Eigenvalue Asymptotics for Schrödinger Operators with Oscillating Decaying Potentials

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February 3rd, 2015

1. Introduction

We will consider the Schrödinger operator

$$H_V := -\Delta + V,$$

self-adjoint in $L^2(\mathbb{R}^d)$, $d \geq 1$. Here

$$-\Delta := -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

is the kinetic-energy operator, and the multiplier V by the function $V : \mathbb{R}^d \to \mathbb{R}$ is the potential-energy operator for a d-dimensional spinless non relativistic quantum particle.

We choose such units that the mass of the particle is equal to $1/2$, and the Planck constant $\hbar = 1$.

Assume that the multiplier V is $-\Delta$ relative compact in the sense of the quadratic forms, i.e. that the operator

$$|V|^{1/2}(-\Delta + 1)^{-1/2}$$

is compact in $L^2(\mathbb{R}^d)$. A simple sufficient condition is that for each $\varepsilon > 0$ we have

$$V = V_{1,\varepsilon} + V_{2,\varepsilon}$$

where $V_{1,\varepsilon} \in L^p(\mathbb{R}^d)$ with $p = 1$ if $d = 1$, $p > 1$ if $d = 2$, $p = d/2$ if $d \geq 3$, and

$$\|V_{2,\varepsilon}\|_{L^\infty(\mathbb{R}^d)} \leq \varepsilon.$$

An even simpler sufficient condition is that $V \in L^p_{\text{loc}}(\mathbb{R}^d)$ and $\lim_{|x|\to\infty} V(x) = 0$.

Then we have

$$\sigma_{\text{ess}}(H_V) = \sigma_{\text{ess}}(H_0) = [0, \infty),$$

so that the possible discrete spectrum of H_V is negative, and it can accumulate only at 0.

Moreover, if $V \geq 0$ almost everywhere in $\mathbb{R}^d$, then $\sigma_{\text{disc}}(H_V) = \emptyset$.

Let us consider the problem about the finiteness of $\sigma_{\text{disc}}(H_V)$. For $E \in (-\infty, 0]$ set

$$N(E; V) := \operatorname{Tr} \mathbf{1}_{(-\infty, E)}(H_V).$$

Thus, $N(E; V)$ is the number of the eigenvalues of H_V smaller than E, and counted with the multiplicities.

Theorem 1. (Cwikel-Lieb-Rozenblum bound) *Let $d \geq 3$, and $V_- \in L^{d/2}(\mathbb{R}^d)$. Then there exists a constant c'_d such that*

$$N(0;V) \leq$$

$$c'_d \left| \left\{ (x,\xi) \in \mathbb{R}^{2d} \,|\, |\xi|^2 + V(x) < 0 \right\} \right| =$$

$$c_d \int_{\mathbb{R}^d} V(x)_-^{d/2} dx \tag{1}$$

where $|\cdot|$ is the Lebesgue measure, $c_d := c'_d \tau_d$, and $\tau_d := \frac{\pi^{d/2}}{\Gamma(1+d/2)}$ is the volume of the unit ball in $\mathbb{R}^d$.

Let $d = 1$. If $V_- \in L^1(\mathbb{R}; (1+|x|)dx)$, then

$$N(0;V) \leq 1 + \int_{\mathbb{R}} |x| V_-(x) dx.$$

If $d = 2$, some more complicated estimates are available as well.

For any $d \geq 1$ these estimates imply

Corollary 2. *Let*

$$V_{-}(x) = O((1 + |x|)^{-\rho}), \quad x \in \mathbb{R}^d,$$

with $\rho \in (2, \infty)$. *Then*

$$N(0; V) < \infty.$$

On the other hand, we have the following

Theorem 3. *Let $d \geq 1$, $\rho \in (0,2)$. Assume that*

$$|V(x)| \leq C(1+|x|)^{-\rho}, \quad x \in \mathbb{R}^d,$$

$$|\nabla V(x)| \leq C(1+|x|)^{-\rho-1}, \quad x \in \mathbb{R}^d,$$

$$V(x) \leq -C|x|^{-\rho}, \quad x \in \mathbb{R}^d, \quad |x| \geq R,$$

for some $C \in (0,\infty)$, $R \in (0,\infty)$. Then

$$N(E;V) =$$

$$\frac{\tau_d}{(2\pi)^d} \int_{\mathbb{R}^d} (V(x)-E)_-^{d/2} dx \, (1+o(1)) \asymp$$

$$|E|^{d(\frac{1}{2}-\frac{1}{\rho})}$$

as $E \uparrow 0$.

The border-line case $\rho = 2$ is handled in the following

Theorem 4. *Let $d \geq 1$. Assume that $V \in L^\infty(\mathbb{R}^d)$ and there exists $L \in \mathbb{R}$ such that $\lim_{|x|\to\infty} |x|^2 V(x) = L$. Then we have*

$$\lim_{E\uparrow 0}(|\ln|E||)^{-1} N(E;V) = \mathcal{C}_d(L)$$

where

$$\mathcal{C}_d(L) :=$$

$$\begin{cases} \frac{1}{\pi}\left(L + \frac{1}{4}\right)_-^{1/2} & \text{if} \quad d = 1, \\ \frac{1}{2\pi}\sum_{q=0}^{\infty}\left(L + \frac{(d-2)^2}{4} + \lambda_q\right)_-^{1/2} & \text{if} \quad d \geq 2, \end{cases}$$

where $\{\lambda_q\}_{q\in\mathbb{Z}_+}$ is the non decreasing sequence of the eigenvalues of the Beltrami-Laplace operator, self-adjoint in $L^2(\mathbb{S}^{d-1})$, counted with their multiplicities. If, moreover,

$$L > -\frac{(d-2)^2}{4}, \quad d \geq 1,$$

then

$$N(0;V) < \infty.$$

3. Main result

Problem considered: Investigate the properties of $\sigma_{\text{disc}}(H_V)$ in the case $V = \eta W$ where η is a (non constant) almost periodic function, and W decays regularly at infinity.

Some notations and definitions:

We will write $W \in \mathcal{S}_{m,\rho}(\mathbb{R}^d)$, $m \in \mathbb{Z}_+$, $\rho \in (0, \infty)$, if $W \in C^m(\mathbb{R}^d)$, and there exists a constant $C \in (0, \infty)$ such that

$$|D^\alpha W(x)| \leq C(1 + |x|)^{-\rho-|\alpha|}, \quad x \in \mathbb{R}^d,$$

for each $\alpha \in \mathbb{Z}_+^d$ with $0 \leq |\alpha| \leq m$.

Further, set

$$\eta(x) := \sum_{\ell \in \mathcal{J}} \eta_\ell e^{i\xi_\ell \cdot x}, \quad x \in \mathbb{R}^d,$$

where:

- $\mathcal{J} := \{0, 1, \ldots, \mathcal{L}\}$, $1 \le \mathcal{L} \le \infty$;

- $\eta_\ell \in \mathbb{C}$ if $\ell \in \mathcal{J}$, $\eta_\ell \neq 0$ if $\ell \in \mathcal{J}_0 := \{1, \ldots, \mathcal{L}\}$, and $\sum_{\ell \in \mathcal{J}} |\eta_\ell| < \infty$;

- $\xi_\ell \in \mathbb{R}^d$ if $\ell \in \mathcal{J}$, $\xi_{\ell_1} \neq \xi_{\ell_2}$ if $\ell_1 \neq \ell_2$, $\xi_0 = 0$, and $r := \inf_{\ell \in \mathcal{J}_0} |\xi_\ell| > 0$.

Then we write $\eta \in \mathcal{A}(\mathbb{R}^d)$. Of course, $\eta \in \mathcal{A}(\mathbb{R}^d)$ implies that $\eta : \mathbb{R}^d \to \mathbb{C}$ is a (uniformly) continuous almost periodic function whose mean value equals η_0.

For $u \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz class in $\mathbb{R}^d$, introduce its Fourier transform

$$(\mathcal{F}u)(\xi) = \widehat{u}(\xi) :=$$

$$(2\pi)^{-1/2} \int_{\mathbb{R}^d} e^{-ix\xi} u(x) dx, \quad \xi \in \mathbb{R}^d.$$

Whenever necessary, the Fourier transform is extended by duality to the dual Schwartz class $\mathcal{S}'(\mathbb{R}^d)$.

Then, $\eta \in \mathcal{A}(\mathbb{R}^d)$ implies that

$$((\operatorname{supp} \widehat{\eta}) \setminus \{0\}) \cap B_r = \emptyset$$

where $B_r := \left\{ \xi \in \mathbb{R}^d \,|\, |\xi| < r \right\}$.

A leading example for $\eta \in \mathcal{A}(\mathbb{R}^d)$ is a function periodic with respect to a non degenerate lattice of periods in $\mathbb{R}^d$, which has an absolutely summable series of Fourier coefficients.

Theorem 5. *Let* $d \geq 1$*,* $\rho \in (0,2]$*, and* $\eta \in \mathcal{A}(\mathbb{R}^d;\mathbb{R})$*,* $W \in \mathcal{S}_{2n,\rho}(\mathbb{R}^d;\mathbb{R})$ *with* $n \in \mathbb{N}$*,*

$$n > \mu(d,\rho) := \begin{cases} \frac{d+2-\rho}{2} & \text{if} \quad d = 1,3, \\ 3 - \rho/2 & \text{if} \quad d = 2, \\ \frac{d-\rho}{2} & \text{if} \quad d \geq 4. \end{cases}$$

(i) *Let* $\rho \in (0,2)$*. Assume that*

$$W(x) \leq -C|x|^{-\rho}, \quad x \in \mathbb{R}^d, \quad |x| \geq R,$$

for certain $C \in (0,\infty)$*,* $R \in (0,\infty)$*. If* $\eta_0 > 0$*, then*

$$N(E;\eta W) =$$

$$\frac{\tau_d}{(2\pi)^d}\int_{\mathbb{R}^d}(\eta_0 W(x) - E)_-^{d/2}\,dx(1+o(1)) \asymp$$

$$|E|^{d(\frac{1}{2}-\frac{1}{\rho})}, \quad E \uparrow 0. \tag{2}$$

If, on the contrary, $\eta_0 < 0$*, then*

$$N(0;\eta W) < \infty. \tag{3}$$

(ii) *Let* $\rho \in (0,2)$. *Assume* $\eta_0 = 0$. *Then*

$$N(E;\eta W) = \begin{cases} O\left(|E|^{\frac{d}{2}(1-\frac{1}{\rho})}\right) & \text{if} \quad \rho \in (0,1), \\ O\left(|\ln|E||\right) & \text{if} \quad \rho = 1, \\ O(1) & \text{if} \quad \rho \in (1,2), \end{cases} \tag{4}$$

as $E \uparrow 0$.

(iii) *Let* $\rho = 2$. *Suppose that there exists* $L \in \mathbb{R}$ *such that*

$$\lim_{|x|\to\infty} |x|^2 W(x) = L.$$

Then

$$\lim_{E\uparrow 0} (|\ln|E||)^{-1} N(E;\eta W) = \mathcal{C}_d(\eta_0 L). \qquad (5)$$

If, moreover, $\eta_0 L > -\frac{(d-2)^2}{4}$, *then* (3) *holds true.*

Remark: If $\eta \in L^\infty(\mathbb{R}^d;\mathbb{R})$, $d \geq 1$, and $W \in \mathcal{S}_{0,\rho}(\mathbb{R}^d;\mathbb{R})$ with $\rho \in (2,\infty)$, then Corollary 2 implies that $N(0;\eta W) < \infty$.

4. Sketch of the proof of Theorem 5

Let X be a separable Hilbert space. We denote by $S_\infty(X)$ the class of linear compact operators $T: X \to X$. Let $T = T^* \in S_\infty(X)$. For $s > 0$ set

$$n_\pm(s; T) := \mathrm{Tr}\, \mathbf{1}_{(s,\infty)}(\pm T);$$

thus, $n_+(s; T)$ (resp., $n_-(s; T)$) is the number of the eigenvalues of T larger than s (resp., smaller than $-s$), and counted with the multiplicities. If $T_j = T_j^* \in S_\infty(X)$, $j = 1, 2$, then the Weyl inequalities

$$n_\pm(s_1 + s_2; T_1 + T_2) \leq n_\pm(s_1; T_1) + n_\pm(s_2; T_2)$$

hold for $s_j > 0$, $j = 1, 2$.

For $T \in S_\infty(X)$ and $s > 0$ put

$$n_*(s; T) := n_+(s^2; T^*T).$$

Thus, $n_*(s; T)$ is the number of the singular values of T larger than s, and counted with the multiplicities.

If $T_j \in S_\infty(X)$, $j = 1, 2$, then the Ky Fan inequalities

$$n_*(s_1 + s_2; T_1 + T_2) \leq n_*(s_1; T_1) + n_*(s_2; T_2)$$

hold for $s_j > 0$, $j = 1, 2$.

Lemma 6. (Birman–Schwinger principle) *Let $T = T^* \geq 0$ be a linear operator in a Hilbert space, and let $S = S^*$ be T-relatively compact in the sense of the quadratic forms. Then*

$$\operatorname{Tr} \mathbf{1}_{(-\infty,-\lambda)}(T - tS) =$$

$$n_+(t^{-1}; (T+\lambda)^{-1/2} S (T+\lambda)^{-1/2})$$

for any $t > 0$ and $\lambda > 0$.

By the unitarity of $\mathcal{F}$ and the Birman–Schwinger principle, for $E < 0$ we have

$$N(E; \eta W) = n_-(1; a(E)\mathcal{F}\eta W \mathcal{F}^* a(E)), \quad (6)$$

where

$$a(\xi; E) := (|\xi|^2 - E)^{-1/2}, \quad \xi \in \mathbb{R}^d, \quad E \leq 0.$$

Denote by χ_1 the characteristic function of the ball

$$B_\delta := \{\xi \in \mathbb{R}^d \,|\, |\xi| < \delta\}$$

with $\delta \in (0, r/2)$ and $r = \inf_{\ell \in \mathcal{J}_0} |\xi_\ell|$. Set $\chi_2 := 1 - \chi_1$ and write

$$a(E)\mathcal{F}\eta W\mathcal{F}^* a(E) =$$

$$\eta_0 a(E)\mathcal{F}W\mathcal{F}^* a(E) +$$

$$\sum_{j=1,2} a(E)\chi_j\mathcal{F}(\eta - \eta_0)W\mathcal{F}^*\chi_j a(E) +$$

$$2\mathrm{Re}\, a(E)\chi_1\mathcal{F}(\eta - \eta_0)W\mathcal{F}^*\chi_2 a(E).$$

For $\nu \in [-1, 1]$ set

$$\omega_\nu(x) := (1 + |x|^2)^{-\rho(1+\nu)/2}, \quad x \in \mathbb{R}^d.$$

Then, for $\nu \in (0,1)$ and $E < 0$, we have

$$a(E)\mathcal{F}(\eta_0 W - \omega_\nu)\mathcal{F}^* a(E) +$$

$$a(E)\chi_1 \mathcal{F}(\eta - \eta_0) W \mathcal{F}^* \chi_1 a(E) -$$

$$(1 + C + 2C^2) a(E)\chi_2 \mathcal{F}\omega_{-\nu}\mathcal{F}^* \chi_2 a(E) \leq$$

$$a(E)\mathcal{F}\eta W \mathcal{F}^* a(E) \leq$$

$$a(E)\mathcal{F}(\eta_0 W + \omega_\nu)\mathcal{F}^* a(E) +$$

$$a(E)\chi_1 \mathcal{F}(\eta - \eta_0) W \mathcal{F}^* \chi_1 a(E) +$$

$$(1 + C + 2C^2) a(E)\chi_2 \mathcal{F}\omega_{-\nu}\mathcal{F}^* \chi_2 a(E),$$

where

$$C := \sup_{x \in \mathbb{R}^d} |\eta(x) - \eta_0| \omega_0(x)^{-1} |W(x)|.$$

By the mini-max principle, the Birman–Schwinger principle, and the Weyl inequalities, we get

$$N(E;(1+s)^{-1}(\eta_0 W+\omega_\nu))-$$

$$n_+(s/2;a(E)\chi_1\mathcal{F}(\eta-\eta_0)W\mathcal{F}^*\chi_1 a(E))-$$

$$n_*(t;\omega_{-\nu}^{1/2}\mathcal{F}^*\chi_2 a(0)) \le$$

$$n_-(1;a(E)\mathcal{F}\eta W\mathcal{F}^* a(E)) \le$$

$$N(E;(1-s)^{-1}(\eta_0 W-\omega_\nu))-$$

$$n_-(s/2;a(E)\chi_1\mathcal{F}(\eta-\eta_0)W\mathcal{F}^*\chi_1 a(E))+$$

$$n_*(t;\omega_{-\nu}^{1/2}\mathcal{F}^*\chi_2 a(0)), \qquad (7)$$

with $s\in(0,1)$ and $t:=\sqrt{s/(2(1+C+2C^2))}$.

In the case of $\eta_0 = 0$ we'll need a slightly different estimate inspired by the article:

O. Safronov, *On the absolutely continuous spectrum of multi-dimensional Schrödinger operators with slowly decaying potentials*, Comm. Math. Phys. **254** (2005), 361-366.

Namely, we have

$$n_-(1; a(E)\mathcal{F}\eta W\mathcal{F}^* a(E)) \leq$$

$$N(E; -C_1\omega_1) +$$

$$n_-(1/4; a(E)\chi_1\mathcal{F}\eta W\mathcal{F}^*\chi_1 a(E)) +$$

$$n_*(1/2; C\omega_0^{1/2}\mathcal{F}^*\chi_2 a(0)). \qquad (8)$$

where $C_1 = C_1(\delta) := 16C^2\delta^{-2}$.

Proposition 7. *Let* $d \geq 1$, $\rho \in (0, \infty)$. *Suppose that* $\eta \in \mathcal{A}(\mathbb{R}^d; \mathbb{R})$, *and* $W \in \mathcal{S}_{2n,\rho}(\mathbb{R}^d; \mathbb{R})$ *with* $n \in \mathbb{N}$, $n > \nu(d, \rho)$. *Then for any* $s > 0$ *we have*

$$n_{\pm}(s; a(E)\chi_1 \mathcal{F}(\eta - \eta_0) W \mathcal{F}^* \chi_1 a(E)) = O_s(1), \tag{9}$$

as $E \uparrow 0$.

In order to prove Proposition 7, we need the following

Lemma 8. *Let* $d \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $\rho \in (0, \infty)$, *and* $n \in \mathbb{Z}_+$, $n > \frac{m+d-\rho}{2}$. *Assume that* $W \in \mathcal{S}_{2n,\rho}(\mathbb{R}^d)$. *Then for each* $\varepsilon > 0$ *we have*

$$\hat{W} \in C^m(\mathbb{R}^d \setminus B_\varepsilon), \tag{10}$$

and

$$\sup_{\xi \in \mathbb{R}^d,\, |\xi| \geq \varepsilon} |(D^\alpha \hat{W})(\xi)| < \infty, \tag{11}$$

for each $\alpha \in \mathbb{Z}_+^d$ *with* $0 \leq |\alpha| \leq m$.

Proof. For $\gamma \in \mathbb{Z}_+^d$ with $0 \le |\gamma| \le m$ set

$$\Phi_{\gamma,n}(x) := x^\gamma \Delta^n W(x), \quad x \in \mathbb{R}^d.$$

We have:

$$W \in \mathcal{S}_{2n,\rho}(\mathbb{R}^d),\ n > \frac{m+d-\rho}{2} \Rightarrow \Phi_{\gamma,n} \in L^1(\mathbb{R}^d).$$

Therefore,

$$\hat{\Phi}_{\gamma,n} \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \tag{12}$$

On the other hand, for each $\xi \in \mathbb{R}^d \setminus \{0\}$, and $\alpha \in \mathbb{Z}_+^d$, $0 \le |\alpha| \le m$, we have

$$D^\alpha \widehat{W}(\xi) = D^\alpha \left(|\xi|^{-2n} |\xi|^{2n} \widehat{W}(\xi) \right) =$$

$$\sum_{\beta+\gamma=\alpha} (-i)^{|\gamma|} (-1)^n \frac{\alpha!}{\beta!\gamma!} D^\beta (|\xi|^{-2n}) \hat{\Phi}_{\gamma,n}(\xi). \tag{13}$$

Now (10) - (11) follow from (13) and (12). $\square$

Proof of Proposition 7:
Let $\varkappa \in (\delta, r/2)$, and let $\Theta \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\Theta = \overline{B}_{2\varkappa}$ and $\operatorname{supp}(1-\Theta) \subset \mathbb{R}^d \setminus B_{2\delta}$, be a real even function. We have

$$a(E)\chi_1\mathcal{F}(\eta-\eta_0)W\mathcal{F}^*\chi_1 a(E) =$$

$$a(E)\chi_1\mathcal{F}\mathcal{V}\mathcal{F}^*\chi_1 a(E)$$

where $\mathcal{V} := (2\pi)^{-d/2}(((\eta-\eta_0)W)*\hat{\Theta})$. Hence, for $s > 0$ and $E < 0$ we have

$$n_\pm(s; a(E)\chi_1\mathcal{F}(\eta-\eta_0)W\mathcal{F}^*\chi_1 a(E)) \leq$$

$$N(E; \mp s^{-1}\mathcal{V}). \tag{14}$$

Therefore, it suffices to show that

$$N(0; \mp s^{-1}\mathcal{V}) < \infty, \quad s > 0. \tag{15}$$

For any $d \geq 1$ we have

$$\hat{\mathcal{V}}(\xi) = \sum_{\ell \in \mathcal{J}_0} \eta_\ell \hat{W}(\xi - \xi_\ell)\Theta(\xi), \quad \xi \in \mathbb{R}^d.$$

Since the series $\{\eta_\ell\}_{\ell \in \mathcal{J}_0}$ is absolutely convergent, $\operatorname{supp} \Theta \subset B_{2\varkappa}$, and $\inf_{\ell \in \mathcal{J}_0} |\xi_\ell| = r > 2\varkappa$, Lemma 8 implies that

$$\hat{\mathcal{V}} \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d).$$

Moreover, evidently, $\operatorname{supp} \hat{\mathcal{V}}$ is compact. Hence,

$$\hat{\mathcal{V}} \in L^p(\mathbb{R}^d), \quad p \in [1, \infty],$$

and

$$\|\hat{\mathcal{V}}\|_{L^p(\mathbb{R}^d)} \leq$$

$$\sum_{\ell \in \mathcal{J}_0} |\eta_\ell| \sup_{\xi \in \mathbb{R}^d,\, |\xi| \geq r - 2\varkappa} |\hat{W}(\xi)| \, \|\Theta\|_{L^p(\mathbb{R}^d)}.$$

Let $d \geq 4$. By the Hausdorff–Young inequality,

$$\|\mathcal{V}\|_{L^{d/2}(\mathbb{R}^d)} \leq c''_d \|\widehat{\mathcal{V}}\|_{L^{d/(d-2)}(\mathbb{R}^d)} < \infty, \qquad (16)$$

where c''_d depends only on d. Now, (15) in the case $d \geq 4$ follows from (14), the CLR estimate (1), and (16).

Assume $d = 1$ or $d = 3$. In these cases Lemma 8 implies that $\widehat{\mathcal{V}} \in C^2(\mathbb{R}^2)$, $\Delta\widehat{\mathcal{V}} \in L^1(\mathbb{R}^d)$, and

$$|x|^2\mathcal{V}(x) = -(2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\Delta\widehat{\mathcal{V}}(\xi)\,d\xi.$$

In particular,

$$\lim_{|x|\to\infty}|x|^2\mathcal{V}(x) = 0. \qquad (17)$$

Thus, (15) for the cases $d = 1, 3$ follows from (14), (17), and Theorem 4.

Assume finally $d = 2$. By Lemma 8 we have $\widehat{\mathcal{V}} \in C^4(\mathbb{R}^2)$, $\Delta^2 \widehat{\mathcal{V}} \in L^1(\mathbb{R}^d)$. In particular,

$$|x|^4 \mathcal{V}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \Delta^2 \widehat{\mathcal{V}}(\xi)\, d\xi,$$

and

$$\sup_{x\in\mathbb{R}^d} |x|^4 |\mathcal{V}(x)| < \infty. \tag{18}$$

Hence, (9) for the case $d = 2$ follows from (14), (18), and Corollary 2. $\square$

Furthermore, for $\mu < 1$ and $t > 0$, we have the estimate

$$n_*(t; \omega_{-\mu}^{1/2} \mathcal{F}^* \chi_2 a(0)) < \infty, \qquad (19)$$

which follows from

Lemma 9. *Let* $f, g \in L^\infty(\mathbb{R}^d)$, $d \geq 1$, *and* $\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} g(x) = 0$. *Then the operator* $f\mathcal{F}^* g$ *is compact in* $L^2(\mathbb{R}^d)$.

Now parts (i) and (iii) of Theorem 5 follow from estimates (6), (7), (9), and (19), as well as Theorem 3, Corollary 2, and Theorem 4.

In the case of part (ii) of Theorem 5, (7) should be replaced by the Safronov-type estimate (8).