Reflectionless property and related problems on 1D Schrödinger operators

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Almost-Periodic and Other Ergodic Problems
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Weyl-functions (m-functions)

- 1D Schrödinger op on $\mathbb{R}$: $L = L^q = -d^2/dx^2 + q$ for $q \in \mathcal{Q}$:

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- For $\forall \lambda \in \mathbb{C} \setminus \mathbb{R}$, $\exists 1 f_\pm = f_\pm(x, \lambda, q)$ satisfying

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$$\left(L^q - \lambda\right)^{-1}(x, y) = g_\lambda(x, y, q) = -\frac{f_+(x, \lambda, q) f_-(y, \lambda, q)}{m_+(\lambda, q) + m_-(\lambda, q)}$$

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- $m_\pm(\lambda)$: Herglotz fts.: $\exists \alpha_\pm \in \mathbb{R}$, $\beta_\pm \geq 0$, $\sigma_\pm$

$$m_\pm(\lambda) = \alpha_\pm + \beta_\pm \lambda + \int_{-\infty}^{\infty} \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1 + \xi^2}\right) \sigma_\pm(d\xi).$$

Spectrum

- $E\left(d\xi\right)$: resolution of Identity for a self adjoint op. A on H

  Lebesgue decomp.: $E\left(d\xi\right) = E_{ac}\left(d\xi\right) + E_{sc}\left(d\xi\right) + E_{p}\left(d\xi\right)$

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- Spectral components:

spec.	$\Sigma = \mathrm{supp}E(d\xi)$	a.c. sp	$\Sigma_{ac} = \mathrm{supp}E_{ac}(d\xi)$
s.c. sp	$\Sigma_{sc} = \mathrm{supp}E_{sc}(d\xi)$	point sp	$\Sigma_p = \mathrm{supp}E_p(d\xi)$

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- For L^q

$$-(m_+ + m_-)^{-1} + \left(m_+^{-1} + m_-^{-1}\right)^{-1} \Longleftrightarrow \sigma$$

$\Longrightarrow$

spec.	$\Sigma(q) = \mathrm{supp}\sigma$	a.c. sp	$\Sigma_{ac}(q) = \mathrm{supp}\sigma_{ac}$
s.c. sp	$\Sigma_{sc}(q) = \mathrm{supp}\sigma_{sc}$	point sp	$\Sigma_p(q) = \mathrm{supp}\sigma_p$

Reflectionless property

Definition (reflectionless set)

$$\Sigma_{re}(q) = \left\{ \xi \in \mathbb{R};\ \ m_+(\xi + i0, q) = -\overline{m_-(\xi + i0, q)} \right\}$$

- $\Sigma_{re}(q) \subset \Sigma_{ac}(q)$ and

  $f_+(x, \xi + i0, q) = \overline{f_-(x, \xi + i0, q)}$ for $x \in \mathbb{R}$ and $\xi \in \Sigma_{re}(q)$.

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  $f_+(x, \xi + i0, q) = \overline{f_-(x, \xi + i0, q)}$ for $x \in \mathbb{R}$ and $\xi \in \Sigma_{re}(q)$.

- Set $g(\lambda) = m_+(\lambda, q) + m_-(\lambda, q)$.
  $\Rightarrow \mathrm{Re}\, g(\xi + i0) = 0$ a.e. $\xi \in (\Sigma_{re}(q))^\circ$ (the interior)
  $\Rightarrow g$ has analytic extension through $(\Sigma_{re}(q))^\circ$.

2 typical potentials with reflectionless property

- **(Algebro-geometric potentials)**
  $\Sigma(q) = \Sigma_{re}(q) = \bigcup_{j=1}^{l} \left[\lambda_j^-, \lambda_j^+\right]$

$$\text{and} \quad q(x) = c - 2\partial_x^2 \log \Theta(x\boldsymbol{\alpha} + \boldsymbol{\beta}), \quad \text{where}$$
$$\Theta(\boldsymbol{z}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^l} \exp\left\{\frac{1}{2}(B\boldsymbol{m}, \boldsymbol{m}) + i(\boldsymbol{z}, \boldsymbol{m})\right\}, \quad \boldsymbol{z} \in \mathbb{C}^l$$

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- **(Soliton type)** $\Sigma(q) = \bigcup_{1\leq j\leq l} \{\lambda_j\} \cup [0, \infty) \ (\lambda_j < 0)$ and
  $\Sigma_{re}(q) = [0, \infty)$
  $q(x) = -2\partial_x^2 \log(I + A(x)), \ A(x) = \left(\frac{\sqrt{m_i m_j}}{\eta_i + \eta_j} e^{-(\eta_i+\eta_j)x}\right)$

Examples

- If q is periodic, then $\Sigma(q) = \Sigma_{re}(q) = \bigcup_j \left[\lambda_j^-, \lambda_j^+\right]$.

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- If q is almost periodic, then $\Sigma_{re}(q) = \Sigma_{ac}(q)$.
  However the existence or non-existence of $\Sigma_{ac}(q)$ is an issue of this workshop.
- Generally for $q^\omega(x) = f(T_x\omega)$ ($\{T_x\}_{x\in\mathbb{R}}$ is an ergodic dynamical system on $(\Omega, \mathcal{F}, P)$)

$$\Sigma_{re}(q^\omega) = \Sigma_{ac}(q^\omega) \quad \text{for a.e.} \quad \omega \in \Omega.$$

Preservation of spectral components

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- If $\exists U$ a unitary op. on H s.t. $U^{-1}AU = B$, then A and B have the same spectral components.
- $\Sigma_{re}(q)$ is not preserved under conjugation by unitary operators.
- However, for the shift $\{S_t\}_{t\in\mathbb{R}}$ defined by $(S_t q)(\cdot) = q(\cdot + t)$, $\exists$holomorphic $U_q(t,z) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ s.t.

$$m_+(z, S_t q) = U_q(t,z) \cdot m_+(z,q), \quad \left(U_q(t,z) \cdot w \equiv \frac{\alpha w + \beta}{\gamma w + \delta} \right),$$

$$\begin{cases} U_q(t,z) \in SL(2,\mathbb{C}) & \text{for } z \in \mathbb{C} \\ U_q(t,\xi) \in SL(2,\mathbb{R}) & \text{for } \xi \in \mathbb{R}. \end{cases}$$

hold, and we have

$$\Sigma_{re}(q) = \Sigma_{re}(S_t q).$$

Remling theorem

Let $\Sigma^+(q), \Sigma^+_{ac}(q)$ be the spectra for L^q on $[0,\infty)$ with Dirichlet bdry cond. at 0.

Theorem (C. Remling 2011)

Let $\widetilde{q}$ be any limit point for $\{S_t q\}_{t\to\infty}$. Then $\Sigma^+_{ac}(q) \subset \Sigma_{re}(\widetilde{q})$.

Corollary

If $\Sigma^+(q) = \Sigma^+_{ac}(q) = \bigcup_j [\lambda_{-,j}, \lambda_{+,j}]$

$$\widetilde{q}(x) = c - 2\partial_x^2 \log \Theta(x\boldsymbol{\alpha} + \boldsymbol{\beta}).$$

If $\Sigma^+(q) = \bigcup_{1\le j\le l} \{\lambda_j\} \cup [0,\infty)\ (\lambda_j < 0)$ *and* $\Sigma^+_{ac}(q) = [0,\infty)$

$$\widetilde{q}(x) = -2\partial_x^2 \log(I + A(x)),\quad A(x) = \left(\frac{\sqrt{m_i m_j}}{\eta_i+\eta_j} e^{-(\eta_i+\eta_j)x}\right)$$

Essential points in the proof

1. **(Transfer matrix)** $\exists$holomorphic $U_q(t,z) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ s.t.

$$m_+(z, S_t q) = U_q(t,z) \cdot m_+(z,q),$$

satisfying

$$\begin{cases} U_q(t,z) \in SL(2,\mathbb{C}) : \mathbb{C}_+ \to \mathbb{C}_+ \text{ for } z \in \mathbb{C}_+ \\ U_q(t,\xi) \in SL(2,\mathbb{R}) \quad \text{for } \xi \in \mathbb{R}. \end{cases}$$

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2. **(Reflection)** $U_q(t,z) \cdot w = -U_{Rq}(t,z) \cdot (-w)$, where $(Rq)(x) \equiv q(-x)$.
3. **(Uniform contraction)** For $z \in \mathbb{C}_+$, $w_1, w_2 \in \mathbb{C}_+$

$$\lim_{t\to\infty} \gamma\left(U_q(t,z) \cdot w_1, U_q(t,z) \cdot w_2\right) = 0 \text{ uniformly.}$$

$$\text{where} \quad \gamma(w_1,w_2) = \frac{|w_1 - w_2|}{\sqrt{\operatorname{Im} w_1}\sqrt{\operatorname{Im} w_2}}.$$

The KdV hierarchy

1 s.t. eq.	shift	$\partial_t q = \partial_x q$
2 and eq.	KdV eq.	$\partial_t q = \partial_x^3 q - 6q\partial_x q$
	$\vdots$	
n th eq.		$\partial_t q = \partial_x^{2n+1} q + p\left(q, \partial_x q, \cdots, \partial_x^{2n} q\right)$
	$\vdots$	

Isospectrum property is common: $L^{q(t,\cdot)}$ is unitarily equivalent to $L^{q(0,\cdot)}$

Problem

To extend Remling's theorem to solutions to the KdV hierarchy. That is,
to obtain asymptotic behavior as $t \to \infty$ to solutions of the KdV hierarchy with initial profile q by using of the spectral data of L^q.

Properties of KdV equation

Lemma

(A Rybkin (2008)): Let $q(t,x)$ *be a solution to the KdV equation*

$$\partial_t q = \partial_x^3 q - 6q\partial_x q.$$

Assume $q(t,x)$ *is bounded w.r.t.* x *for any fixed* t*. Then,* $\exists$*holomorphic* $U_q(t,z) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ *s.t.*

$$m_+(z, q(t,\cdot)) = U_q(t,z) \cdot m_+(z, q(0,\cdot)),$$

and

$$\begin{cases} U_q(t,z) \in SL(2,\mathbb{C}) & \text{for } z \in \mathbb{C} \\ U_q(t,\xi) \in SL(2,\mathbb{R}) & \text{for } \xi \in \mathbb{R}. \end{cases}$$

- We have $\Sigma_{re}(q) = \Sigma_{re}(q(t,\cdot))$.

Analogue of Remling's theorem for KdV

Theorem

Let $q(t,x)$ *be a solution to the Kdv equation*

$$\partial_t q = \partial_x^3 q - 6q\partial_x q.$$

Assume $q(t,x)$ *is bounded w.r.t.* x *for any fixed* t*. Let* $\widetilde{q}$ *be any limit point of* $\{q(t,\cdot)\}_{t\to\infty}$*. Set*

$$\widetilde{m} = \inf_{x\in\mathbb{R}} \widetilde{q}(x)$$

Then

$$\Sigma_{ac}^{-}(q(0,\cdot)) \cap [\widetilde{m}/4, \infty) \subset \Sigma_{re}(\widetilde{q}).$$

Construction of KdV flow

For $\lambda_0 < \lambda_1$ set

$$\Omega_{\lambda_0,\lambda_1} = \{q \in \mathcal{Q};\ \Sigma(q) \subset [\lambda_0, \infty),\ \ \Sigma_{re}(q) \supset [\lambda_1, \infty)\}.$$

- Marchenko-Lundina: $\Omega_{\lambda_0,\lambda_1} \ni q$ is holmorphic on

$$\left\{z \in \mathbb{C};\ \ |\mathrm{Im}\, z| < \sqrt{\gamma}^{-1}\right\} \quad (\gamma = \lambda_1 - \lambda_0)$$

with uniform bound

$$|q(z) - \lambda_1| \le 2\gamma\left(1 - \sqrt{\gamma}\,|\mathrm{Im}\, z|\right)^{-2} \Longrightarrow \Omega_{\lambda_0,\lambda_1} \text{ compact.}$$

Elements of

$\Omega_{\lambda_0,\lambda_1}$

1. algebro-geometric functions

$$q(x) = c - 2\partial_x^2 \log \Theta(x\boldsymbol{\alpha} + \boldsymbol{\beta})$$

2. n-solitons: Let $\lambda_0 < 0 = \lambda_1$. For $0 < \eta_i < \sqrt{-\lambda_0}$, $m_i > 0$

$$\begin{aligned} q(x) &= -2\partial_x^2 \log \det(I + A(x)), \text{ where} \\ A(x) &= \left(\frac{\sqrt{m_i m_j}}{\eta_i + \eta_j} e^{-x\left(\eta_i + \eta_j\right)} \right) \end{aligned}$$

Theorems

Theorem

There exists a flow $\{K_t\}_{t\in\mathbb{R}}$ *on* $\Omega_{\lambda_0,\lambda_1}$ *such that* $q(t,x) = K_t\left(q\right)\left(x\right)$ *for* $q \in \Omega_{\lambda_0,\lambda_1}$ *solves the Cauchy problem for the KdV equation*

$$\partial_t q = \partial_x^3 q - 6q\partial_x q, \quad q\left(0,x\right) = q\left(x\right).$$

Theorem

Let $\widetilde{q}$ be any limit point of $\{K_t(q)(\cdot)\}_{t\to\infty}$. Set

$$\widetilde{m} = \inf_{x\in\mathbb{R}} \widetilde{q}(x). \ \left(\text{Generally } 2\lambda_0 - \lambda_1 \leq \widetilde{m} \leq \lambda_1\right)$$

Then

$$\Sigma_{ac}^{-}(q(0,\cdot)) \cap [\widetilde{m}/4, \infty) \subset \Sigma_{re}(\widetilde{q}).$$

Since $K_t(q) \in \Omega_{\lambda_0,\lambda_1}$ we have $\widetilde{q} \in \Omega_{\lambda_0,\lambda_1}$ if $q \in \Omega_{\lambda_0,\lambda_1}$, hence

$$[\lambda_1, \infty) \subset \Sigma_{re}(\widetilde{q})$$

is always valid. Therefore, the theorem is meaningful only when

$$\frac{\widetilde{m}}{4} \leq \lambda_1,$$

which holds if $\lambda_1 \geq 0$.

Remarks

- Construction of the KdV flow is non- trivial if initial profiles are neither decaying nor periodic, since Functional analysis will not be available. Therefore, we employed an algebraic method discovered by Sato.

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- Construction of the KdV flow is non- trivial if initial profiles are neither decaying nor periodic, since Functional analysis will not be available. Therefore, we employed an algebraic method discovered by Sato.
- Almost any function q can be approximated by functions of

$$\bigcup_{\lambda_0<\lambda_1} \Omega_{\lambda_0,\lambda_1}$$

which may help to define a KdV flow in a general function space.

Open question 1: Construction of KdV flow by iterating Darboux transformation

The construction of the KdV flow $\{K_t\}_{t\in\mathbb{R}}$ on $\Omega_{-1,0}$ is done as follows. Let Γ be

$$\Gamma = \left\{ g(z) = e^{h(z)};\ h \in H^2\left(|z| < 1\right) \text{ and } h'\left(e^{i\theta}\right) \in L^2,\ h(0) = 0 \right\}.$$

Then Γ is an Abelian group by usual multiplication. It is proved that there exists a smooth flow $\left\{ \widetilde{K}\left(g\right) \right\}_{g\in\Gamma}$ on $\Omega_{\lambda_0,\lambda_1}$ such that

$$\begin{aligned}
\text{if } g_t\left(z\right) &= e^{-tz} \Longrightarrow \left(\widetilde{K}\left(g_t\right) q\right)\left(x\right) = \left(S_t q\right)\left(x\right) = q\left(x+t\right) \text{ shift} \\
\text{if } g_t\left(z\right) &= e^{-4tz^3} \Longrightarrow \left(\widetilde{K}\left(g_t\right) q\right)\left(x\right) = \left(K_t q\right)\left(x\right) \text{ KdV flow} \\
&\ \vdots
\end{aligned}$$

$$\lim_{n\to\infty}\left\{\left(1-\frac{z^3}{n\zeta}\right)\left(1-\frac{z^3}{n\overline{\zeta}}\right)\right\}^n = \exp\left(-\left(\frac{1}{\zeta}+\frac{1}{\overline{\zeta}}\right)z^3\right) = e^{-tz^3}$$

$$t = \frac{1}{\zeta}+\frac{1}{\overline{\zeta}}$$

Let $\{\omega_j\}_{j=1,2,3}$ be distinct solutions to $z^3=\zeta$. Since

$$\left(1-\frac{z^3}{n\zeta}\right)\left(1-\frac{z^3}{n\overline{\zeta}}\right) = \prod_j \left(1-\frac{z}{n^{1/3}\omega_j}\right)\left(1-\frac{z}{n^{1/3}\overline{\omega_j}}\right),$$

$$\widetilde{K}\left(e^{-tz^3}\right) = \lim_{n\to\infty}\left\{\widetilde{K}\left(1-\frac{z}{n^{1/3}\omega_j}\right)\widetilde{K}\left(1-\frac{z}{n^{1/3}\overline{\omega_j}}\right)\right\}^n$$

For $\alpha \in \mathbb{C} \backslash \mathbb{R}$ s.t. $|\alpha| > 1$ and $q \in \Omega_{-1,0}$

$$\widetilde{q} \equiv \widetilde{K}\left(1 - \frac{z}{\alpha}\right) \widetilde{K}\left(1 - \frac{z}{\overline{\alpha}}\right) q \in \Omega_{-1,0}$$

$\widetilde{K}\left(1 - \frac{z}{\alpha}\right) q$ is the Darboux transformation of q, and one can show that

$$m_{+}(z, \widetilde{q}) = \Delta_{\overline{\alpha}} \Delta_{\alpha} m_{+}(\cdot, q).$$

where

$$(\Delta_{\alpha} f)(z) \equiv -\frac{z - \alpha}{f(z) - f(\alpha)} - f(\alpha).$$

Therefore, it might be possible to construct a KdV flow on a general function space by showing the convergence

$$\lim_{n \to \infty} \left(\Delta_{\overline{n^{1/3}\omega_j}} \Delta_{n^{1/3}\omega_j}\right)^{n} m_{+}(\cdot, q).$$

Open question 2: Study of KdV flow from the point of view of ergodic theory

We have constructed a KdV flow $\{K_t\}_{t\in\mathbb{R}}$ on a compact space $\Omega_{\lambda_0,\lambda_1}$. It is reasonable to predict the following property holds.

Problem

Let μ be any probability measure on $\Omega_{\lambda_0,\lambda_1}$ which is shift invariant. Then the integrated density of state is preserved under K_t ?

Thank you !