Exponential dynamical localization
in N-particle Anderson models with long-range interaction on graphs
via Fractional Moment Analysis

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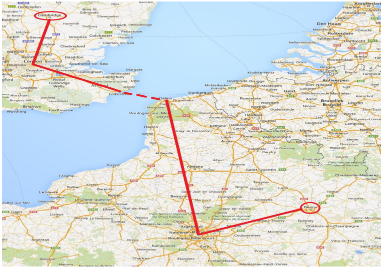

Periodic and ergodic spectral problems, INI, Cambridge, 18.02.2013

1 The model

▶The Hamiltonian: $\mathbf{H}(\omega) = \mathbf{H}_g(\omega) = -\mathbf{\Delta} + g\mathbf{V}(\mathbf{x};\omega) + \mathbf{U}(\mathbf{x})$
$= \sum_{j=1}^{N} \left(\Delta^{(1;j)} + gV(x_j;\omega)\right) + \mathbf{U}(\mathbf{x})$

$\mathbf{x} \in \mathcal{X}^N$; $\mathcal{X}$ is a countable graph with polynomial rate of growth of balls $\mathrm{B}_L(x) := \{y \in \mathcal{X} : \mathrm{d}(x,y) \le L\}$:

$$\sup_{x\in\mathcal{X}} |\mathrm{B}_L(x)| \le CL^d.$$

$$\mathbf{V}(\mathbf{x};\omega) = \sum_{j=1}^{N} V(x_j;\omega), \quad \mathbf{U}(\mathbf{x}) = \sum_{1\le i<j\le N} U^{(2)}(|x_i - x_j|)$$

$U^{(2)} : \mathbb{N} \to \mathbb{R}$ a two-body interaction potential satisfying

$$U^{(2)}(r) \le \mathrm{e}^{-cr}.$$

$V : \mathcal{X} \times \Omega \to \mathbb{R}$ is an IID random field on $\mathcal{X}$ relative to $(\Omega, \mathfrak{B}, \mathbb{P})$, bdd with $\mathbb{P}\{\, V \in [0,1] \,\} = 1$, and with bdd density ($dF_V(t) = p_V(t)\,dt$, $\|p_V\|_\infty < \infty$).
First results on the MPFMM (for $U^{(2)}$ compactly supported): **Aizenman & Warzel, '08, reported in the INI**.

The main result:

Theorem 1 (VC, '14)

For any $N > 1$ *there exists* $g^* = g^*(N)$ *such that for all* $|g| \geq g^*$*:*

- *with probability one,* $\mathbf{H}_g(\omega)$ *has p.p. spectrum with exponentially decaying eigenfunctions;*
- *the eigenfunction correlators for* $\mathbf{H}_g$ *decay exponentially at infinity, thus*

$$\forall\, \mathbf{x} \quad \mathbb{E}\left[\sup_{t\in\mathbb{R}} \| \mathrm{e}^{c\mathbf{X}} \mathrm{e}^{it\mathbf{H}_g(\omega)} \delta_{\mathbf{x}} \| \right] < +\infty.$$

Here $(\mathbf{X}\psi)(\mathbf{x}) := \|\mathbf{x}\|\, \psi(\mathbf{x})$ *for* $\mathcal{X} = \mathbb{Z}^d$*, or* $(\mathbf{X}\psi)(\mathbf{x}) := \mathrm{d}(\mathbf{x}, \mathbf{x}_\circ)\, \psi(\mathbf{x})$ *for a graph* $\mathcal{X}$ *rooted at* $\mathbf{x}_\circ$*.*

(Complete exponential spectral and strong dynamical localization; cf. arXiv:math-ph/1410.1079.)

The main technique: an extension to the infinite-range interactions of the Aizenman-Warzel method developed for the interactions of finite range.

The main novelty: exponential decay of the EFCs (sub-exponential decay of EFCs in presence of exponential or sub-exponential interaction was proved earlier [VC & Yuri Suhov].

Is the N-particle (say, with $N = 2$) localization not obvious, given the results and techniques of the single-particle theory?

Consider single-site "cubes" at $\mathbf{x} = (0, 0)$ and $\mathbf{y} = (R, 0)$.

$$\lambda(\omega) = V(0;\omega) + V(0;\omega), \quad \mu(\omega) = V(0;\omega) + V(R;\omega), \;\; R \gg 1.$$

$\lambda(\omega)$ and $\mu(\omega)$ are correlated via $V(0;\omega)$, no matter how large is R, measuring the distance $\|\mathbf{x} - \mathbf{y}\|$ (or sym. distance $\mathrm{d_S}(\mathbf{x}, \mathbf{y}) = \min[\|\mathbf{x} - \mathbf{y}\|, \|\mathbf{x} - S(\mathbf{y})\|]$, $S(y_1, y_2) = (y_2, y_1)$). The correlation does not depend upon the distance.

A severe deficit of Independence (or decay of dependence) At Distance (IAD).

Solution: conditioning on $V(0;\cdot)$, which renders $V(0;\cdot) = v_0$ non-random, so

$$\lambda(\omega) = 2v_0, \quad \mu(\omega) = v_0 + V(R;\omega).$$

Now $|\lambda - \mu| = |V(R;\omega) - const|$ has a regular distribution, so it is small ("near-resonance") with small probability.

In essence, we used here that $\mathbf{x}$ and $\mathbf{y}$ are separated with respect to the **Hausdorff distance**: with $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$

$$\mathrm{d}_{\mathcal{H}}(X, Y) := \max\left[\max_{x \in X} \mathrm{d}(x, Y), \max_{y \in Y} \mathrm{d}(y, X)\right]$$

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(Below,the norms correspond to the case where the graph is $\mathbb{Z}^d$; in the general case, they are to be replaced by suitable graph-distances.)
Some elementary geometric inequalities:

$$\mathrm{d}_{\mathcal{H}}(\mathbf{x},\mathbf{y}) \le \mathrm{d}_{\mathrm{S}}(\mathbf{x},\mathbf{y}) \le \|\mathbf{x}-\mathbf{y}\| \le \mathrm{d}_{\mathcal{H}}(\mathbf{x},\mathbf{y}) + \min\left[\operatorname{diam}\mathbf{x}, \operatorname{diam}\mathbf{y}\right],$$

with $\operatorname{diam}\mathbf{x} := \operatorname{diam} X = \operatorname{diam}\{x_1,\dots,x_N\}$.
An important point:

$$\mathrm{d}_{\mathcal{H}}(\mathbf{x},\mathbf{y}) \ge \|\mathbf{x}-\mathbf{y}\| - \operatorname{diam}\mathbf{x}.$$

(proof: $|x_i - y_j| \ge |x_j - y_j| - |y_i - y_j|$).
Therefore, for any $\mathbf{x}$ there is a ball $\mathbf{B}_{R_{\mathbf{x}}}(\mathbf{x})$ such that for all $\mathbf{y} \notin \mathbf{B}_{R_{\mathbf{x}}}(\mathbf{x})$

$$\mathrm{d}_{\mathcal{H}}(\mathbf{x},\mathbf{y}) \ge \|\mathbf{x}-\mathbf{y}\| - \operatorname{diam}\mathbf{x} \ge \frac{1}{2}\|\mathbf{x}-\mathbf{y}\|$$

so the LHS $\to +\infty$ along with the RHS, and $\mathrm{e}^{-m\mathrm{d}_{\mathcal{H}}(\mathbf{x},\mathbf{y})} \le \mathrm{e}^{-m\|\mathbf{x}-\mathbf{y}\|/2}$.
CONCLUSION: the decay bounds in the Hausdorff (pseudo-)distance **suffice** for the decay in the natural distance (but only in an **infinite** configuration space; localization in **finite** (**now matter how large**) volumes is a much harder problem. Exception: $N = 2$.)

1 The model

a A priory bounds on the FMs

Such results are fundamental to any variant of the FMM: before ever proving the fractional moments - i.e., some expectations - are small or decay in any way, the first question is :

ARE THE EXPECTATIONS **FINITE** ?

Lemma 2

Let $\{x_1, \ldots, x_N\} \ni u_1$, $\{y_1, \ldots, y_N\} \ni u_2$ *(possibly* $u_1 = u_2$*), then*

$$\mathbb{E}\left[\,|\mathbf{G}(\mathbf{x},\mathbf{y};E)|^s \,\middle|\, \mathfrak{F}_{\neq u_1,u_2}\right] \leq Cg^{-s}.$$

From the analitical point of view, the key ingredients of the proof are:

- the Birman-Schwinger relation,
- Boole's identity.

[Birman, Schwinger and Boole are thus the three musketeers, but, as everybody knows, this means 4 persons; the forth one here is **Hausdorff**, whose **metric** appears inevitably in the applications of the above *a priori*

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This kind of spectral reductions have been developed in parallel in the FMM and in MSA. An FMM-specific result was proved by Aizenman and Warzel (for $N > 1$). Earlier, Elgart et al. ['10]; Martinelly and Scoppola ['83].
Denote $\mathbf{F}_z(E) = \max_{y:\|y-z\|=L} |G(z,y;E)|$.

Theorem 3

Let be given positive numbers $a_L, b_L, c_L, u(L)$ *s. t.* $b_L \le \min\left(a_L c_L^2, c_L\right]$ *and*

$$\max_{z\in\{x,y\}} \sup_{E\in I} \mathbb{P}\left\{ \mathbf{F}_z > a_L \right\} \le u(L).$$

Assume $\mathbb{P}\left\{ \operatorname{dist}\left(\Sigma(\mathbf{H}_{\mathbf{B}_L(x)}, \Sigma(\mathbf{H}_{\mathbf{B}_L(y)})\right) \le s \right\} \le f(s).$ *Then*

$$\mathbb{P}\left\{ \sup_{E\in I} \min\left[\mathbf{F}_x(E), \mathbf{F}_y(E)\right] > a_L \right\} \le 2|I|b^{-1} + f(4c_L).$$

$$a_L = \mathrm{e}^{-\frac{1}{3}mL}, b_L = \mathrm{e}^{-\frac{2}{3}mL}, c_L = \mathrm{e}^{-\frac{1}{8}mL}, u(L) = \mathrm{e}^{-mL}.$$

[Essentially due to Germinet–Klein, '01]
(Different approaches: Germinet–De Bièvre '98, Damanik–Stollmann '01)

Theorem 4

Assume that for a pair of disjoint balls $\mathbf{B}_L(x), \mathbf{B}_L(y)$ *and some positive* a_L, h_L

$$\mathbb{P}\left\{ \sup_{E\in I} \min\left[\mathbf{F}_x(E), \mathbf{F}_y(E)\right] > a_L \right\} \leq h_L.$$

Then for any connected subset $\Lambda \supset \mathrm{B}_L(x) \cup \mathrm{B}_L(y)$ *one has*

$$\mathbb{E}\left[\sup_{\mathcal{C}(\mathbb{R})\ni\phi:\|\phi\|\leq 1} \left|\langle \mathbf{1}_x|\, \phi(H_\Lambda)\, |\mathbf{1}_y\rangle\right| \} \right] \leq 4a_L + h_L.$$

The induction on the number of particles requires the MPFMM to be organised as a logical "spiral":

$$\text{EFC}(N-1) \rightsquigarrow \text{GF}(N) \rightsquigarrow \text{EFC}(N).$$

The next lemma shows explains the relations between the GFs and EFCs:

Lemma 5

$$\int_I |G(x,y;E)|^s \, dE \le \frac{2|I|^{1-s}}{1-s} \big(Q(x,y) \big)^s .$$

Key points of the proof – spectral averaging and Boole's identity: given positive numbers $c_1, \dots, c_n$ and distinct real numbers $\lambda_1, \dots, \lambda_n$, one has

$$\operatorname{mes} \left\{ E \in \mathbb{R} : \left| \sum_k \frac{c_k}{E - \lambda_k} \right| > t \right\} = \frac{2 \sum_k c_k}{t} .$$

3 Structure of [existing] proofs of N-particle AL

The MPFMM has to face a severe lack of IAD. To cut the “Gordian Knot”, Aizenman and Warzel put forward an unexpected solution: trading off the valuable mono-scale technology of the single-particle FMM for the flexibility of a special multi-scale induction.

As a result, today the toolkit of the N-particle localization theory contains not one, but two kinds of **multi-scale** inductions: the one (MPFMM) based on the expectations of the (fractional powers of) GFs, and the other on bounds in probability for the GFs (MPMSA).

There was a price to pay: for example, the Bethe lattices, as well as other graphs of exponential growth, are off-limits for both approaches.

Both approaches (MPMSA and MPFMM) employ an **induction on the number of particles**.

3 Structure of [existing] proofs of N-particle AL

A double induction:
The external inductive loop: Induction on N.

- $N = 1$: the conventional, 1-particle theory (FMM/MSA).
- Each step $N - 1 \rightsquigarrow N$:
  Induction on scales:
  - L_0: assume that one of the mechanisms powered by disorder is strong enough.
    This turns out to be an easy task.

    Pictorially, one can treat the N-particle system as a "single-particle" one, but with a composite quantum object playing the role of the "particle". [No hand-waving: a simple technical observation.]
  - Scaling step $L_k \rightsquigarrow L_{k+1}$: decomposition into non-interacting (resp., weakly interacting) sub-clusters + using information on subsystems of $1 \leq n \leq N - 1$ particles from the previous step of the first induction loop.

4 Decay of the fractional moments of the GFs

It is convenient to use the following length scales:

$$L_{k+1} := 2(L_k + 1),\ k = 0, 1, \dots,$$

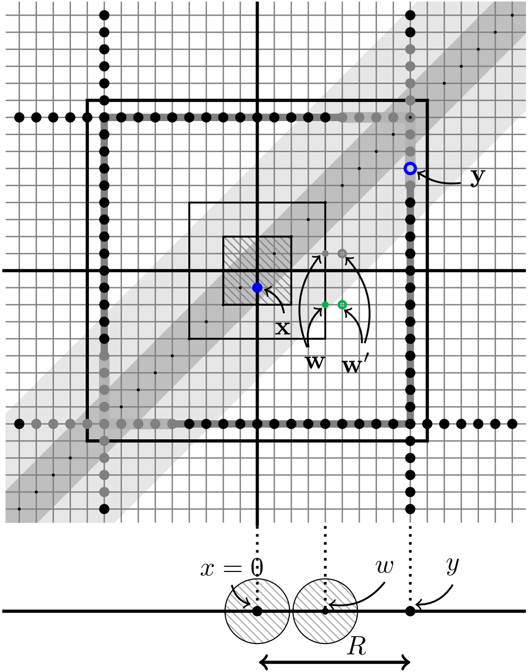
or explicitly,

$$L_k = 2^k (L_0 + 2) - 2.$$

Further, it suffices to assess only Hausdorff-distant pairs $(\mathbf{x}, \mathbf{y})$, with $\mathrm{d}_{\mathcal{H}}(\mathbf{x}, \mathbf{y}) =: R > L_0$.

Then there exists a unique $k \in \mathbb{N}$ such that

$$L_k < R \le L_{k+1}.$$



y
x
w
w′
x = 0
w
y
R

Classification of configurations $\mathbf{x}$, $\mathbf{y}$ by their **diameters**:

Lemma 6

If in addition to the general assumption that $\mathrm{d}_{\mathcal{H}}(\mathbf{x}, \mathbf{y}) > L_k$ *one has* $\max(\operatorname{diam} \mathbf{x}, \operatorname{diam} \mathbf{y}) > L_k/2$, *then*

$$\mathbb{E}\left[\,|\mathbf{G}(\mathbf{x}, \mathbf{y}; E)|^s\,\right] \leq A\mathrm{e}^{-\frac{m}{2} L_k}.$$

The proof does not any scale induction, so the case of two Hausdorff-distant configurations, of which at least one is wide-split, can be addressed independently of the main inductive procedure.

Key points of the proof:

- decay of the EFCs in subsystems (induction on N)
- interaction $\mathbf{U}$ as perturbation, far away from the support of the interaction (split configs !).

$$\mathrm{d}_{\mathcal{H}}(\mathbf{x},\mathbf{y}) \geq \max\left[\mathrm{d}(x_0,\mathbf{y}),\mathrm{d}(y_0,\mathbf{x})\right] \geq d(x_0,y_0) - \operatorname{diam}\mathbf{x}.$$

In our case, with $x_0 = 0, y_0 = y$,

$$\mathrm{d}_{\mathcal{H}}(\mathbf{x},\mathbf{y}) \geq d(0,y) - \operatorname{diam}\mathbf{x} > R - L_k/2,$$

thus

$$\mathrm{d}_{\mathcal{H}}(\Pi\mathbf{B}_{L_k/2}(\mathbf{x}),\Pi\mathbf{B}_{L_k/2}(\mathbf{y})) > R - L_k \geq 1.$$

By the FGRI for the ball $\mathbf{B}_{L_k/2}(\mathbf{0})$,

$$\begin{aligned}
&\mathbb{E}\left[\,|\mathbf{G}(\mathbf{x},\mathbf{y})|^s\,\right] \\
&\leq |\partial\mathbf{B}_{L_k/2}(\mathbf{0})\| \cdot \max_{(\mathbf{w},\mathbf{w}')\in\partial\mathbf{B}_{L_k/2}(\mathbf{0})} \mathbb{E}\left[\,|\mathbf{G}_{\mathbf{B}_{L_k/2}(\mathbf{0})}(\mathbf{x},\mathbf{w})|^s|\mathbf{G}(\mathbf{w}',\mathbf{y})|^s\,\right]
\end{aligned}$$

and with some $u_1 \in \Pi\mathbf{w}'$, $u_2 \in \Pi\mathbf{y}$

$$\begin{aligned}
&\mathbb{E}\left[\,|\mathbf{G}_{\mathbf{B}_{L_k/2}(\mathbf{0})}(\mathbf{x},\mathbf{w})|^s|\mathbf{G}(\mathbf{w}',\mathbf{y})|^s\,\right] \\
&= \mathbb{E}\left[\,|\mathbf{G}_{\mathbf{B}_{L_k/2}(\mathbf{0})}(\mathbf{x},\mathbf{w})|^s\mathbb{E}\left[\,|\mathbf{G}(\mathbf{w}',\mathbf{y})|^s|\mathfrak{F}_{\neq u_1,u_2}\,\right]\,\right]
\end{aligned}$$

By the a priori bound on the FMs,

$$\mathbb{E}\left[\,|\mathbf{G}(\mathbf{w}',\mathbf{y})|^s|\mathfrak{F}_{\neq u_1,u_2}\,\right] \leq C_s g^{-s},$$

thus

$$\mathbb{E}\left[\,|\mathbf{G}(\mathbf{x},\mathbf{y})|^s\,\right] \leq C_s g^{-s}\mathbb{E}\left[\,|\mathbf{G}_{\mathbf{B}_{L_k/2}(\mathbf{0})}(\mathbf{x},\mathbf{w})|^s\,\right].$$

Assessing the above expectation is the most tedious task, and it will be entrusted to the multi-scale induction.

Now $\mathbf{x}$ and $\mathbf{y}$ are Hausdorff-distant ($R \geq L_k$) and we consider first the case where $\mathbf{w}$ is at least $L_k/2$-split.

Apply Lemma on distant **and split** configurations:

$$\mathbb{E}\left[\left| \mathbf{G}_{\mathbf{B}_{L_k/2}(\mathbf{0})}(\mathbf{x}, \mathbf{w}) \right|^s \right] \leq \mathrm{Const}\, \mathrm{e}^{-mL_k/2}.$$

This has been an easy case . . .

Before going further:

Expectations in the energy-disorder space: (cf. Martinelli–Scoppola '83)

$$\mathbb{E}^I\left[f(E,\omega) \right] = \frac{1}{|I|} \int_I \mathbb{E}\left[f(E,\omega) \right] dE$$

Why:

1. convenient for switching between the EFCs and FMs of the GFs;
2. perfectly adapted for the reductions:

   fixed $E \rightsquigarrow$ interval $I \rightsquigarrow$ dynamical localization.

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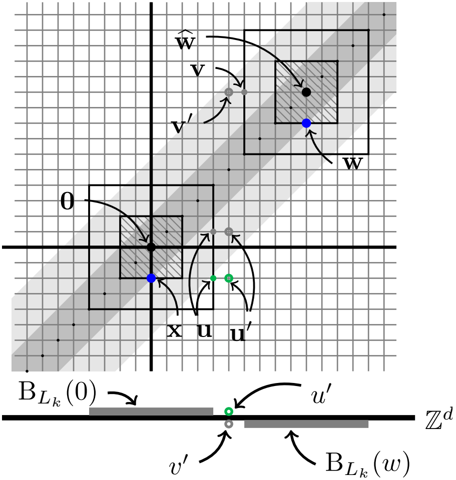
$\widehat{\mathbf{w}}$
$\mathbf{v}$
$\mathbf{v}'$
$\mathbf{w}$
$\mathbf{0}$
$\mathbf{x}$
$\mathbf{u}$
$\mathbf{u}'$
$\mathrm{B}_{L_k}(0)$
u'
$\mathbb{Z}^d$
v'
$\mathrm{B}_{L_k}(w)$

We shall establish a recursive scaling relation for the EFCs of the form

$$\Upsilon(L) = |\partial \mathbf{B}_L| \sup_{|I|\geq 1} \sup_{\Lambda\subset \mathrm{B}_L(0)} \sum_{\substack{\mathrm{d}(0,u)=L \\ \mathbf{x}\in \mathcal{X}^{\Lambda}_{L/2}(0) \\ \mathbf{w}\in \mathcal{X}^{\Lambda}_{L/2}(u)}} \mathbb{E}^I \left[\, |\mathbf{G}_\Lambda(\mathbf{x},\mathbf{w})|^s \, \right]$$

Technically, it turns out to be more convenient to work with restricted correlators

$$\widetilde{\Upsilon}(L_{k+1}) = |\partial \mathbf{B}_{L_{k+1}}| \sup_{|I|\geq 1} \sup_{\Lambda\subset \mathrm{B}_{L_{k+1}}(0)} \sum_{\substack{\mathrm{d}(0,u)=L_{k+1} \\ \mathbf{x}\in \mathcal{X}^{\Lambda}_{\mathbf{L_k/2}}(0) \\ \mathbf{w}\in \mathcal{X}^{\Lambda}_{\mathbf{L_k/2}}(u)}} \mathbb{E}^I \left[\, |\mathbf{G}_\Lambda(\mathbf{x},\mathbf{w})|^s \, \right]$$

(*notice the radius* $\mathbf{L_k/2}$ *in* $\widetilde{\Upsilon}(L_{k+1})$, *replacing* $\mathbf{L_{k+1}/2}$ *that would be used in unrestricted EFCs* $\Upsilon(L_{k+1})$.)

Lemma 7 (Approximation bound)

$$0 \leq \Upsilon(L_{k+1}) - \widetilde{\Upsilon}(L_{k+1}) \leq 2AL_{k+1}^2 \mathrm{e}^{-mL_k/2}$$

Sketch of the proof:

• presence of a given term in the sum $\Upsilon(L_{k+1})$ but not in the restricted sum $\widetilde{\Upsilon}(L_{k+1})$ implies the configurations involved in the term are at least $L_k/2$-split, so we can apply the lemma on split configs: these terms are exponentially small in L_k.

• Count these terms: polynomial in L_k, so the exponent takes over.

This has been an easy case . . .

[A sad] **remark**: **This** is where the hope is lost to treat Bethe lattices and other graphs with exponential growth of balls ... The above mentioned combinatorial factors become exponential and ruin all localization bounds.

Lemma 8

$$\widetilde{\Upsilon}(L_{k+1}) \le C g^{-s} \left(\Upsilon(L_k) + C' L_k^q \mathrm{e}^{-mL_k/2} \right)^2 .$$

Consequently, taking into account the approximation error (Υ vs. $\widetilde{\Upsilon}$),

$$\Upsilon(L_{k+1}) \le C g^{-s} \left(\Upsilon(L_k) + C' \mathrm{e}^{-mL_k/2} \right)^2 + A L_{k+1}^p \mathrm{e}^{-mL_k/2} .$$

The proof is a bit more technical than in the previous lemma. The key tool is a decoupling inequality for the fractional moments of the GFs, close in spirit to the one used already by Aizenman and Molchanov ('93), but more involved due to the lack of independence at distance.

The proof actually reveals **two important points**: it shows

- why the Hausdorff distance is (substantially!) more **convenient** than the symmetrized distance; this is the pivot of the entire procedure;
- why one has to abandon the valuable and much appreciated **mono-scale** technology of the Aizenman–Molchanov approach.

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4 Decay of the fractional moments of the GFs

g Rescaling: Conclusion, and end of the scale induction

Lemma 9 (Perturbed quadratic dynamics)

Assume

$$\beta_{k+1} \le \frac{1}{2} e^{-2\nu} \beta_k^2 + \frac{1}{2} e^{-2\nu L_k} \beta_k .$$

Then for all $k \ge 1$,

$$\beta_k \le \max \left[e^{-\nu L_k}, e^{-\mu L_k} \right]$$

with

$$\mu := \frac{\nu + \ln \beta_0^{-1}}{1 + \frac{L_0}{2}}$$

The proof is elementary; essentially, reduction to the recursion $\beta_{k+1} \le \left(e^{-\nu}\beta_k\right)^2$, yielding $\beta_k \le \left(e^{-\nu}\beta_0\right)^{2^k}$, unless the perturbation term $e^{-2\nu L_k}$ interferes and becomes leading, thus limiting the decay rate.

It remains only to apply this lemma to the EFCs $\Upsilon(L_k)$ and conclude.

This marks the end of the scaling step and of the multi-scale induction.

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5 FMs for split configurations

Now we shall see how the MPFMM extends to the interactions of infinite range.

Lemma 10

Suppose that

$$\min\left[\mathrm{d}_{\mathcal{H}}(\mathbf{x},\mathbf{y}), \mathrm{diam}\,\mathbf{x} \vee \mathrm{diam}\,\mathbf{y}\right] \geq R > 0,$$

and assume that the EF correlators in n*-particle systems for all* $n \in [1, N-1]$ *feature exponential decay with exponent* $m > 0$*. Then for some* $c > 0$*, one has*

$$\mathbb{E}\left[\left| \mathbf{G}^{(N)}(\mathbf{x},\mathbf{y}) \right|^s \right] \leq A\mathrm{e}^{-cmR}.$$

Remark: for $N > 1$ arbitrary, $c = c'/(N-1)$.

Proof *(Sketch, for $N = 2$)* in three steps.

I. Setting up a **perturbation analysis**:

$$\mathbf{H} = \underbrace{H^{(1)} \otimes \mathbf{1}^{(2)} + \mathbf{1}^{(1)} \otimes H^{(2)}}_{\text{non-interacting}} + \underbrace{\mathbf{U}^{(1,2)}}_{\text{perturbation}} .$$

II. Assessing the **non-interacting** system: induction on the number of particles; subsystems contain $\leq N - 1$ particles and are far apart (at least one of the configurations, e.g., $\mathbf{x}$, is wide-split by assumption $\mathrm{diam}(\mathbf{x}) \geq R$).

III. Assessing the **perturbation**:

(i) by assumption, $\mathrm{d}_{\mathcal{H}}(\mathbf{x}, \mathbf{y}) \geq R$, and

(ii) by (II), the non-interacting GFs decay exponentially. (+ some geom. inequalities for the Hausdorff distance).

Stage (II) in more detail.
The subsystem EF corralators decay exponentially (induction on the number of particles, but no scale induction here):

$$\mathbb{E}\left[Q^{(1)}(x_1, y_1) \right] \le A\mathrm{e}^{-m\mathrm{d}(x_1,y_1)},$$
$$\mathbb{E}\left[Q^{(2)}(x_2, y_2) \right] \le A\mathrm{e}^{-m\mathrm{d}(x_2,y_2)}$$

(in general, (x_1, x_2) and (y_1, y_2) are decompositions in $n_1 + n_2$ particle subsystems, and $\mathrm{d}(\cdot,\cdot)$ becomes $\mathrm{d}_{\mathcal{H}}(\cdot,\cdot)$). Both $Q^{(1)}$ and $Q^{(2)}$ are bounded by 1, so it suffices to write

$$\begin{aligned} \mathbb{E}\left[Q^{(1)}\, Q^{(2)} \right] &\le \mathrm{e}^{-m\max(\mathrm{d}(x_1,y_1),\mathrm{d}(x_2,y_2))} \\ &\le \mathrm{e}^{-m\mathrm{d}_{\mathcal{H}}(\mathbf{x},\mathbf{y})} \end{aligned}$$

(for $N = 2$, the symmetrized product-max-distance is more natural to use, and it is equivalent to $\mathrm{d}_{\mathcal{H}}$ in this particular case – for $N = 2$).

To conclude the analysis of the non-interacting system assess the FMs of the Green functions:
Using the spectral averaging inequality (Lemma 5)

$$\int_I |\mathbf{G}(\mathbf{x},\mathbf{y};E)|^s \, dE \le \frac{2|I|^{1-s}}{1-s} \big(Q(\mathbf{x},\mathbf{y})\big)^s,$$

(Birman-Schwinger relation, Boole's identity ...), we come to the bound

$$\begin{aligned}\mathbb{E}^I \left[|\mathbf{G}^{(1,2)}(\mathbf{x},\mathbf{y})|^s \right] &\le \frac{C}{1-s}\mathbb{E}\left[|Q^{(1,2)}(\mathbf{x},\mathbf{y})|^s \right] \\ &\le \frac{C}{1-s} A \mathrm{e}^{-m \mathrm{d}_{\mathcal{H}}(\mathbf{x},\mathbf{y})}\end{aligned}$$

The main point here was an appropriate choice of the Birman–Schwinger operators (Aizenman–Warzel, '09), closely related to the choice of the suitable metric (Hausdorff distance). The "trick" does **not** work for the symmetrized norm- (or graph-) distance.

The situation with spectral averaging was substantially simpler in the case $N = 1$, where, in the discrete systems, the rank-2 Krein formula had been used by Aizenman et al. ('93 – '01). Rank-1 goes back to Kotani and Simon–Wolff.

Let $\ell = \ell(\mathbf{w}) = \mathrm{d}(w_1, w_2)$ (the size of the inter-cluster gap) We have

$$\mathbb{E}\left[|\mathbf{G}^{(1,2)}\mathbf{U}\mathbf{G}|^s \right] \le \|\mathbf{U}\| S_1 + S_2$$

where

$$S_1 = \sum_{\ell(\mathbf{w}) \le R'} \mathbb{E}\left[|\mathbf{G}^{(1,2)}(\mathbf{x}, \mathbf{w})|^s \; |\mathbf{G}(\mathbf{w}, \mathbf{y})|^s \right]$$

$$S_2 = \sum_{\ell(\mathbf{w}) > R'} |\mathbf{U}(\mathbf{w})| \cdot |\mathbb{E}\left[|\mathbf{G}^{(1,2)}(\mathbf{x}, \mathbf{w})|^s \mathbf{G}(\mathbf{w}, \mathbf{y})|^s \right]$$

S_1 is exponentially small, since $\mathbf{w}$ must be far from $\mathbf{x}$ (use $\mathbf{G}^{(1,2)}(\mathbf{x}, \mathbf{w})$).

S_2 is exponentially small, because of exponential decay of $\mathbf{U}(\mathbf{w})$ for $\mathbf{w}$ far from the "diagonal".

□

6 Sub-exponentially decaying interactions

Discrete systems:

- $U^{(2)}(r) \le e^{-r^\zeta}$, $\zeta < 1$ close to 1: Sub-exponential decay of the EFCs with some $\Upsilon(r) \le e^{-r^\delta}$ with **some** $\delta > 0$;
  EFs decay exponentially: $\Psi_j(\mathbf{x}) \le e^{-m|\mathbf{x}-\widehat{\mathbf{x}}_j(\omega)|}$. [VC, '11; MPMSA]
- $U^{(2)}(r) \le e^{-r^\zeta}$, any $\zeta > 0$ (no matter how small): Sub-exponential decay of the EFCs with **some** $\delta > 0$;
  EFs decay exponentially: $\Psi_j(\mathbf{x}) \le e^{-m|\mathbf{x}-\widehat{\mathbf{x}}_j(\omega)|}$.
  [VC & Y. Suhov, '14; MPMSA]

Continuous systems (in $\mathbb{R}^d$):

- Sub-exponential decay of the EFCs (hence, of the EFs). Decay is exponential if $U^{(2)}$ decays exponentially [Fauser & Warzel, '14; MPFMM]
- Sub-exponential decay of the EFCs, but still a genuine **exponential** decay of the EFs). [VC, '14; MPMSA]