{1}{z^2}+\mathcal{O}\left(\frac{1}{z^3}\right)\right) \\
= \; & F\left(\frac{1}{3}\right)-\frac{U_t}{3}F'\left(\frac{1}{3}\right)\frac{1}{z}+\left(\frac{t}{3}F'\left(\frac{1}{3}\right)+\frac{U_t^2}{3^2\cdot 2}F''\left(\frac{1}{3}\right)\right)\frac{1}{x^2}+\mathcal{O}\left(\frac{1}{z^3}\right) \\
= \; & F\left(\frac{1}{3}\right)-\frac{1}{z}\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}\frac{3^{\frac{1}{3}}}{2^{\frac{2}{3}}}U_t-\frac{1}{z^2}\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}\frac{1}{3^{2/3}2^{5/3}}\left(U_t^2-6t\right)+\mathcal{O}\left(\frac{1}{z^3}\right)
\end{aligned}
$$

– We deduce that U_t and U_t^2-6t are martingales

- How do we prove that Cardy's formula is a martingale? The idea is to show that crossing probabilities behaves as discrete-time martingales as the percolation interface grows. After that, we pass everything to the scaling limit.
- The crossing probability is a hitting probability for the percolation interface, so, if we look at this probability, conditionally on the first n steps of the interface and make one more step, averaging over the two steps, we get the original probability.