Lectures on Gaussian Multiplicative Chaos

RÉMI RHODES[1] & VINCENT VARGAS[2]

[1]Université Paris-Est Marne la Vallée, LAMA, Champs sur Marne, France.
[2]ENS Ulm, DMA, 45 rue d'Ulm, 75005 Paris, France.

Contents

To the reader

We have tried to gather in these lectures notes some basic elements of the theory of Gaussian multiplicative chaos as well as further comments corresponding to questions that we are often asked. Therefore these lectures will evolve in time as we will take into consideration (and be grateful to) any suggestion given by a reader who will have the kindness to contact us.

1 Introduction

It is rather customary in science to measure the importance of a theory through the number of occurrences of the theory in different and apparently non related areas of science. This short introduction is devoted to mentioning (non exhaustively) the various areas of probability theory that the theory of Gaussian multiplicative chaos has spread over, ranging from turbulence to Liouville quantum gravity through mathematical finance. Some other works related to random matrix theory, glassy phase of disordered systems or decaying Burger turbulence will be also mentioned.

Lognormal multiplicative martingales, the ancestor of Gaussian multiplicative chaos, were introduced by Mandelbrot [35] in order to build random measures describing energy dissipation and contribute explaining intermittency effects in Kolmogorov-Obukhov's KO62 theory of fully developed turbulence (see [31, 39]). Recall that a flow is said turbulent when the source of kinetic energy making the fluid move is much greater than the viscosity forces of the fluid: such flows are characterized by a highly irregular aspect, an unpredictable behaviour and the existence of many time or space scales. However, his model was difficult to define mathematically and this is why he proposed in [36] the simpler well known model of discrete multiplicative cascades. In 1985, Kahane came back to Mandelbrot's original suggestion [35] and laid the foundations of what is nowadays known under the name of Gaussian multiplicative chaos theory (GMC). Roughly speaking, in this lecture where we will mainly focus on a particular case of the theory, GMC is about making sense of random measures on a domain $D \subset \mathbb{R}^d$ of the type

$$e^{X(x)}\sigma(dx)$$

where σ is a Radon measure on D and $X : D \to \mathbb{R}$ a centered Gaussian distribution (in the sense of Schwartz) with covariance kernel of log-type, namely

$$\mathbb{E}[X(x)X(y)] = \gamma^2 \ln_+ \frac{1}{|x-y|} + g(x,y)$$

where $\gamma > 0$ is a parameter, $\ln_+(x) = \max(\ln x, 0)$ and g is a bounded function over $D \times D$.

One can thus claim that this theory is born in the study of turbulent flows. Since then, it has found applications in various other fields of science. For instance, it is also used in mathematical finance. If $(X(t))_{t\geq 0}$ is the logarithm of the price of a financial asset, the volatility V of the asset on the interval $[0,t]$ is by definition equal to the quadratic variation of X:

$$V[0,t] = \lim_{n\to\infty} \sum_{k=1}^{n} (X(tk/n) - X(t(k-1)/n))^2$$

and can be viewed as a random measure on $\mathbb{R}$. If one chooses V as a GMC, it satisfies many empirical properties measured on financial assets (approximate lognormality of the volatility, highly correlated volatility with long range correlations, etc...) called stylized facts. It is an instance of a larger class of one dimensional measures called log-infinitely divisble multifractal random measures, which were developed mathematically by Barral-Mandelbrot in [8] and Bacry-Muzy in [3]. Given the volatility V, the most natural way to construct a model for the (log) price X is to set:

$$X(t) = B_{V[0,t]} \tag{1}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion independent of V. Formula (1) defines the Multifractal Random Walk (MRW) when V is GMC and was introduced by Bacry, Delour and Muzy in [4, 6] (see [5] for a recent review of financial applications of the MRW model).

More recently, this theory has turned out to be useful in $2d$ Liouville quantum gravity[3] (LQG for short) which was introduced in the 1981 seminal work of Polyakov [41]. Roughly speaking,

[3]In fact, the following discussion will concern Liouville quantum field theory but we stick to the terminology LQG which is now standard in probability theory.

LQG can be seen as a canonical probability measure on the space of random functions $X : M \to \mathbb{R}$ where M is a two dimensional Riemannian manifold. For the sake of simplicity, we will consider the Riemann sphere $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ equipped with the standard round metric

$$\hat{g}(x)dx^2 \quad \text{with} \quad \hat{g}(x) = \frac{4}{(1+|x|^2)^2}.$$

On the Riemann sphere, LQG is the law of a random function $X : \mathbb{S}^2 \to \mathbb{R}$, which takes on the following formal form for all functionals F

$$\mathbb{E}[F(X)] = Z^{-1} \int F(X) e^{-S_L(X)} DX, \tag{2}$$

where Z is a normalization constant, DX stands for some "uniform measure" over random functions $X : M \to \mathbb{R}$ and S_L is the so-called Liouville action given by

$$S_L(X) := \frac{1}{4\pi} \int_{\mathbb{R}^2} \left(|\partial^{\hat{g}} X|^2(x) + Q R_{\hat{g}}(x) X(x) + 4\pi\mu e^{\gamma X(x)} \right) \hat{g}(x) dx \tag{3}$$

where $R_{\hat{g}}(x)$ is the Ricci scalar curvature of $\hat{g}$ (here equal to 2), $\partial^{\hat{g}} X$ is the gradient of X in the round metric and γ, μ, Q are some parameters satisfying

$$\gamma \in]0,2[, \quad Q = \frac{\gamma}{2} + \frac{2}{\gamma}, \quad \mu > 0. \tag{4}$$

Hence, LQG is parametrized by two constants γ and μ (μ is called the cosmological constant). Rigorously constructing the probability measure (2) is non trivial and the theory of Gaussian multiplicative chaos is one of the building blocks of the construction. We will not go into the details of the construction here (see [15] for the details) but let us just say that, after fixing three distinct points $x_1, x_2, x_3 \in \mathbb{S}^2$ called insertion points, there is a canonical way to construct under the probability measure (2) a unit volume random measure Z_L on the Riemann sphere, the so-called Liouville measure, that we will now describe using the language of GMC. We will also relate this object to random planar maps: see conjecture 1 below.

First, the theory of Gaussian multiplicative chaos enables to make sense for all $\gamma \in]0,2[$ of the random measure given formally by

$$M_\gamma(dx) = e^{\gamma X} \hat{g}(x) dx,$$

where X is a Gaussian field, called Gaussian Free Field (GFF), with correlation structure of the form

$$\mathbb{E}[X(x)X(y)] = G(x,y)$$

and G is the Green function on the Riemann sphere with vanishing mean, i.e.

$$G(x,y) = \ln \frac{(1+|x|^2)^{1/2}(1+|y|^2)^{1/2}}{|y-x|}. \tag{5}$$

Consider the measure

$$Z_\gamma(dx) = e^{\gamma^2 \sum_{i=1}^3 G(x,x_i)} M_\gamma(dx).$$

One can show that (see [15])

$$Z_\gamma(\mathbb{R}^2) < \infty \text{ a.s.} \quad \text{and} \quad \mathbb{E}[\frac{1}{Z_\gamma(\mathbb{R}^2)^\beta}] < \infty, \ \forall \beta > 0.$$

The unit volume Liouville measure Z_L is then defined for all $A_1, \cdots, A_k \subset \mathbb{S}^2$ by the following formula

$$\mathbb{E}^{x_1,x_2,x_3}[F(Z_L(A_1), \cdots, Z_L(A_k))] = \frac{\mathbb{E}\left[F\left(\frac{Z_\gamma(A_1)}{Z_\gamma(\mathbb{R}^2)}, \cdots, \frac{Z_\gamma(A_k)}{Z_\gamma(\mathbb{R}^2)}\right) Z_\gamma(\mathbb{R}^2)^{-3+2\frac{Q}{\gamma}}\right]}{\mathbb{E}[Z_\gamma(\mathbb{R}^2)^{-3+2\frac{Q}{\gamma}}]}, \tag{6}$$

where Q satisfies (4).

Following Polyakov's work [41], it was soon acknowledged by physicists that one should recover LQG as the limit of some kind of discretized 2d quantum gravity given by finite triangulations of size N as N goes to infinity (see [2] for example). From now on, we assume that the reader is familiar with the definition of a triangulation of the sphere: further details on triangulations of the sphere can be found in Miermont's lecture. More precisely, let $\mathcal{T}_N$ be the set of triangulations of $\mathbb{S}^2$ with N faces and $\mathcal{T}_{N,3}$ be the set of triangulations with N faces and 3 marked faces or points (called roots). Let us now explain how to embed a triangulation $T \in \mathcal{T}_{N,3}$ onto the sphere $\mathbb{S}^2$ and define a random measure on $\mathbb{S}^2$ out of it. Following [25] (see also [14, section 2.2]), we can equip such a triangulation with a conformal structure (where each face has the geometry of an equilateral triangle). The uniformization theorem tells us that we can then conformally map the triangulation onto the sphere $\mathbb{S}^2$ and the conformal map is unique if we demand the map to send the three marked points to the fixed points x_1, x_2, x_3. We denote by $\nu_{T,N}$ the corresponding deterministic measure where each triangle of the sphere is given a volume $\frac{1}{N}$. Concretely, the uniformization provides for each face $t \in T$ a conformal map $\psi_t : \Delta \to \mathbb{S}^2$ where Δ is an equilateral triangle of volume 1. Then $\nu_{T,N}(dz) = \frac{1}{N}|(\psi_t^{-1})'(z)|^2 dz$ on the image of the triangle $\psi_t(\Delta)$. In particular, the volume of the total space $\mathbb{S}^2$ is $N \times \frac{1}{N} = 1$. Now, we consider the random measure ν_N defined by

$$\mathbb{E}^N[F(\nu_N)] = \frac{1}{Z_N} \sum_{T \in \mathcal{T}_{N,3}} F(\nu_{T,N}), \tag{7}$$

for positive bounded functions F where Z_N is a normalization constant. We denote by $\mathbb{P}^N$ the probability law associated to $\mathbb{E}^N$.

We can now state a precise mathematical conjecture:

Conjecture 1 ([15]). *Under $\mathbb{P}^N$, the family of random measures $(\nu_N)_{N\geq 1}$ converges in law as $N \to \infty$ in the space of Radon measures equipped with the topology of weak convergence towards the law of the Liouville measure of LQG with parameter $\gamma = \sqrt{\frac{8}{3}}$ given by (6).*

Let us also mention that another (and related) approach of LQG has been developed in a series of works by Duplantier-Miller-Sheffield: see [17, 20, 37, 48]. These works rely heavily on the coupling of the field X and the associated GMC measure with SLE (see [16, 48] for the coupling theorems): for more on this approach, see the course by Miller. On the LQG side, it is not quite clear to us what this approach corresponds to in terms of the action which enters the definition of (2). On the random planar map side, this should correspond to studying the limit of ν_N defined by (7) as N goes to infinity while zooming around an insertion point, say x_1. In this context, one can rely on extra symmetries (scale invariance and isotropy around x_1).

Finally, let us mention that GMC also showed up in the study of decaying Burgers turbulence [23], of the extremes of log-correlated Gaussian fields [10, 11, 33], of the glassy phase of disordered systems [12, 21, 22, 34] or of the eigenvalues of Haar distributed random matrices [49]. For more on these topics (and also LQG), one can also have a look at the reviews [24, 43].

2 Prerequisite and notations

Notations

We equip $\mathbb{R}^d$ with the inner product $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$. Recall that a stochastic process $(X_t)_{t\in T}$ (where T is an arbitrary index set) is said Gaussian if and only if for all $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ and $t_1, \cdots, t_n \in T$ the variable $\sum_{i=1}^n \lambda_i X_i$ is Gaussian, in which case we denote its covariance kernel by $k_X(u, v) = \mathrm{cov}(X_u, X_v)$ for $u, v \in T$.

A stochastic process $(X_t)_{t\in\mathbb{R}^d}$ is said stationary if and only if, for every $z \in \mathbb{R}^d$, the processes $(X_t)_{t\in\mathbb{R}^d}$ and $(X_{z+t})_{t\in\mathbb{R}^d}$ have the same law. When X is Gaussian, X is stationary if and only if it has constant mean and its covariance kernel $k_X(u, v)$ only depends on the difference $u-v$. In this case, we can write $k_X(u, v) = g(u - v)$ for some function g, which is even and of positive type.

Finally, recall that a Gaussian random distribution X over a domain $D \subset \mathbb{R}^d$ is a Gaussian process indexed by the test functions $\varphi \in C_c^\infty(D)$. Here the word "distribution" has to be understood in the sense of Schwartz.

We will use the notation $\ln_+$ throughout the paper: it is defined by $\ln_+(x) = \max(\ln x, 0)$.

We define $\mathcal{M}_+(D)$ as the space of positive Radon measures on a locally compact metric space (D, ρ) and we equip $\mathcal{M}_+(D)$ with the topology of vague convergence of measures. Recall that this means that a sequence $(\mu_n)_n$ of elements in $\mathcal{M}_+(D)$ converges towards a measure $\mu \in \mathcal{M}_+(D)$ if for every continuous function f with compact support in D we have

$$\lim_{n\to\infty} \mu_n(f) = \mu(f).$$

We will denote by $\mathcal{B}(D)$ the Borel subsets of D.

Convexity inequalities

We recall the following comparison principle proved in [30] and which is a rather straightforward consequence of Lemma 22 in the appendix:

Theorem 2. Convexity inequalities. [Kahane, 1985]. *Let $(X_i)_{1\le i\le n}$ and $(Y_i)_{1\le i\le n}$ be two centered Gaussian vectors such that:*

$$\forall i, j, \quad \mathbb{E}[X_i X_j] \le \mathbb{E}[Y_i Y_j].$$

Then for all combinations of nonnegative weights $(p_i)_{1\le i\le n}$ and all convex (resp. concave) functions $F : \mathbb{R}_+ \to \mathbb{R}$ with at most polynomial growth at infinity

$$\mathbb{E}\Big[F\big(\sum_{i=1}^n p_i e^{X_i - \frac{1}{2}\mathbb{E}[X_i^2]}\big)\Big] \le (\textit{resp.} \ \ge)\, \mathbb{E}\Big[F\big(\sum_{i=1}^n p_i e^{Y_i - \frac{1}{2}\mathbb{E}[Y_i^2]}\big)\Big]. \tag{8}$$

Proof. If F is convex and smooth, then the result is a simple consequence of lemma 22. Indeed, with the notations of lemma 22, we have $\varphi'(t) \le 0$ for all $t \in [0, 1]$ since $F'' \ge 0$ and $\mathbb{E}[X_i X_j] \le \mathbb{E}[Y_i Y_j]$ for all i, j. Hence $\varphi(1) \le \varphi(0)$ which is precisely the desired inequality (8). If F is not smooth, one can approximate F by smooth convex functions to get the result. $\square$

3 Gaussian multiplicative chaos: Kahane's method

Let us consider a locally compact metric space (D,ρ). The main purpose of what follows is to explain how to construct random measures formally denoted by

$$e^{\gamma X(x)}\,\sigma(dx) \tag{9}$$

where $\sigma \in \mathcal{M}_+(D)$, X is a Gaussian random distribution and $\gamma > 0$ is some fixed constant. Of course, we could absorb the dependence in γ into the field X but we choose not to do so for reasons which will become apparent thereafter. When X is almost surely a continuous function of the variable $x \in D$ then the above measure perfectly makes sense and is absolutely continuous with respect to σ. It turns out that, in applications, the most interesting measures of the form 9 are those where X is a Gaussian distributions which does not live almost surely in the space of continuous functions and therefore it is not straightforward to make sense of (9): indeed, how do you define the exponential of a distribution?

Usually, the method when facing a badly behaved function consists in regularizing the function and see what we get by using some limiting procedure when the regularization is removed. Here, they are plenty of natural ways of regularizing the field X: for instance, one could think of making a convolution of X with some smooth function. We will discuss them all along this lecture. For the time being, we begin with Kahane's strategy as the construction is quite elegant, covers a large class of regularizations and is the cornerstone to treat the most general situation.

3.1 Kernels of σ-positive type

Kahane built a theory relying on the notion of kernel of σ-positive type. Consider a locally compact metric space (D,ρ). A function $K : D \times D \to \mathbb{R}_+ \cup \{\infty\}$ is of σ-positive type if there exists a sequence $(K_k)_k$ of continuous nonnegative and positive definite kernels $K_k : D \times D \to \mathbb{R}_+$ such that:

$$\forall x, y \in D, \quad K(x,y) = \sum_{k\geq 1} K_k(x,y). \tag{10}$$

Note that the sum perfectly makes sense as the functions $(K_k)_k$ are nonnegative.

Remark. *Basically, in what follows, K will play the part of the covariance kernel of the (centered) Gaussian distribution X, namely we will be in the situation where*

$$\mathbb{E}[X(\varphi)X(\psi)] = \int_D \int_D \varphi(x)\psi(y)K(x,y)\,dxdy,$$

for all test functions φ, ψ such that the above quantities make sense.

We further stress that Kahane's strategy uses nonnegativity of the kernels $(K_k)_k$ to establish some kind of universality of the law of measures (see below). One can skip this nonnegativity assumption if we are only interested in statements about existence of the measures. Furthermore, we will see in the next section a criterion to ensure uniqueness in law that does not rely on the nonnegativity of the kernels $(K_k)_k$.

If we are given a kernel K of σ-positive type with decomposition (10) and if we assume that the kernels $(K_k)_k$ are Hölder, one can consider a sequence of independent centered continuous

Gaussian processes $(Y_k)_{k\geq 1}$ with covariance kernels $(K_k)_k$. Then the Gaussian process

$$X_n = \sum_{k=1}^{n} Y_k$$

has covariance kernel $\sum_{k=1}^{n} K_k$, and thus may be thought of as an approximation of a centered Gaussian random distribution with covariance kernel K.

3.2 Gaussian multiplicative chaos

Given a Radon measure $\sigma \in \mathcal{M}_+(D)$, we consider the sequence of random measures $(M_{n,\gamma})_n$ given by:

$$\forall A \in \mathcal{B}(D), \quad M_{n,\gamma}(A) = \int_A e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\sigma(dx). \tag{11}$$

The reader may easily check that for each compact set A, the sequence $(M_{n,\gamma}(A))_n$ is a nonnegative martingale. Therefore it converges almost surely. This easily entails the almost sure convergence of the family $(M_{n,\gamma})_n$ in $\mathcal{M}_+(D)$. The limiting measure, denoted by M_γ, is called Gaussian multiplicative chaos[4] with kernel K acting on σ. This martingale structure ensuring the almost sure convergence of (11) at low cost is the main motivation for considering kernels of σ-positive type.

Then Kahane established a whole set of properties of this GMC that he derived from the comparison principle (Theorem 2). This ingenious inequality sheds light on the mechanism of GMC. For instance, let us stress that a kernel K of σ-positive type admits infinitely many decompositions of the form (10): one can obtain other decompositions by changing the order of the kernels K_k, by gathering them, etc... so there are possibly plenty of different families $(K'_k)_k$ whose sum is K. The important question thus is: does the law of the limiting measure M_γ depend on the choice of the decomposition $(K_k)_k$ in (10)?

Theorem 3. Uniqueness. [Kahane, 1985]. *The law of the limiting measure M_γ does not depend on the sequence of nonnegative and positive definite kernels $(K_k)_{k\geq 1}$ used in the decomposition (10) of K.*

Thus, the theory enables to give a unique and mathematically rigorous definition to a random measure M_γ in D defined formally by:

$$\forall A \in \mathcal{B}(D), \quad M_\gamma(A) = \int_A e^{\gamma X(x) - \frac{\gamma^2}{2}E[X(x)^2]}\,\sigma(dx). \tag{12}$$

where $(X(x))_{x\in D}$ is a centered "Gaussian field" whose covariance K is a kernel of σ-positive type. To show the usefulness of Theorem 2, it is worth giving a few words about the proof of Theorem 3.

[4]Private communication with J.P. Kahane: The terminology multiplicative chaos was adopted since this theory may be seen as a multiplicative counterpart of the additive Wiener chaos theory. Actually, it is Paul Lévy himself who suggested to J.P. Kahane in the seventies to construct a multiplicative theory of random variables, arguing that this should be as fundamental as the additive theory of random variables. It took Kahane almost ten years to build his theory of Gaussian multiplicative chaos.

(Short) proof of Theorem 3. Assume that you have two decompositions $(K_k)_{k\geq 1}$ and $(K'_k)_{k\geq 1}$ of K with associated Gaussian process sequences $(X_n)_n$ and $(X'_n)_n$ and associated measures $(M_n)_n$ and $(M'_n)_n$. Both sequences $(\sum_{k=1}^n K_k)_n$ and $(\sum_{k=1}^n K'_k)_n$ converge pointwise towards K in a nondecreasing way. Therefore, if we choose a compact set $T \subset D$ then, for each fixed $p \geq 1$ and $\epsilon > 0$, the Dini theorem entails that

$$\sum_{k=1}^{p} K_k \leq \epsilon + \sum_{k=1}^{q} K'_k$$

for q large enough on $T \times T$. Since $\sum_{k=1}^q K'_k$ (resp. $\sum_{k=1}^p K_k$) is the covariance kernel of X'_q (resp. X_p), we can apply Kahane's convexity inequalities and get, for each bounded convex function $F : \mathbb{R}_+ \to \mathbb{R}$:

$$\mathbb{E}[F(M_p(A))] \leq \mathbb{E}[F(e^{\sqrt{\epsilon}\gamma Z - \frac{\epsilon\gamma^2}{2}} M'_q(A))],$$

where Z is a standard Gaussian random variable independent of M'. By taking the limit as q tends to ∞, and then $p \to \infty$, we obtain

$$\mathbb{E}[F(M(A))] \leq \mathbb{E}[F(e^{\sqrt{\epsilon}\gamma Z - \frac{\epsilon\gamma^2}{2}} M'(A))].$$

Since $\epsilon > 0$ can be chosen arbitrarily small, we deduce

$$\mathbb{E}[F(M(A))] \leq \mathbb{E}[F(M'(A))].$$

The converse inequality is proved in the same way, showing $\mathbb{E}[F(M(A))] = \mathbb{E}[F(M'(A))]$ for each bounded convex function F. By choosing $F(x) = e^{-\lambda x}$ for $\lambda > 0$, we deduce that the measures M and M' have the same law. $\square$

However, the simplicity of the argument to get the almost sure convergence does not solve the question of non-degeneracy of the limiting measure M_γ: it is possible that M_γ identically vanishes. It seems difficult to state a general decision rule to decide whether M_γ is degenerate or not. It depends in an intricate way on the covariance structure, i.e. the kernel K, and on the measure σ. So Kahane focused on the situation when the kernel K and the measure σ are intertwined via the metric structure of D. The metric dependence of the kernel K is quantified through the assumption that K can be written as

$$\forall x, y \in D, \quad K(x,y) = \ln_+ \frac{T}{\rho(x,y)} + g(x,y) \tag{13}$$

where $T > 0$, $g : D \times D \to \mathbb{R}$ is a bounded continuous function. The metric dependence of σ is quantified in terms of the class R^+_α:

Definition 4. *For $\alpha > 0$, a Borel measure σ is said to be in the class R^+_α if for all $\epsilon > 0$ there is $\delta > 0$, $C < \infty$ and a compact set $A_\epsilon \subset D$ such that $\sigma(D \setminus A_\epsilon) \leq \epsilon$ and:*

$$\forall O \text{ open set}, \quad \sigma(O \cap A_\epsilon) \leq C_\epsilon \mathrm{diam}_\rho(O)^{\alpha+\delta}, \tag{14}$$

where $\mathrm{diam}_\rho(O)$ is the diameter of O with respect to ρ.

For instance, the Lebesgue measure of $\mathbb{R}^d$, restricted to any bounded domain of $\mathbb{R}^d$ equipped with the Euclidean distance ρ, is in the class R_α^+ for all $\alpha < d$.

The above definition looks like a Hölder condition for measures. It is intimately related to the notion of measure with finite β-energy: a Borel measure σ is said to be of finite β-energy if

$$I_\beta(\sigma) = \int_D \int_D \frac{1}{\rho(x,y)^\beta} \sigma(dx)\sigma(dy) < +\infty. \tag{15}$$

Indeed, if σ has a finite β-energy then $\sigma \in R_\alpha^+$ for all $\alpha < \beta$. Conversely, if σ is in the class R_α^+, then the measure $\sigma_{A_\epsilon}(dx) = \mathbf{1}_{A_\epsilon}(x)\sigma(dx)$ where A_ϵ satisfies (14) has finite β-energy for all $\beta < \alpha + \delta$.

To have a flavor of the forthcoming results, let us treat the following simple situation, which we call the L^2-threshold case. It is about formulating a criterion ensuring that the martingale $(M_{n,\gamma}(A))_n$ is bounded in L^2 for some given bounded set A, and therefore uniformly integrable. A straightforward computation shows that:

$$\begin{aligned}
\mathbb{E}[M_{n,\gamma}(A)^2] =& \int_A \int_A \mathbb{E}[e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]} e^{\gamma X_n(y) - \frac{\gamma^2}{2}\mathbb{E}[X_n(y)^2]}]\, \sigma(dx)\sigma(dy) \\
\leq & \int_A \int_A e^{\gamma^2 K(x,y)}\, \sigma(dx)\sigma(dy) \\
\leq & C \int_A \int_A \frac{1}{\rho(x,y)^{\gamma^2}}\, \sigma(dx)\sigma(dy).
\end{aligned}$$

Therefore, if the measure $\mathbf{1}_A(x)\sigma(dx)$ has finite γ^2-energy, the martingale $(M_{n,\gamma}(A))_n$ is bounded in L^2 and thus converges towards a non trivial limit. In the case when σ is the Lebesgue measure of $\mathbb{R}^d$ and ρ the Euclidean distance, this condition simply reads $\gamma^2 < d$.

Kahane proved the forthcoming highly deeper results.

Definition 5. *We will say that the GMC M_γ associated to a kernel K of σ-positive type and a Radon measure σ is non-degenerate if for each compact set A, the martingale $(M_{n,\gamma}(A))_n$ is uniformly integrable.*

We claim

Theorem 6. Non-degeneracy. [Kahane, 1985]. *Assume that the kernel K takes on the form* (13) *and that the measure σ is in the class R_α^+ for some $\alpha > 0$. Then, if $\gamma^2 < 2\alpha$, for each compact set A, the sequence $(M_{n,\gamma}(A))_n$ is a uniformly integrable martingale. Hence*

$$\gamma^2 < 2\alpha \Rightarrow M_\gamma \textit{ is non degenerate}.$$

In particular, for each compact set A, $\mathbb{E}[M_\gamma(A)] = \sigma(A)$.

Proof. To be added soon. ☐

As a by-product of his proof, Kahane also shows the following result concerning the dimension of the carrier of the measure M_γ:

Theorem 7. Structure of the carrier. [Kahane, 1985]. *Assume that the kernel K takes on the form* (13) *and that the measure σ is in the class R_α^+ for some $\alpha > 0$. If $\gamma^2 < 2\alpha$, the measure M_γ is non degenerate and is almost surely in the class $R^+_{\alpha - \frac{\gamma^2}{2}}$.*

Proof. To be added soon. □

Note that Theorem 7 implies that the measure M cannot give positive mass to a set of Hausdorff dimension less or equal than $\alpha - \frac{\gamma^2}{2}$. Therefore the measure M cannot possess atoms if σ is in some class R^+_α.

Remark. *If σ is the Lebesgue measure on some domain $D \subset \mathbb{R}^d$ equipped with the Euclidean metric, then σ is in the class $R^+_{d-\epsilon}$ for all $\epsilon > 0$ hence, if $\gamma^2 < 2d$, the measure M_γ is non degenerate and in the class $R^+_{d-\frac{\gamma^2}{2}-\epsilon}$ for all $\epsilon > 0$.*

It turns out that the Hausdorff dimension of the carrier is exactly $d - \frac{\gamma^2}{2}$ when σ is the Lebesgue measure on a domain of $\mathbb{R}^d$: see the work [9] for the $1d$ case and [13, 43] for the general case (this was also treated in [27] in the case of the two-dimensional Gaussian free field with a circle average cutoff).

Partial converses of Theorem 6 are more intricate. Kahane first gave a general necessary condition:

Theorem 8. Necessary condition of non-degeneracy. [Kahane, 1985]. *Assume (D, ρ) is a locally compact metric space and:*
-the function $(t, s) \mapsto \rho(t, s)^2$ is of negative type,
-σ has the doubling property, namely that there exists a constant C such that

$$\forall x \in D, \forall r > 0, \quad \sigma(B(x, 2r)) \leq C\sigma(B(x, r)),$$

-the kernel K takes on the form (13).
Denote by $\dim(D)$ the Hausdorff dimension of D. If $\gamma^2 > 2\dim(D)$ then M_γ is degenerate.

As pointed out by Kahane, for a squared distance of negative type, the assumptions of the above theorem are satisfied when the triple (D, ρ, σ) admits a Lipschitz immersion into a finite-dimensional space. Let us also stress that the critical situation $\gamma^2 = 2\dim(D)$ is not settled by this theorem. Nevertheless, he reinforced his assumptions to prove:

Theorem 9. Necessary and sufficient condition of non-degeneracy. [Kahane, 1985]. *Assume (D, ρ) is a d-dimensional manifold of class C^1 and let σ be its volume form (or any Radon measure absolutely continuous w.r.t the volume form with a bounded density). Assume that the kernel K takes on the form (13). Then*

$$M_\gamma \text{ is non degenerate} \Leftrightarrow \gamma^2 < 2d.$$

3.3 Examples

In this lecture, we will try to illustrate the use of the theory of Gaussian multiplicative chaos through two examples: translational invariant log-correlated fields and the massless Gaussian free field (GFF) in some domain D (equipped with the Euclidean metric). Of course, we could work with other GFFs as well, like a GFF on D with Neumann boundary conditions or with the GFF on the Riemann sphere $\mathbb{S}^2$ with vanishing mean (with covariance (5)), but we stick to these examples for the sake of clarity. Also, we will choose σ to be the Lebesgue measure. So, we explain below how to apply Kahane's method to these fields by constructing a white noise decomposition. Yet, before treating these important examples, we detail first a very simple example. In all the examples below, the Radon measure σ is chosen to be the Lebesgue measure.

A simple example.

Take any nonnegative covariance kernel $\tilde{K}$ of a centered continuous Gaussian process Z on $\mathbb{R}^d$, which is invariant under translation, normalized such that $\mathbb{E}[Z(0)^2]=1$ and satisfying

$$\forall x\in\mathbb{R}^d,\quad \tilde{K}(x)\leq \frac{C}{(1+|x|)^{\alpha}},\quad |\tilde{K}(x)-\tilde{K}(0)|\leq C|x|. \tag{16}$$

Notice that it is straightforward to construct such a process: take a white noise W on $\mathbb{R}^d$ and any smooth nonnegative function φ with compact support and such that $\int_{\mathbb{R}^d}\varphi^2(x)\,dx=1$ and then set

$$Z(x)=\int_{\mathbb{R}^d}\varphi(x-y)\,W(dy).$$

The idea is now to stack the process Z at different scales. More precisely, we consider a sequence $(Z_k)_k$ of independent processes with the same law as Z. For all $k,n\geq 1$ we set

$$Y_k(x)=k^{-1/2}Z(kx),\quad X_n(x)=\sum_{k=1}^{n}Y_k(x).$$

Then Y_k is a Gaussian process with covariance kernel $K_k(x)=k^{-1}\tilde{K}(kx)$. We define the kernel

$$K(x,y)=\sum_{k=1}^{\infty}K_k(x-y),$$

which is by construction of σ-positive type. The reader may check that there exists a bounded function g over $\mathbb{R}^d\times\mathbb{R}^d$ such that

$$K(x,y)=\ln_+\frac{1}{|x-y|}+g(x,y). \tag{17}$$

The normalization $\mathbb{E}[Z(0)^2]=1$ ensures that the term $\ln_+$ is normalized, i.e. the factor is 1. If we do not normalize the variance we would get $K(x,y)=\mathbb{E}[Z(0)^2]\ln_+\frac{1}{|x-y|}+g(x,y)$.

Then Kahane's theory applies and we can construct the GMC measure

$$M_\gamma(dx)=\lim_{n\to\infty}e^{\gamma X_n(x)-\frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\,dx,$$

where the limit holds almost surely in the sense of weak convergence of measures. The limit is non trivial if and only if $\gamma^2<2d$.

Translationally invariant log-correlated fields.

Here we describe a general procedure to treat the important case of translationally invariant log-correlated fields (TILCF). Such fields can be characterized by the Fourier transform of their covariance kernel, which decays like $\frac{1}{|u|^d}$ in dimension d. We are going to use this Fourier transform to construct a white noise decomposition of TILCF as this description appears rather naturally in physics. We stress here that there are easier way to produce a decomposition à la Kahane of such kernels but our approach has the main advantage that it produces cutoff approximations that are measurable functions of the original log-correlated Gaussian distribution.

Consider a continuous strictly positive even function φ defined on $\mathbb{R}^d$ such that for some constants $C, \alpha > 0$

$$|1-\varphi(x)| \leq \frac{C}{(1+|x|)^\alpha}. \tag{18}$$

In particular, we see that $\varphi(x) \to 1$ as $x \to \infty$.

Let us begin with our framework and assumptions. We consider the following kernel of positive type

$$\forall x, y \in \mathbb{R}^d, \quad K(x,y) = \frac{1}{V_d} \int_{\mathbb{R}^d} e^{-i\langle u, x-y\rangle} \frac{\varphi(u)}{1+|u|^d}\, du, \tag{19}$$

where V_d is the surface of the unit sphere of $\mathbb{R}^d$. We have (see the proof below)

$$K(x,y) = \ln_+ \frac{1}{|x-y|} + g(x,y) \tag{20}$$

for some bounded function g over $\mathbb{R}^d \times \mathbb{R}^d$. We further stress that all translationally invariant kernels of log type take on the form (19) up to the fact that the convergence of φ towards 1 is not necessarily polynomial. Our condition (18) is not necessary but it covers most of the practical applications.

Now we construct a centered Gaussian distribution that has covariance kernel K. Let us introduce the following bilinear form on the Schwartz space $\mathcal{S}$

$$\langle f, g\rangle_K = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(y)K(x,y)\, dxdy = \frac{(2\pi)^d}{V_d}\int_{\mathbb{R}^d} \hat{f}(u)\bar{\hat{g}}(u)\frac{\varphi(u)}{1+|u|^d}\, du,$$

where $\hat{f}(u) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-i\langle u,x\rangle} f(x)\, dx$ is the Fourier transform of f. $\mathcal{S}'$ stands for the standard topological dual of $\mathcal{S}$. Notice that the mapping $f \mapsto \langle f, f\rangle_K$ is continuous on $\mathcal{S}$. By the Minlos theorem ($\mathcal{S}$ is nuclear), there exists a (centered) random Gaussian distribution X living almost surely in $\mathcal{S}'$ such that

$$\forall f \in \mathcal{S}, \quad \mathbb{E}[e^{iX(f)}] = e^{-\frac{1}{2}|f|_K^2}.$$

Notice that for all $f, g \in \mathcal{S}$

$$\mathbb{E}[X(f)X(g)] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(y)K(x,y)\, dx\, dy.$$

Now we construct the Kahane approximations of X in a way that makes them measurable with respect to the distribution X, so that the Gaussian multiplicative chaos obtained will be a measurable function of X. We consider any strictly increasing sequence $(c_k)_{k\geq 0}$ such that $c_0 = 0$ and $\lim_{k\to\infty} c_k = \infty$. Then we define for $k \geq 1$ the kernel

$$\forall x, y \in \mathbb{R}^d, \quad K_k(x,y) = \frac{1}{V_d}\int_{C_k} e^{-i\langle u, x-y\rangle}\frac{\varphi(u)}{1+|u|^d}\, du, \tag{21}$$

where C_k stands for the annulus $\{x \in \mathbb{R}^d; c_{k-1} \leq |x| < c_k\}$. Obviously

$$K(x,y) = \sum_{k\geq 1} K_k(x,y).$$

It remains to construct a sequence of independent centered Gaussian process $(Y_k)_k$ each of which with covariance kernel K_k. Given $f \in \mathcal{S}$, we denote by f_e and f_o the even and odd part of f

$$f_e(x) = \frac{f(x)+f(-x)}{2} \quad \text{and} \quad f_e(x) = \frac{f(x)-f(-x)}{2}.$$

Let us define the two following centered real-valued Gaussian distributions

$$\forall f \in \mathcal{S}, \quad W_e(f) = X(\hat{f_e}^{-1}) \quad \text{and} \quad W_o(f) = X(-i\hat{f_o}^{-1})$$

where $\hat{f}^{-1}$ stands for the inverse Fourier transform of f. Setting

$$\langle f, g\rangle_\varphi = \frac{(2\pi)^d}{V_d}\int_{\mathbb{R}^d} f(u)g(u)\frac{\varphi(u)}{1+|u|^d}\,du.$$

The reader may check that W_e and W_o satisfy

- W_e and W_o are independent
- W_e is symmetric (meaning $W_e(du) = W_e(-du)$), W_o is antisymmetric (meaning $W_o(du) = -W_o(-du)$)
- for all functions $f, g \in \mathcal{S}$

$$\mathbb{E}[W_e(f_e)W_e(g_e)] = \mathbb{E}[W_o(f_o)W_o(g_o)] = \langle f, g\rangle_\varphi.$$

Therefore W_e and W_o stand respectively for a symmetric (resp. antisymmetric) white noise with intensity measure $\frac{(2\pi)^d}{V_d}\frac{\varphi(u)}{1+|u|^d}\,du$, and are independent. Now we can construct Y_k as

$$Y_k(x) = \int_{\mathbb{R}^d} \cos(\langle u.x)\mathbf{1}_{C_k}(u)\,W_e(du) + \int_{\mathbb{R}^d} \sin(\langle u.x\rangle)\mathbf{1}_{C_k}(u)\,W_o(du).$$

The reader can check that this is a Gaussian centered process with covariance kernel K_k. Notice that it is also almost surely continuous: indeed, it is readily seen that

$$|K_k(x) - K_k(0)| \leq D|x|$$

for some constant D (eventually depending on k) and the Kolmogorov criterion then ensures that Y_k admits a continuous modification. Furthermore, the processes $(Y_k)_k$ are independent because the annuli $(C_k)_k$ are disjoint.

Now we conclude about the non-triviality of the GMC: the main issue here is that the kernels $(K_k)_k$ are not necessarily nonnegative so that we need to do some more work to apply Kahane's strategy. However, it is straightforward to apply Kahane's theory as prescribed in the following. Notice that we have the relation

$$\forall n \geq 1, \quad \sum_{k=1}^n K_k(x,y) \leq \ln_+ \frac{1}{|x-y|+\epsilon_n} + A \tag{22}$$

for some constant constant $A > 0$ and $\epsilon_n = e^{-\sum_{k=1}^n K_k(0)}$ (same proof as (20)). Notice that $\epsilon_k \to 0$ as $k \to \infty$. Take any open ball $B \subset \mathbb{R}^d$ and notice that the sequence

$$M_{n,\gamma}(B) = \int_B e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\,dx$$

is a positive martingale and thus converges almost surely. The question is now to know whether it is uniformly integrable. Let us consider $\gamma^2 < 2d$. Take now the "simple example" as constructed in the previous subsection, call $\tilde{X}_n$ (resp. $\tilde{M}$, $\tilde{K}_k$, $\tilde{K}$) the corresponding approximations (resp. limiting measure, truncated kernel, kernel). As $\gamma^2 < 2d$, we know that the martingale

$$\tilde{M}_{n,\gamma}(B) = \int_B e^{\gamma \tilde{X}_n(x) - \frac{\gamma^2}{2}\mathbb{E}[\tilde{X}_n(x)^2]}\, dx$$

is uniformly integrable. From the De la Vallée-Poussin theorem, there exists a convex function $F : \mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{|x|\to\infty} F(x)/|x| = +\infty$ and

$$\sup_n \mathbb{E}[F(\tilde{M}_{n,\gamma}(B))] < +\infty.$$

We can further require F to satisfy $F(\lambda x) \leq \lambda^\theta F(x)$ for some $\theta > 0$ and all $\lambda > 0$.

Because of (17)+(22), there exists a constant A such that for all fixed n, we can find n' large enough such that

$$\sum_{k=1}^{n} K_k(x,y) \leq A + \sum_{k=1}^{n'} \tilde{K}_k(x,y).$$

Let G be a centered Gaussian random variable with variance A independent of the sequence $(\tilde{X}_n)_n$. By the comparison principle (Theorem 2), we deduce that

$$\mathbb{E}[F(M_{n,\gamma}(B))] \leq \mathbb{E}[F(e^{G-A/2}\tilde{M}_{n',\gamma}(B))].$$

Then notice that

$$\mathbb{E}[F(e^{G-A/2}\tilde{M}_{n',\gamma}(B))] \leq \mathbb{E}[e^{\theta G - \theta A/2}] \sup_n \mathbb{E}[F(\tilde{M}_{n,\gamma}(B))] < +\infty.$$

We deduce that the sequence $(M_{n,\gamma}(B))_n$ is uniformly integrable by the De la Vallée-Poussin theorem.

To sum up, for all $\gamma^2 < 2d$, the sequence of measures $(e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\, dx)_n$ on $\mathbb{R}^d$ almost surely converges in the sense of vague convergence of measures towards a random measure M_γ, which is non trivial, diffuse and measurable with respect to the distribution X (as it is an almost sure limit of measurable functions of the field X). Furthermore, this measure M_γ is invariant under translations in law because the distribution X and the Lebesgue measure are.

Example 10. *One could for instance mention that the important example of the massive Green function in dimension* 2 *with mass* $m > 0$*. More precisely, if we consider the problem with data* f *and unknown function* u

$$m^2 u - \triangle u = 2\pi f \text{ over } \mathbb{R}^2$$

then

$$u = \int K(x-y) f(y)\, dy$$

with K *given by*

$$K(x) = \int_0^\infty e^{-\frac{m^2}{2}v} e^{-\frac{|x|^2}{2v}} \frac{dv}{2v}.$$

We claim

$$K(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\langle u, x-y\rangle} \frac{1}{|u|^2 + m^2}\, du.$$

Indeed, using the Fourier transform of the Gaussian law, we get

$$\begin{aligned}\int_0^\infty e^{-\frac{m^2}{2}v}e^{-\frac{|x|^2}{2v}}\,\frac{dv}{2v}=&\int_0^\infty e^{-\frac{m^2}{2}v}\Big(\int_{\mathbb{R}^2}e^{i\langle u,x\rangle}e^{-\frac{|u|^2v}{2}}\frac{vdu}{2\pi}\Big)\,\frac{dv}{2v}\\ =&\frac{1}{4\pi}\int_{\mathbb{R}^2}e^{i\langle u,x\rangle}\Big(\int_0^\infty e^{-\left(\frac{m^2}{2}+\frac{|u|^2}{2}\right)v}\,dv\Big)\,du\\ =&\frac{1}{2\pi}\int_{\mathbb{R}^2}e^{i\langle u,x\rangle}\frac{1}{m^2+|u|^2}\,du\end{aligned}$$

Therefore, the massive Green function fits into our framework with $\varphi(u)=\frac{1+|u|^2}{|u|^2+m^2}$.

Proof of (20). We introduce the superficial measure ds on the unit sphere $\mathbb{S}$ of $\mathbb{R}^d$. After a change of variables in polar coordinates, we decompose the expression of K as follows

$$\begin{aligned}K(x,y)=&\int_0^\infty\Big(\frac{1}{V_d}\int_{\mathbb{S}}e^{-i\langle rs,x-y\rangle}\varphi(rs)\,ds\Big)\frac{r^{d-1}}{1+r^d}\,dr \qquad (23)\\ =&\int_0^{\frac{1}{|x-y|}}(H_r(x-y)-1)\frac{r^{d-1}}{1+r^d}\,dr+\int_0^{\frac{1}{|x-y|}}\frac{r^{d-1}}{1+r^d}\,dr+\int_{\frac{1}{|x-y|}}^\infty H_r(x-y)\frac{r^{d-1}}{1+r^d}\,dr\end{aligned}$$

where we have set

$$H_r(x)=\frac{1}{V_d}\int_{\mathbb{S}}e^{-i\langle rs,x\rangle}\varphi(rs)\,ds.$$

Observe that we have the following expression for H_r

$$\begin{aligned}H_r(x)=&\frac{1}{V_d}\int_{\mathbb{S}}e^{-i\langle rs,x\rangle}(\varphi(rs)-1)\,ds+\frac{1}{V_d}\int_{\mathbb{S}}\cos(r|x|s_1)\,ds\\ =&\frac{1}{V_d}\int_{\mathbb{S}}e^{-i\langle rs,x\rangle}(\varphi(rs)-1)\,ds+\frac{1}{\pi}\int_0^\pi\cos(r|x|\cos\theta)\,d\theta\\ =&\frac{1}{V_d}\int_{\mathbb{S}}e^{-i\langle rs,x\rangle}(\varphi(rs)-1)\,ds+\frac{1}{\pi}\int_{-1}^1\cos(r|x|u))\frac{du}{\sqrt{1-u^2}}. \qquad (24)\end{aligned}$$

where we have used spherical coordinates and a change of variables. The first term above can be estimated with (18)

$$\Big|\frac{1}{V_d}\int_{\mathbb{S}}e^{-i\langle rs,x\rangle}(\varphi(rs)-1)\,ds\Big|\leq\frac{C}{(1+|r|)^\alpha}. \qquad (25)$$

Hence

$$\begin{aligned}|H_r(x)-1|\leq&\frac{C}{(1+|r|)^\alpha}+\frac{1}{\pi}\int_0^\pi\big(1-\cos(r|x|u)\big)\frac{du}{\sqrt{1-u^2}}\\ \leq&\frac{C}{(1+|r|)^\alpha}+r|x|. \qquad (26)\end{aligned}$$

We deduce that the first term in the right-hand side of (23) can be estimated by

$$\Big|\int_0^{\frac{1}{|x-y|}}(H_r(x-y)-1)\frac{r^{d-1}}{1+r^d}\,dr\Big|\leq C\int_0^{\frac{1}{|x-y|}}\frac{r^{d-1}}{(1+r^d)(1+r)^\alpha}\,dr+1. \qquad (27)$$

This function is obviously bounded in x,y.

The second term in (23) gives

$$\int_0^{\frac{1}{|x-y|}} \frac{r^{d-1}}{1+r^d}\,dr = \frac{1}{d}\ln\big(1+\frac{1}{|x-y|^d}\big). \tag{28}$$

Concerning the third term, we use once again the relation (24) to see that it is enough to show that

$$\Big|\frac{1}{\pi}\int_{-1}^{1}\cos(r|x|u)\frac{du}{\sqrt{1-u^2}}\Big| \leq C'(1+r|x|)^{-\eta}$$

for some $\eta\in]0,1/2[$, which is a simple exercise. Gathering the above relations, this proves the claim. □

Gaussian free field with Dirichlet boundary condition

Let D be a simply connected domain $D\subsetneq\mathbb{R}^2$ (say with smooth boundary). Denote by $H^1_0(D)$ the closure of $C^\infty_c(D)$ with respect to the Dirichlet inner product

$$\langle f,g\rangle_{H^1_0} = \int_D \partial f\cdot\partial g\,dx.$$

and by $H^{-1}(D)$ its topological dual. A Gaussian Free Field with Dirichlet boundary condition on D is a centered Gaussian random distribution X (it lives almost surely in $H^{-1}(D)$, the dual of the Sobolev space $H^1_0(D)$, see [16, 47] for instance), with covariance kernel given by the Green function G of the Laplacian on D with Dirichlet boundary condition. More precisely, given a bounded function f, the solution u of the problem

$$-\triangle u = 2\pi f,\quad u_{|\partial D}=0$$

is given by $u(x)=\int_D G(x,y)f(y)\,dy$, which we denote by $Gf(x)$. Recall that the kernel G possesses a singularity of log-type as it can be written as

$$G(x,y)=\ln\frac{1}{|x-y|}+g(x,y)$$

where g is a continuous function on $D\times D$ (possibly blowing up along the boundary).

We will present several constructions of the GMC associated to the GFF X in this lecture: eigenvalue expansion, white noise representation or convolution. We will see in the next sections that all the corresponding GMC are actually the same. But here we focus on the constructions via eigenvalue expansion or white noise representation.

The $H^1_0(D)$ expansion

This construction is based on similar arguments to those presented in the previous subsection, the main difference is that the Fourier transform is exchanged with expansion along an $H^1_0(D)$ basis. Let $(e_n)_n$ be an orthonormal basis of $H^1_0(D)$ consisting of smooth functions. Consider an i.i.d. sequence $(\alpha_n)_n$ of Gaussian random variables with mean 0 and variance 1. Formally, we get (actually the series converges in $H^{-1}(D)$, see [16, 47])

$$X=\sqrt{2\pi}\sum_n \alpha_n e_n$$

in such a way that for f a test function

$$X(f)=\sqrt{2\pi}\sum_n \alpha_n\int_D f e_n dx=\frac{1}{\sqrt{2\pi}}\sum_n \langle Gf,e_n\rangle_{H^1_0}.$$

The variance of $X(f)$ can be straightforwardly evaluated by

$$\mathbb{E}[X(f)^2]=\frac{1}{2\pi}\sum_n \langle Gf,e_n\rangle^2_{H^1_0}=\frac{1}{2\pi}\langle Gf,Gf\rangle^2_{H^1_0}=\int_D fGf\,dx.$$

Let us also stress that the sequence $(\alpha_n)_n$ can be then recovered as a function of X by the relation

$$\alpha_k=-\frac{1}{\sqrt{2\pi}}X(\triangle e_k).$$

For $n\geq 1$, we set

$$X_n(x)=\sqrt{2\pi}\sum_{k=1}^n \alpha_k e_k(x),$$

which plays the role of a cut-off family for the GFF X. Notice that this does not exactly fall into the framework of Kahane since the intermediate kernels

$$\mathbb{E}[X_n(x)X_n(y)]=2\pi\sum_{k=1}^n e_k(x)e_k(y)$$

are not necessarily non negative but this can be easily overcome by the following simple argument. Consider a smooth mollifier θ with compact support: for all function f and $\epsilon>0$, denote by $f_\epsilon=\frac{1}{\epsilon^2}(f*\theta(\frac{\cdot}{\epsilon}))$ the convolution of f with θ (at scale ϵ). Given a compact set $A\subset D$, we have for all $n\leq p$ and $\epsilon>0$.

$$\int_A e^{\gamma X_{n,\epsilon}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{n,\epsilon}(x)^2]}dx=\mathbb{E}\Big[\int_A e^{\gamma X_{p,\epsilon}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{p,\epsilon}(x)^2]}dx|\,\alpha_1,\cdots,\alpha_n\Big].$$

Therefore, if F is some positive and convex function, we get for all $n\leq p$ and $\epsilon>0$

$$\mathbb{E}\Big[F\Big(\int_A e^{\gamma X_{n,\epsilon}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{n,\epsilon}(x)^2]}dx\Big)\Big]\leq\mathbb{E}\Big[F\Big(\int_A e^{\gamma X_{p,\epsilon}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{p,\epsilon}(x)^2]}dx\Big)\Big]$$

Therefore, letting $p\to\infty$, we get

$$\mathbb{E}\Big[F\Big(\int_A e^{\gamma X_{n,\epsilon}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{n,\epsilon}(x)^2]}dx\Big)\Big]\leq\mathbb{E}\Big[F\left(\int_A e^{\gamma X_{\epsilon}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{\epsilon}(x)^2]}dx\right)\Big].\tag{29}$$

Now we use the comparison principle (Theorem 2). Choose any $2d$ "simple example" studied above and restricted to D, call it $\tilde{X}$. Since $\gamma^2<2d=4$, the De La Vallée Poussin theorem tells us that we can find a positive convex function $F:\mathbb{R}_+\to\mathbb{R}$ such that $\lim_{|x|\to\infty}F(x)/|x|=+\infty$ and

$$\mathbb{E}\Big[F\Big(\int_A e^{\gamma\tilde{X}(x)-\frac{\gamma^2}{2}\mathbb{E}[\tilde{X}(x)^2]}dx\Big)\Big]<+\infty$$

and $F(\lambda x)\leq\lambda^\theta F(x)$ for some $\theta>0$ and all $\lambda>0$. Furthermore we can find a constant $C>0$ such that for all $\epsilon>0$

$$\mathbb{E}[X_\epsilon(x)X_\epsilon(y)]\leq\mathbb{E}[\tilde{X}(x)\tilde{X}(y)]+C.$$

Let us consider a centered Gaussian random variable N with variance 1, Theorem 2 implies that

$$\sup_{\epsilon>0}\mathbb{E}\Big[F\left(\int_A e^{\gamma X_\epsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]}dx\right)\Big]\leq \mathbb{E}[e^{\gamma C^{1/2}N-\frac{\gamma^2}{2}C}]\mathbb{E}\Big[F\Big(\int_A e^{\gamma\tilde{X}(x)-\frac{\gamma^2}{2}\mathbb{E}[\tilde{X}(x)^2]}dx\Big)\Big]<+\infty.$$

Therefore the right-hand side of (29) is bounded independently of ϵ and we get

$$\sup_{\epsilon>0}\sup_{n\geq 1}\mathbb{E}\Big[F\Big(\int_A e^{\gamma X_{n,\epsilon}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{n,\epsilon}(x)^2]}dx\Big)\Big]\leq \mathbb{E}\Big[F\Big(\int_A e^{\gamma\tilde{X}(x)-\frac{\gamma^2}{2}\mathbb{E}[\tilde{X}(x)^2]}dx\Big)\Big].$$

We conclude that, once again, the sequence of measures $(e^{\gamma X_n(x)-\frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\,dx)_n$ on D almost surely converges in the sense of weak convergence of measures towards a random measure M_γ, which is non trivial, diffuse and measurable with respect to the distribution X. Let us mention that the specific choice of the basis $(e_n)_n$ given by the eigenfunctions of the Laplacian is of particular interest because we have an explicit expression for this basis (for instance write it explicitly on the unit square and use the Riemann uniformizing theorem to find the expression of the eigenfunctions on other simply connected domains). The case of the GFF will be further discussed in section 5 in order to treat the case of convolution cutoffs.

White noise decomposition of the GFF

Another possible decomposition of the Green function is based on the formula:

$$G(x,y)=2\pi\int_0^\infty p_D(t,x,y)dt.$$

where p_D is the (sub-Markovian) semi-group of a Brownian motion B killed upon touching the boundary of D, namely

$$p_D(t,x,y)=P^x(B_t\in dy,\ T_D>t)$$

with $T_D=\inf\{t\geq 0,\ B_t\notin D\}$. Hence we can write:

$$G(x,y)=\sum_{n\geq 0}K_n(x,y)\quad\text{with}\quad K_n(x,y)=2\pi\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}}p_D(t,x,y)dt,\ n\geq 1 \tag{30}$$

and $K_0(x,y)=2\pi\int_1^\infty p_D(t,x,y)dt$. The continuity of p_D implies that K_n is continuous. The symmetry of p_D implies that K_n is positive definite. Indeed, for each smooth function φ with compact support in D, we have for $n\geq 1$:

$$\begin{aligned}\int_D\int_D\varphi(x)K_n(x,y)\varphi(y)\,dx\,dy&=2\pi\int_D\int_D\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}}\varphi(x)p_D(t,x,y)\varphi(y)\,dt\,dx\,dy\\&=2\pi\int_D\int_D\int_D\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}}\varphi(x)p_D(t/2,x,z)p_D(t/2,z,y)\varphi(y)\,dt\,dz\,dx\,dy\\&=2\pi\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}}\int_D\Big(\int_D\varphi(x)p_D(t/2,x,z)\,dx\Big)^2\,dt\,dz\\&\geq 0.\end{aligned}$$

Since K_n is obviously nonnegative, we can apply Kahane's theory.

We further stress that this argument implies a white noise decomposition of the underlying GFF: the most direct way to construct a GFF is then to consider a white noise W distributed on $D \times \mathbb{R}_+$ and define

$$X(x) = \sqrt{2\pi} \int_{D\times\mathbb{R}_+} p_D(\frac{s}{2}, x, z) W(dz, ds). \tag{31}$$

One can check that $\mathbb{E}[X(x)X(y)] = 2\pi \int_0^\infty p_D(s, x, y)\, ds = G_D(x, x')$. In that case, one can even work with a continous parameter ϵ and define the GMC measure as the almost sure limit as $\epsilon \to 0$ of $M_{\epsilon,\gamma}(dx) = e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]}\, dx$ where the corresponding cut-off approximations X_ϵ are given by (see also [42, 43]):

$$X_\epsilon(x) = \sqrt{2\pi} \int_{D\times[\epsilon^2,\infty[} p_D(\frac{s}{2}, x, z) W(dz, ds). \tag{32}$$

Indeed, within this framework, the sequence $(M_{\epsilon,\gamma}(A))_{\epsilon>0}$ is a positive martingale for all compact set A. Note the following expression for the covariance of X_ϵ:

$$E[X_\epsilon(x)X_\epsilon(y)] = 2\pi \int_{\epsilon^2}^\infty p_D(s, x, y)\, ds \tag{33}$$

Exact kernels

Finally let us mention another important kernel studied in [3] (in fact, [3] studied it in dimension 1 but higher dimensional extensions were studied in [44]).

For $T > 0$ the kernel

$$\forall x, y \in \mathbb{R}^d, \quad K(x, y) = \ln_+ \frac{T}{|x - y|} \tag{34}$$

is of σ-positive type in dimension $d = 1, 2$. The interest lies in the fact that such a kernel exactly coincides with the function $\ln$ over a neighborhood of 0 and therefore possesses exact scaling relations.

To see that this kernel if of σ-positive type, a computation yields:

$$\ln_+ \frac{T}{|x|} = \int_0^{+\infty} (t - |x|)_+ \nu_T(dt)$$

where ν_T is the measure (δ_T is the Dirac mass at T):

$$\nu_T(dt) = \mathbf{1}_{[0,T]}(t) \frac{dt}{t^2} + \frac{1}{T}\delta_T(dt).$$

Hence for any $\mu > 0$, we have:

$$\ln_+ \frac{T}{|x|} = \frac{1}{\mu} \ln_+ \frac{T^\mu}{|x|^\mu} = \frac{1}{\mu} \int_0^{+\infty} (t - |x|^\mu)_+ \nu_{T^\mu}(dt).$$

By using the Chasles relation in the integral of the right-hand side, proving that this kernel is of σ-positive type thus boils down to considering the possible values of $\mu > 0$ such that the function $(1 - |x|^\mu)_+$ is of positive type: this is the Kuttner-Golubov problem (see [26]).

For $d = 1$, it is straightforward to see that $(1 - |x|)_+$ is of positive type (compute the inverse Fourier transform). In dimension 2, Pasenchenko [40] proved that the function $(1 - |x|^{1/2})_+$ is of positive type on $\mathbb{R}^2$. We can thus write

$$\ln_+ \frac{T}{|x|} = \sum_{n \geq 1} K_n(x)$$

with

$$K_n(x) = \frac{1}{\mu} \int_{\frac{1}{n}}^{\frac{1}{n-1}} (t - |x|^\mu)_+ \nu_{T^\mu}(dt)$$

with $\mu = 1$ in dimension 1 and $\mu = 1/2$ in dimension 2.

3.4 Moments and further properties

In this subsection, we give moment estimates in the case when the Radon measure σ is the Lebesgue measure and the kernel K is of the type

$$K(x, y) = \ln_+ \frac{1}{|x - y|} + g(x, y) \tag{35}$$

for some locally bounded function g. Let us recall that, in this case, the non-degeneracy condition reads $\gamma^2 < 2d$. We denote by M_γ the associated chaos measure.

The following estimates on moments were obtained in a variety of contexts (see [3, 8] in dimension 1 and [45] for all dimensions)

Theorem 11. Positive Moments. *If the measure M_γ is non degenerate, that is $\gamma^2 < 2d$, the measure M_γ admits finite positive moments of order p for all $p \in]0, \frac{2d}{\gamma^2}[$. More precisely, for all compact set $A \subset D$ and $p \in]0, \frac{2d}{\gamma^2}[$, we have $\mathbb{E}[M_\gamma(A)^p] < +\infty$.*

Basically, it suffices to prove the finiteness of the moments for your favorite kernel K of the type (35) and deduce that the conclusions remain valid for all the kernels of the type (13) via Theorem 2. A relevant choice for K is then a kernel with finite correlation length to use the technics in [3, 8].

Now we turn to the existence of negative moments which were investigated by Molchan [38] in the case of Mandelbrot's multiplicative cascades. The following theorem is proved in [45] by adapting the argument of [38] and using the comparison principle (Theorem 8).

Theorem 12. Negative Moments. *If the measure M_γ is non degenerate, that is $\gamma^2 < 2d$, the measure M_γ admits finite negative moments of order p for all $p \in] - \infty, 0[$. More precisely, for any compact nonempty Euclidean ball $A \subset D$ and $p \in] - \infty, 0[$, we have $\mathbb{E}[M_\gamma(A)^p] < +\infty$.*

Let us also point out an important result in [7] where the authors compute the tail distributions of the measure M_γ in dimension 1 with the kernel $K(x, y) = \ln_+ \frac{1}{|x-y|}$ (recall that it is of σ-positive type)

Theorem 13. Distribution tails. [Barral, Jin, 2012]. *If A is some nonempty segment of $\mathbb{R}$ then there exists a constant $c > 0$ such that*

$$\lim_{x \to +\infty} x^{\frac{2}{\gamma^2}} \mathbb{P}(M_\gamma(A) > x) = c.$$

(To be added: a word on small deviations)

We conclude this theoretical background by pointing out some further interesting property about the structure of the support of the measure.

Proposition 14. *Assume that K is a kernel of σ-positive type, that it can be written under the form (35), that the Radon measure σ is in the class R_α^+ and take $\gamma \in]0, 2\alpha$. Let us consider the GMC on a domain $D \subset \mathbb{R}^d$*

$$M_\gamma(dx) = e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X^2(x)]}\,\sigma(dx),$$

which is non-degenerate according to Theorem 7. Then, almost surely, the topological support of the measure M_γ coincide with that of σ.

Proof. This results from the $0-1$ law of Kolmogorov: if you consider a ball B, the $0-1$ law tells you that the event $\{M_\gamma(B) > 0\}$ has probability 0 or 1. Indeed, we have

$$\inf_{x \in B} e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n^2(x)]} M_{\gamma,n}^c(B) \leq M_\gamma(B) \leq \sup_{x \in B} e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n^2(x)]} M_{\gamma,n}^c(B), \tag{36}$$

where $M_{\gamma,n}^c(B)$ is the Gaussian multiplicative chaos

$$M_{\gamma,n}^c(dx) = \lim_{k \to \infty} e^{\gamma(X_k - X_n)(x) - \frac{\gamma^2}{2}\mathbb{E}[(X_k - X_n)^2(x)]}\,\sigma(dx).$$

Since for $k > n$:

$$X_k - X_n = \sum_{p=n+1}^{k} Y_p,$$

we may say that we have just removed the dependency on the first n fields $Y_1, \dots, Y_n$. Therefore (36) entails that for any $n \geq 0$

$$\{M_\gamma(B) > 0\} = \{M_{\gamma,n}^c(B) > 0\}.$$

Since the event $\{M_{\gamma,n}^c(B) > 0\}$ is independent of the fields $Y_1, \dots, Y_n$, we deduce that the event $\{M_\gamma(B) > 0\}$ belongs to the asymptotic sigma-algebra generated by the fields $(Y_n)_n$and therefore the event $\{M_{\gamma,n}^c(B) > 0\}$ has probability 0 or 1. Since M_γ is non-degenerate, we have $\mathbb{E}[M_\gamma(B)] = \sigma(B)$. Hence the event $\{M_{\gamma,n}^c(B) > 0\}$ has probability 1 if $\sigma(B) > 0$ and probability 0 otherwise. This argument can be reproduced for every ball chosen among a countable family of balls $(B_n)_n$ generating the open sets of D. Therefore, almost surely, for all n we have $M_\gamma(B_n) > 0$ if and only if $\sigma(B) > 0$, proving that almost surely the support of M_γ is that of σ. □

3.5 Multifractality, stochastic scale invariance

In this subsection, we consider the Euclidian framework. More precisely, we consider an open set D of $\mathbb{R}^d$ and a Gaussian multiplicative chaos of the type

$$\forall A \in \mathcal{B}(D), \quad M_\gamma(A) = \int_A e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}\,dx \tag{37}$$

where the Gaussian distribution X has a covariance kernel of the form:

$$K(x,y) = \ln_+ \frac{T}{|x-y|} + g(x,y) \tag{38}$$

for some bounded function g over $D \times D$ and $\gamma^2 < 2d$. Then the power-law spectrum of such Gaussian multiplicative chaos presents some interesting features, such as non-linearity (in the parameter q in the following theorem):

Theorem 15. *Assume that the kernel K takes on the form (38) with $\gamma^2 < 2d$. For each $q \in]-\infty, \frac{2d}{\gamma^2}[$, there exists a constant c_q such that for all $r \in]0,1]$ and $x \in D$*

$$c_q^{-1} r^{\xi(q)} \leq \mathbb{E}\big[M_\gamma(B(x,r))^q\big] \leq c_q r^{\xi(q)}, \tag{39}$$

where ξ is the structure exponent of the measure M_γ:

$$\forall q \in]-\infty, \frac{2d}{\gamma^2}[, \quad \xi(q) = \big(d + \frac{\gamma^2}{2}\big)q - \frac{\gamma^2}{2}q^2.$$

Heuristic proof. For simplicity, assume that $T = 1$ in (38). By making a change of variables, we get:

$$\begin{aligned} M_\gamma(B(x,r)) &= \int_{B(x,r)} e^{\gamma X(y) - \frac{\gamma^2}{2}\mathbb{E}[X(y)^2]}\, dy \\ &= r^d \int_{B(0,1)} e^{\gamma X(x+ry) - \frac{\gamma^2}{2}\mathbb{E}[X(x+ry)^2]}\, dy. \end{aligned}$$

Then we observe that the field $(X(x+ry))_y$ has a covariance structure approximatively given for $r \leq 1$ by:

$$\begin{aligned} \mathbb{E}[X(x+ry)X(x+rz)] &\simeq \ln_+ \frac{1}{r|y-z|} \\ &= \ln \frac{1}{r} + \ln_+ \frac{1}{|y-z|} \\ &\simeq \ln \frac{1}{r} + \mathbb{E}[X(x+y)X(x+z)]. \end{aligned}$$

The above relation gives us the following (good) approximation in law

$$(X(x+ry))_{y \in B(0,1)} \simeq \Omega_r + (X(x+y))_{y \in B(0,1)} \tag{40}$$

where Ω_r is a centered Gaussian random variable independent of the field $(X(x+y))_{y \in B(0,1)}$ and with variance $\ln \frac{1}{r}$. Therefore

$$\begin{aligned} M_\gamma(B(x,r)) &= r^d \int_{B(0,1)} e^{\gamma X(x+ry) - \frac{\gamma^2}{2}\mathbb{E}[X(x+ry)^2]}\, dy \\ &\simeq r^d e^{\gamma \Omega_r - \frac{\gamma^2}{2}\mathbb{E}[\Omega_r^2]} \int_{B(0,1)} e^{\gamma X(x+y) - \frac{\gamma^2}{2}\mathbb{E}[X(x+y)^2]}\, dy \\ &= r^d e^{\gamma \Omega_r - \frac{\gamma^2}{2}\mathbb{E}[\Omega_r^2]} M_\gamma(B(x,1)). \end{aligned}$$

By taking the q-th power and integrating, we get as $r \to 0$

$$\mathbb{E}\big[M_\gamma(B(x,r))^q\big] \simeq r^{\xi(q)}\mathbb{E}[M_\gamma(B(x,1)^q],$$

thus explaining the theorem. Actually, the rigorous proof of this result is very close to the heuristic developed here: this is a proof if we consider the kernel (34) because the relation (40) is exact for this kernel (i.e. we have $=$ instead of $\simeq$ in (40), see below). Then we only need to use Theorem (2) to extend the result to other kernels. $\square$

Notice that the quadratic structure of the structure exponent is intimately related to the Gaussian nature of the random distribution X. Random measures with a non-linear power-law spectrum are often called *multifractal*. That is why Gaussian multiplicative chaos (and other possible extensions) are sometimes called Multifractal Random Measures (MRM for short) in the literature. It is also natural to wonder if some specific choice of the covariance kernel K may lead to replacing inequality (39) in Theorem 15 by the symbol $=$. It turns out that this question is related to some specific scaling relation, which we describe below.

Theorem 16. Exact stochastic scale invariance [3, 45]. *Let K be the covariance kernel given by (34) in dimension $d = 1$ or $d = 2$. The associated Gaussian multiplicative chaos M_γ is exactly stochastic scale invariant:*

$$\forall \lambda \in]0,1], \quad (M(\lambda A))_{A\subset B(0,T/2)} \overset{law}{=} \lambda^d e^{\gamma\Omega_\lambda - \frac{\gamma^2}{2}\mathbb{E}[\Omega_\lambda^2]}(M(A))_{A\subset B(0,T/2)}, \tag{41}$$

where Ω_λ is a Gaussian random variable, independent of the measure $(M(A))_{A\subset B(0,T/2)}$, with mean 0 and variance $\ln\frac{1}{\lambda}$.

4 Convergence in law for general cut-offs

Kahane's theory of GMC relies on the notion of kernels of σ-positive type: when a kernel is of σ-positive type, you can construct the measure almost surely and the theory states uniqueness in law within this framework. However, on the one hand it is not always straightforward to check that a kernel is of σ-positive type and on the other hand there are many other natural ways of constructing cut-off approximations. In this section, we give a general criterion to get convergence in law of a Gaussian multiplicative chaos: this section contains (in a more efficient way) the ideas of [43, 45]. This section will also be the building block of the theorems for convergence in probability when working on a fixed probability space established in the next section.

Let D be a bounded open set of $\mathbb{R}^d$. For all $\delta > 0$, we set:

$$D^{(\delta)} = \{x \in D;\ \mathrm{dist}(x, \partial D) \geq \delta\}$$

We consider a positive definite kernel K (non necessarily stationary) satisfying:

$$\forall x, y \in D, \quad K(x,y) = \ln^+ \frac{T}{|x-y|} + g(x,y) \tag{42}$$

where $\gamma^2 < 2d$ and g is a bounded continuous function over $D^{(\delta)}$ for all $\delta > 0$. Let X be a Gaussian distribution living almost surely in the topological dual H' of some space of functions H with $C_c^\infty(D)$ densely contained in H and assume that X has covariance kernel given by (42). We introduce the following notion of K-Gaussian approximation

Definition 17. *K-Gaussian approximations.* *We say that a sequence of centered Gaussian fields $(X_\epsilon)_{\epsilon>0}$ is a K-Gaussian approximation with K given by (42) if for all $\epsilon>0$ the field $x\mapsto X_\epsilon(x)$ is measurable on D, the function $(x,y)\mapsto \mathbb{E}[X_\epsilon(x)X_\epsilon(y)]$ is measurable on $D\times D$ and if for all $\delta>0$ the quantity*

$$C_A=\varlimsup_{\epsilon\to 0}\sup_{\substack{x,y\in D^{(\delta)},\\ |x-y|\geq A\epsilon}}\Big|\mathbb{E}[X_\epsilon(x)X_\epsilon(y)]-\ln^+\frac{T}{|x-y|+\epsilon}-g(x,y)\Big| \tag{43}$$

is such that $A\mapsto C_A$ is bounded on $[0,\infty[$ and converges to 0 as A goes to infinity.

The typical example the reader can have in mind is the convolution. Take, for $\alpha>0$, any α-Hölder function $\theta\in H$ with compact support over $\mathbb{R}^d$ and such that $\int_{\mathbb{R}^d}\theta(x)\,dx=1$. Define

$$X_\epsilon(x)=X(\theta_\epsilon(x+\cdot)),\quad \theta_\epsilon=\epsilon^{-d}\theta(\cdot/\epsilon).$$

Then $(X_\epsilon)_{\epsilon>0}$ is a K-Gaussian approximation of K.

Now, we claim the following universality result:

Theorem 18. *Assume that we are given two K-Gaussian approximations $(X_\epsilon)_{\epsilon>0}$ and $(\bar{X}_\epsilon)_{\epsilon>0}$ and set*

$$M_{\epsilon,\gamma}(dx):=e^{\gamma X_\epsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]}dx,\quad \bar{M}_{\epsilon,\gamma}(dx)=e^{\gamma\bar{X}_\epsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[\bar{X}_\epsilon(x)^2]}dx \tag{44}$$

Let $\gamma^2<2d$ and $\delta>0$. If $\alpha\in[0,1]$ then for all closed balls $A_1,\cdots,A_k\subset D^{(\delta)}$ and $\lambda_1,\cdots,\lambda_k\in\mathbb{R}_+$

$$\varlimsup_{\epsilon\to 0}\Big|\mathbb{E}[(\sum_{i=1}^k\lambda_i\bar{M}_{\epsilon,\gamma}(A_i))^\alpha]-\mathbb{E}[(\sum_{i=1}^k\lambda_i M_{\epsilon,\gamma}(A_i))^\alpha]\Big|=0$$

In particular, if the couple $(X_\epsilon,M_{\epsilon,\gamma})$ converges in law (possibly along some subsequence) towards some couple (X,M_γ) in $H'\times\mathcal{M}_+$ then the couple $(\bar{X}_\epsilon,\bar{M}_{\epsilon,\gamma})$ converges in law towards the same limit.

Without loss of generality, we may show the result in dimension 1 when $k=1$ and $A_1=[0,1]\subset D^{(\delta)}$. Let us define for $t\in[0,1]$

$$Z_\epsilon(t,x)=\sqrt{t}\bar{X}_\epsilon(x)+\sqrt{1-t}X_\epsilon(x),$$

where $\bar{X}_\epsilon, X_\epsilon$ are independent (we can always consider a probability space where are defined two independent copies of the two processes).

Let us write $K_\epsilon(x,y)=\mathbb{E}[X_\epsilon(x)X_\epsilon(y)]$ and $\bar{K}_\epsilon(x,y)=\mathbb{E}[\bar{X}_\epsilon(x)\bar{X}_\epsilon(y)]$. By using a continuous version of lemma 22 in the appendix, we deduce that

$$\mathbb{E}[\bar{M}_{\epsilon,\gamma}(B)^\alpha]-\mathbb{E}[M_{\epsilon,\gamma}(B)^\alpha]=\frac{\alpha(\alpha-1)}{2}\int_0^1\varphi_\epsilon(t)\,dt \tag{45}$$

with φ_ϵ defined by

$$\varphi_\epsilon(t)=\int_{[0,1]^2}(\bar{K}_\epsilon(t_2,t_1)-K_\epsilon(t_2,t_1))\mathbb{E}[\chi_\epsilon(t,t_1,t_2)]\,dt_1dt_2$$

where $\chi_\epsilon(t,t_1,t_2)$ is given by

$$\chi_\epsilon(t,t_1,t_2)=\frac{e^{\gamma Z_\epsilon(t,t_1)+\gamma Z_\epsilon(t,t_2)-\frac{\gamma^2}{2}\mathbb{E}[\gamma Z_\epsilon(t,t_1)^2]-\frac{\gamma^2}{2}\mathbb{E}[\gamma Z_\epsilon(t,t_2)^2]}}{\big(\int_0^1 e^{\gamma Z_\epsilon(t,u)-\frac{\gamma^2}{2}\mathbb{E}[\gamma Z_\epsilon(t,u)^2]}\,du\big)^{2-\alpha}}.$$

We can then decompose (45) in the following way

$$\mathbb{E}[\bar{M}_{\epsilon,\gamma}(B)^\alpha]-\mathbb{E}[M_{\epsilon,\gamma}(B)^\alpha]=\frac{\alpha(\alpha-1)}{2}\int_0^1\varphi_\epsilon^A(t)\,dt+\frac{\alpha(\alpha-1)}{2}\int_0^1\overline{\varphi}_\epsilon^A(t)\,dt$$

where

$$\begin{aligned}\varphi_\epsilon^A(t)&=\int_{[0,1]^2,|t_2-t_1|\le A\epsilon}(\bar{K}_\epsilon(t_2,t_1)-K_\epsilon(t_2,t_1))\mathbb{E}[\chi_\epsilon(t,t_1,t_2)]\,dt_1dt_2\\ \overline{\varphi}_\epsilon^A(t)&=\int_{[0,1]^2,|t_2-t_1|>A\epsilon}(\bar{K}_\epsilon(t_2,t_1)-K_\epsilon(t_2,t_1))\mathbb{E}[\chi_\epsilon(t,t_1,t_2)]\,dt_1dt_2.\end{aligned}$$

We have

$$\begin{aligned}|\overline{\varphi}_\epsilon^A(t)|\le&\gamma^2C_A\int_{[0,1]^2}\mathbb{E}[\chi_\epsilon(t,t_1,t_2)]\,dt_1dt_2\\ \le&\gamma^2C_A\mathbb{E}\big[\big(\int_0^1 e^{\gamma Z_\epsilon(t,u)-\frac{\gamma^2}{2}\mathbb{E}[Z_\epsilon(t,u)^2]}\,du\big)^\alpha\big]\end{aligned}$$

By Jensen's inequality, the last expectation can be bounded independently of ϵ,t by 1. Therefore

$$\overline{\lim_{\epsilon\to0}}\big|\mathbb{E}[\bar{M}_{\epsilon,\gamma}(B)^\alpha]-\mathbb{E}[M_{\epsilon,\gamma}(B)^\alpha]\big|\le\gamma^2\frac{\alpha(1-\alpha)}{2}C_A+\frac{\alpha(1-\alpha)}{2}\overline{\lim_{\epsilon\to0}}\int_0^1\varphi_\epsilon^A(t)\,dt.$$

Now we show that, for each fixed A, $\int_0^1\varphi_\epsilon^A(t)\,dt$ converges to 0 as $\epsilon\to0$ by using the dominated convergence theorem. Now, the difference $|\bar{K}_\epsilon(t_2,t_1)-K_\epsilon(t_2,t_1)|$ is uniformly bounded by C_0, i.e. C_A for $A=0$. Therefore we get

$$\begin{aligned}&|\varphi_\epsilon^A(t)|\\ &\le C_0\mathbb{E}\left[\frac{\int_0^1\int_{t_2-A\epsilon}^{t_2+A\epsilon}e^{\gamma Z_\epsilon(t,t_1)+\gamma Z_\epsilon(t,t_2)-\frac{\gamma^2}{2}\mathbb{E}[\gamma Z_\epsilon(t,t_1)^2]-\frac{\gamma^2}{2}\mathbb{E}[\gamma Z_\epsilon(t,t_2)^2]}dt_1dt_2}{\big(\int_0^1 e^{\gamma Z_\epsilon(t,u)-\frac{\gamma^2}{2}\mathbb{E}[\gamma Z_\epsilon(t,u)^2]}\,du\big)^{2-\alpha}}\right]\\ &\le C_0\mathbb{E}\left[\frac{\sup_{v\in[0,1]}\int_{v-A\epsilon}^{v+A\epsilon}e^{\gamma Z_\epsilon(t,u)-\frac{\gamma^2}{2}\mathbb{E}[\gamma Z_\epsilon(t,u)^2]}du}{\big(\int_0^1 e^{\gamma Z_\epsilon(t,u)-\frac{\gamma^2}{2}\mathbb{E}[\gamma Z_\epsilon(t,u)^2]}\,du\big)^{1-\alpha}}\right]\\ &\le C_0\mathbb{E}\left[\frac{\sup_{0\le i\le\frac{1}{2A\epsilon}}\int_{2Ai\epsilon}^{2A(i+1)\epsilon}e^{\gamma Z_\epsilon(t,u)-\frac{\gamma^2}{2}\mathbb{E}[\gamma Z_\epsilon(t,u)^2]}du}{\big(\int_0^1 e^{\gamma Z_\epsilon(t,u)-\frac{\gamma^2}{2}\mathbb{E}[\gamma Z_\epsilon(t,u)^2]}\,du\big)^{1-\alpha}}\right]\\ &\le C_0\mathbb{E}\left[\Big(\sup_{0\le i\le\frac{1}{2A\epsilon}}\int_{2Ai\epsilon}^{2A(i+1)\epsilon}e^{\gamma Z_\epsilon(t,u)-\frac{\gamma^2}{2}\mathbb{E}[\gamma Z_\epsilon(t,u)^2]}du\Big)^\alpha\right]\end{aligned}$$

Let us denote by M_ϵ^t the measure $e^{\gamma Z_\epsilon(t,u)-\frac{\gamma^2}{2}\mathbb{E}[Z_\epsilon(t,u)^2]}\,du$. Hence

$$|\varphi_\epsilon^A(t)| \leq C_0 \mathbb{E}[(\sup_{0\leq i\leq \frac{1}{2A\epsilon}} M_\epsilon^t[2Ai\epsilon, 2A(i+1)\epsilon])^\alpha]$$

Now, using Theorem 2 and Theorem 15, M_ϵ^t is tight and every limit measure M is such that there exists some $C>0$ such that for all $q<\frac{2}{\gamma^2}$

$$\mathbb{E}[M[a,b]^q] \leq C(b-a)^{\xi(q)}, \quad a<b$$

where $\xi(q)=(1+\frac{\gamma^2}{2})q-\frac{\gamma^2}{2}q^2$. In particular, this proves that M has no atoms. Indeed, choose $q>1$ such that $\xi(q)>1$; then, we have for all $\eta>0$

$$\mathbb{P}(\cup_{1\leq i\leq 2^n}\{M[\frac{i-1}{2^n},\frac{i}{2^n}]>\eta\}) \leq \frac{C}{\eta^q}\frac{1}{(2^n)^{\xi(q)-1}}$$

and therefore

$$\mathbb{P}(\cup_{n\geq 1}\cap_{1\leq i\leq 2^n}\{M[\frac{i-1}{2^n},\frac{i}{2^n}]\leq\eta\})=1,$$

hence establishing that M has no atoms. Hence, we get

$$\mathbb{E}[(\sup_{0\leq i\leq \frac{1}{2A\epsilon}} M_\epsilon^t[2Ai\epsilon, 2A(i+1)\epsilon])^\alpha] \underset{\epsilon\to 0}{\to} 0.$$

Now, we turn to the proof of the second statement of theorem 18 and thus we suppose that the couple $(X_\epsilon, M_{\epsilon,\gamma})$ converges in law to (X, M_γ). We know by (44) that $\bar{M}_{\epsilon,\gamma}$ converges in law to M_γ. Now, we show how one can transfer the convergence in law of $\bar{M}_{\epsilon,\gamma}$ to the convergence in law of the couple $(\bar{X}_\epsilon, \bar{M}_{\epsilon,\gamma})$. Take any function f in $C_c^\infty(D)$. For any bounded continuous functionals F on $\mathbb{R}$ and g with compact support in D, we have by using the Girsanov transform

$$\mathbb{E}[e^{\bar{X}_\epsilon(f)}F(\bar{M}_{\epsilon,\gamma}(g))] = e^{\frac{1}{2}\mathrm{Var}[\bar{X}_\epsilon(f)]}\mathbb{E}[F(\bar{M}_{\epsilon,\gamma}(T_\epsilon(g))]$$

where $T_\epsilon(g)$ is the function on D defined by

$$T_\epsilon(g)(x) = e^{\gamma\int_D \overline{K}_\epsilon(x,y)f(y)\,dy}g(x).$$

It is readily seen that the mapping $x\mapsto\int_D \overline{K}_\epsilon(x,y)f(y)\,dy$ converges uniformly over $\mathrm{Supp}(g)$ towards $x\mapsto\int_D K(x,y)f(y)\,dy$. Similarly, $\mathrm{Var}[\bar{X}_\epsilon(f)]$ converges towards $\int_D\int_D f(x)K(x,y)f(y)\,dxdy$. The same properties hold for $(X_\epsilon, M_{\epsilon,\gamma})$. We deduce

$$\lim_{\epsilon\to 0}\mathbb{E}[e^{\bar{X}_\epsilon(f)}F(\bar{M}_{\epsilon,\gamma}(g))] = \mathbb{E}[e^{X(f)}F(M_\gamma(g))],$$

which completes the proof. □

5 Convergence in probability for general cut-offs

In this section, the setup is the following. We assume that we are given a centered Gaussian distribution X on $C_c^\infty(D)$ for some domain D of $\mathbb{R}^d$ defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ with covariance kernel K of the type (13). Such a random distribution can be constructed via the Minlos theorem as prescribed in section 3. We treat now the general theory of convergence in probability of the associated Gaussian multiplicative chaos measures. For the sake of simplicity, we will focus here on the cases that we have considered in this course, namely stationary log-correlated fields and the GFF with Dirichlet boundary conditions. The results we will present here were first proved by Shamov [46] and fall under the scope of his general theory of approximation of GMC measures. The paper [46] is elegant and works in great generality (beyond the log-correlated case of this course) however it is also very abstract. Here we will give a proof that relies heavily on the previous section on convergence in law whereas Shamov's proof does not rely on theses results. This proof has one advantage: it can in fact be adapted to prove a similar universality result for the so-called Seneta Heyde construction of the critical Gaussian multiplicative chaos introduced in [18, 19]: see remark below. Finally, we mention that the proof we propose is not quite independent of [46] as it relies on a simple and general observation of [46] (Lemma 49) . The next lemma is in fact a slight variant of lemma 49 in [46] that we will use in the proof:

Lemma 19. *Let $(\bar{M}_\epsilon)_{\epsilon>0}$ be a sequence of measures on D which are functions of X (or of a white noise W) and such that the couple $(X, \bar{M}_\epsilon)_{\epsilon>0}$ (or $(W, \bar{M}_\epsilon)_{\epsilon>0}$) converges in law as ϵ goes to 0 to a couple $(\tilde{X}, \tilde{M})$ (or $(\tilde{W}, \tilde{M})$) where $\tilde{M}$ is a function of $\tilde{X}$ of the form $F(\tilde{X})$ (or $F(\tilde{W})$). Then, the family $(X, \bar{M}_\epsilon)_{\epsilon>0}$ (or $(W, \bar{M}_\epsilon)_{\epsilon>0}$) converges in probability to $F(X)$ (or $F(W)$).*

Proof. We will suppose that $(X, \bar{M}_\epsilon)_{\epsilon>0}$ converges in law as ϵ goes to 0 to a couple $(\tilde{X}, \tilde{M})$ where $\tilde{M}$ is a function of $\tilde{X}$ of the form $F(\tilde{X})$ (the proof is exactly the same if we replace X by a white noise W). The triplet $(X, \bar{M}_\epsilon, F(X))_{\epsilon>0}$ is tight and let $(\tilde{X}, \tilde{M}, \tilde{\tilde{M}})$ be a limit along a subsequence. The law of $(X, \tilde{M})$ is $(\tilde{X}, F(\tilde{X}))$ and the law of $(\tilde{X}, \tilde{\tilde{M}})$ is also $(\tilde{X}, F(\tilde{X}))$ hence $\tilde{M} = \tilde{\tilde{M}}$. Therefore $\bar{M}_\epsilon - F(X)$ converges in law towards 0 hence also in probability.

□

Now, we can state the main result of this section:

Theorem 20. *Let X be the GFF with Dirichlet boundary conditions or a log-correlated field with covariance K of the form (19). We fix $\gamma^2 < 2d$ (hence $\gamma < 2$ if X is the GFF). Let us assume that we are given a smooth nonnegative function ρ with compact support, say contained in the ball $B(0,1)$, such that $\int_{\mathbb{R}^d} \rho(x)\,dx = 1$. For $\epsilon \in]0,1]$, we set $\rho_\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon)$. Then we define the cut-off approximation*

$$\bar{X}_\epsilon(x) = X(\rho_\epsilon(x+\cdot)). \tag{46}$$

The family of random measures

$$\bar{M}_\epsilon(dx) = e^{\gamma \bar{X}_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[\bar{X}_\epsilon(x)^2]}\,dx \tag{47}$$

converges in probability as $\epsilon \to 0$ towards a random measure, which does not depend on the mollifier ρ.

Proof. We first consider the case where K of the form (19). We consider X_ϵ the cut-off associated to (21). The associated chaos measure converges almost surely to some measure M_γ which is a function of X, i.e. of the form $F(X)$. We can apply theorem 18 since X_ϵ and $\bar{X}_\epsilon$ are both K-approximations: in fact, a look at the proof of theorem 18 shows that it remains valid with the couple (X, M_ϵ) (and $(\bar{X}, \bar{M}_\epsilon)$) insted of (X_ϵ, M_ϵ) (and $(\bar{X}_\epsilon, \bar{M}_\epsilon)$. Hence, the couple $(X, \bar{M}_\epsilon)_{\epsilon>0}$ converges in law as ϵ goes to 0 to a couple $(\tilde{X}, F(\tilde{X}))$. Now, we conclude by lemma 19.

Next, we consider the case where X is the GFF and we suppose that X is constructed as a function of some white noise along (31). This is no restriction as the convergence in probability result we will establish does not depend on how you construct the GFF. We consider X_ϵ the cut-off associated to (32). For each $\epsilon > 0$, X_ϵ is a function of some white noise W. The associated chaos measure converges almost surely to some measure M_γ which is a function of W, i.e. of the form $F(W)$. Notice here also that X_ϵ and $\bar{X}_\epsilon$ are both K-approximations. Now, we can apply a variant of theorem 18 where instead of the couple (X_ϵ, M_ϵ) (and $(X_\epsilon, \bar{M}_\epsilon)$) we consider the couple (W, M_ϵ) (and $(W, \bar{M}_\epsilon)$); indeed, the proof works the same. Hence, we get that the couple $(W, \bar{M}_\epsilon)_{\epsilon>0}$ converges in law as ϵ goes to 0 to a couple $(\tilde{W}, F(\tilde{W}))$. Now, we conclude by lemma 19.

□

Remark. *In fact, when X is the GFF, the techniques of this section along with section 4 can be adapted to the so-called critical case (see [28] for the details) to show the exact analog of theorem 20 where one replaces the measure (47) by the following measure*

$$\bar{M}_\epsilon(dx) = \sqrt{\ln \frac{1}{\epsilon}} e^{2\bar{X}_\epsilon(x) - 2\mathbb{E}[\bar{X}_\epsilon(x)^2]} \, dx$$

6 Appendix

6.1 Gaussian integration by parts formula and a useful interpolation lemma

We start with the following classical lemma which can be found in any classical textbook on Gaussian processes (see [29]):

Lemma 21 (Integration by parts). *Let $(X, X_1, \cdots, X_n)$ be a centered Gaussian vector and G some smooth function with polynomial growth at infinity. Then, we have*

$$E[XG(X_1, \cdots, X_n)] = \sum_{i=1}^{n} \mathbb{E}[XX_i]\mathbb{E}[\frac{\partial G}{\partial x_i}(X_1, \cdots, X_n)]$$

We now recall and prove here the following interpolation lemma first derived in [30]:

Lemma 22. *Let $(X_i)_{1\leq i \leq n}$ and $(Y_i)_{1\leq i \leq n}$ be two independent centered Gaussian vectors and $(p_i)_{1\leq i \leq n}$ a sequence of nonnegative numbers. If $F : \mathbb{R}_+ \to \mathbb{R}$ is some smooth function with polynomial growth at infinity, we define:*

$$\varphi(t) = E[F(\sum_{i=1}^{n} p_i e^{Z_i(t) - \frac{1}{2}E[Z_i(t)^2]})], \tag{48}$$

with $Z_i(t) = \sqrt{t}X_i + \sqrt{1-t}Y_i$. Then, we have the following formula for the derivative:

$$\varphi'(t) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j (E[X_i X_j] - E[Y_i Y_j]) E[e^{Z_i(t) + Z_j(t) - \frac{1}{2}E[Z_i(t)^2] - \frac{1}{2}E[Z_j(t)^2]} F''(W_{n,t})], \tag{49}$$

where:

$$W_{n,t} = \sum_{k=1}^{n} p_k e^{Z_k(t) - \frac{1}{2}E[Z_k(t)^2]}.$$

Proof. By taking the derivative under the expectation in (48), we get

$$\begin{aligned}
\varphi'(t) &= \frac{1}{2}E\left[\left(\sum_{i=1}^{n} p_i\Big(\frac{X_i}{\sqrt{t}} - \frac{Y_i}{\sqrt{1-t}} - (\mathbb{E}[X_i^2]-\mathbb{E}[Y_i^2])\Big)\right)F'(W_{n,t})\right] \\
&= \sum_{i=1}^{n}\frac{p_i}{2}E\left[\Big(\frac{X_i}{\sqrt{t}} - \frac{Y_i}{\sqrt{1-t}}\Big)e^{Z_i(t)-\frac{1}{2}E[Z_i(t)^2]}F'(W_{n,t})\right] \\
&- \sum_{i=1}^{n}\frac{p_i}{2}(\mathbb{E}[X_i^2]-\mathbb{E}[Y_i^2])E\left[e^{Z_i(t)-\frac{1}{2}E[Z_i(t)^2]}F'(W_{n,t})\right]
\end{aligned}$$

Now, for each i, if we apply lemma 21 then we get

$$\begin{aligned}
&E\left[\Big(\frac{X_i}{\sqrt{t}} - \frac{Y_i}{\sqrt{1-t}}\Big)e^{Z_i(t)-\frac{1}{2}E[Z_i(t)^2]}F'(W_{n,t})\right] \\
&= (E[X_i^2]-E[Y_i^2])E\left[e^{Z_i(t)-\frac{1}{2}E[Z_i(t)^2]}F'(W_{n,t})\right] \\
&+ \sum_{j=1}^{n} p_j(\mathbb{E}[X_iX_j]-\mathbb{E}[Y_iY_j])E\left[e^{Z_i(t)+Z_j(t)-\frac{1}{2}E[Z_i(t)^2]-\frac{1}{2}E[Z_j(t)^2]}F''(W_{n,t})\right],
\end{aligned}$$

where we have used the fact that $\mathbb{E}[(\frac{X_i}{\sqrt{t}} - \frac{Y_i}{\sqrt{1-t}})Z_j(t)] = E[X_iX_j] - E[Y_iY_j]$. Gathering the above considerations gives formula (49).

□

References

[1] Allez R., Rhodes R., Vargas V.: Lognormal $\star$-scale invariant random measures, to appear in *Probability Theory and Related Fields*, arXiv:1102.1895v1.

[2] Ambjorn, J., Durhuus B., Jonsson T.: Quantum Geometry: a statistical field theory approach, *Cambridge Monographs on Mathematical Physics*, 2005.

[3] Bacry E., Muzy J.F.: Log-infinitely divisible multifractal processes, *Comm. Math. Phys.*, **236** no.3, 449-475 (2003).

[4] Bacry, E., Delour, J., and Muzy, J.F.: Multifractal random walks, *Phys. Rev. E*, **64**, 026103-026106 (2001).

[5] Bacry E., Kozhemyak, A., Muzy J.-F.: Continuous cascade models for asset returns, available at www.cmap.polytechnique.fr/ bacry/biblio.html, to appear in *Journal of Economic Dynamics and Control*.

[6] Bacry E., Muzy J.F.: Log-infinitely divisible multifractal process, *Communications in Mathematical Physics*, **236**, 449-475 (2003).

[7] Barral J., Jin X.: On exact scaling log-infinitely divisible cascades, arXiv:1208.2221 [math.PR].

[8] Barral, J., Mandelbrot, B.B.: Multifractal products of cylindrical pulses, *Probab. Theory Relat. Fields* **124**, 409-430 (2002).

[9] Barral, J., Mandelbrot, B.B.: Random Multiplicative Multifractal Measures I, II, III, in M. Lapidus, M. van Frankenhuijsen, eds., Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Proceedings of Symposia in Pure Mathematics, vol 72., n. 2, 3-90 (2004).

[10] Bramson M., Ding J., Zeitouni O.: Convergence in law of the maximum of the two-dimensional discrete Gaussian Free Field, arXiv:1301.6669v1.

[11] Biskup M., Louidor O.: Extreme local extrema of two-dimensional discrete Gaussian free field, arXiv:1306.2602.

[12] Carpentier D., Le Doussal P.: Glass transition of a particle in a random potential, front selection in nonlinear RG and entropic phenomena in Liouville and Sinh-Gordon models, *Phys. Rev. E* **63**:026110 (2001).

[13] Cipriani A., Hazra R.S.: Thick points for Gaussian free fields with different cutoffs, arXiv:1407.5840.

[14] Curien N.: A glimpse of the conformal structure of random planar maps, arXiv:1308.1807, to appear in *Commun. Math. Phys.*

[15] David F., Kupiainen A., Rhodes R., Vargas V.: Liouville Quantum Gravity on the Riemann sphere, *Preprint*, arXiv:1410.7318.

[16] Dubédat J.: SLE and the Free Field: partition functions and couplings, *Journal of the AMS*, **22** (4), 995-1054 (2009).

[17] Duplantier B., Miller J., Sheffield S.: Liouville quantum gravity as a mating of trees, arXiv:1409.7055.

[18] Duplantier B., Rhodes R., Sheffield S., Vargas V.: Critical Gaussian multiplicative chaos: convergence of the derivative martingale, to appear in *Annals of probability*, arXiv:1206.1671.

[19] Duplantier B., Rhodes R., Sheffield S., Vargas V.: Renormalization of Critical Gaussian Multiplicative Chaos and KPZ, to appear in *Communications in Mathematical Physics*, arXiv:1212.0529.

[20] Duplantier, B., Sheffield, S.: Liouville Quantum Gravity and KPZ, *Inventiones mathematicae*, **185** (2), 333-393 (2011).

[21] Fyodorov Y. and Bouchaud J.P.: Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential, *J. Phys. A* **41**, 372001 (2008).

[22] Fyodorov Y, Le Doussal P., Rosso A.: Statistical Mechanics of Logarithmic REM: Duality, Freezing and Extreme Value Statistics of $1/f$ Noises generated by Gaussian Free Fields, *J. Stat. Mech.*, P10005 (2009).

[23] Fyodorov Y, Le Doussal P., Rosso A.: Freezing transition in decaying Burgers turbulence and random matrix dualities, *Europhysics Letters*, 90, 60004 (2010).

[24] Garban C.: Quantum gravity and the KPZ formula, Bourbaki seminar, **1052**, (2012).

[25] Gill J. Rhode S.: On the Riemann surface type of random planar maps, *Revista Mat. Iberoamericana* **29**, 1071-1090 (2013).

[26] Gneiting, T.: Criteria of Polya type for radial positive definite functions, *Proceedings of the American Mathematical Society*, **129** no. 8, 2309-2318 (2001).

[27] Hu X., Miller J., Peres Y.: Thick points of the Gaussian free field, *Annals of Probability*, **38**, 896-926 (2010).

[28] Huang Y., Rhodes R., Vargas V.: Liouville Quantum Gravity on the unit disk, in preparation.

[29] Janson, S.: Gaussian Hilbert Spaces, *Cambridge University press* (1997).

[30] Kahane, J.-P.: Sur le chaos multiplicatif, *Ann. Sci. Math. Québec*, **9** no.2, 105-150 (1985).

[31] Kolmogorov A.N.: A refinement of previous hypotheses concerning the local structure of turbulence, *J. Fluid. Mech.*, **13**, 83-85 (1962).

[32] Ledoux, M., Talagrand, M.: *Probability in Banach Spaces*, Springer-Verlag (1991).

[33] Madaule T.: Maximum of a log-correlated Gaussian field, to appear in Annales de l'Institut Henri Poincaré, arXiv:1307.1365.

[34] Madaule T., Rhodes R., Vargas V.: Glassy phase and freezing of log-correlated Gaussian potentials, to appear in Annals of Applied Probability, arXiv:1310.5574.

[35] Mandelbrot B.B.: A possible refinement of the lognormal hypothesis concerning the distribution of energy in intermittent turbulence, *Statistical Models and Turbulence*, La Jolla, CA, Lecture Notes in Phys. no. 12, Springer, 333-351 (1972).

[36] B.B. Mandelbrot, Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire, I and II. *Comptes Rendus* (Paris): **278A**, 289-292 and 355-358 (1974).

[37] Miller J., Sheffield S.: Quantum Loewner Evolution, arXiv:1312.5745.

[38] Molchan, G.M.: Scaling exponents and multifractal dimensions for independent random cascades, *Comm. Math Physics* **179**, 681-702 (1996).

[39] Obukhov A.M.: Some specific features of atmospheric turbulence, *J. Fluid. Mech.*, **13**, 77-81 (1962).

[40] Pasenchenko, O. Yu.: Sufficient conditions for the characteristic function of a two-dimensional isotropic distribution, *Theory Probab. Math. Statist.*, **53**, 149-152 (1996).

[41] Polyakov A.M., Quantum geometry of bosonic strings, *Phys. Lett.* **103B** 207 (1981).

[42] Rhodes R., Vargas V.: KPZ formula for log-infinitely divisible multifractal random measures, *ESAIM: Probability and Statistics*, **15**, 358-371 (2011).

[43] Rhodes R., Vargas, V.: Gaussian multiplicative chaos and applications: a review, to appear in *Probability Surveys*, arXiv:1305.6221v1.

[44] Rhodes R., Vargas, V.: Multidimensional multifractal random measures, *Electronic Journal of Probability*, **15**, 241-258 (2010).

[45] Robert, R., Vargas, V.: Gaussian multiplicative chaos revisited, *Annals of Probability*, **38** 2, 605-631 (2010).

[46] Shamov A.: On Gaussian multiplicative chaos, arXiv:1407.4418.

[47] Sheffield S.: Gaussian free fields for mathematicians, *Probab. Th. Rel. Fields*, **139**, 521-541 (2007).

[48] Sheffield S.: Conformal weldings of random surfaces: SLE and the quantum gravity zipper, arXiv:1012.4797.

[49] Webb C.: The characteristic polynomial of a random unitary matrix and Gaussian multiplicative chaos - the L^2-phase, arXiv:1410.0939.