Partition Relation Perspectives

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The Role of the Higher Infinite
in Mathematics and Other Disciplines
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Mirna’s doctoral work

In 1993, Mirna Dzamonja received her PhD under the direction of Ken Kunen at the University of Wisconsin, Madison.
Mirna’s early career

- From 1993-1995, Mirna was a postdoctoral fellow at the Hebrew University of Jerusalem.
- From 1995-1998, she was a visiting assistant professor at the University of Wisconsin, Madison.
- I first met her at an Annual Association of Symbolic Logic Meeting held in Madison in March 1996.
Mirna has been at the University of East Anglia since 1998.
Mirna’s students
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- Katherine Thompson, University of East Anglia (UEA) 2003.
- Grace Piper, UEA, 2003
- Firoz Shaikh, UEA, 2007
- Alexander Primavesi, UEA, 2011
- Omar Selim, 2012
- Gregor Dolinar, University of Ljubljana, 2013
- Sharifa Al Mahrouqi, UEA, 2013
Her co-authors

Uri Abraham  
Arthur Apter  
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Piotr Borodulin-Nadzieja  
James Cummings  
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Ken Kunen  
Jean Larson  
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Source: MathSciNet

Justin Tatch Moore  
Charles Morgan  
Eva Murtinová  
Grzegorz Plebanek  
Anatoli Plichko  
Saharon Shelah  
Katherine Thompson  
Jouko Väänänen
Balanced Baumgartner-Hajnal-Todorcevic (BHT) Theorem, 1993

For every regular uncountable \( \kappa, \xi < \kappa, \) and \( k < \omega, \)

\[(2^\kappa)^+ \rightarrow (\kappa + \xi)^2_k.\]

This theorem has a very nice proof using ideals that come from chains of elementary submodels.
Balanced BHT Theorem

Ari Brodsky’s Balanced BHT Theorem for Trees, 2014

For every infinite regular cardinal \( \kappa, \xi \) with \( 2^{\|\xi\|} < \kappa \), and \( k < \omega \),

\[
\text{non-} \left( 2^{<\kappa} \right) \text{-special tree } \rightarrow (\kappa + \xi)^2_k.
\]

Brodsky developed a theory of stationary subsets of trees in order to prove this theorem and also used ideals that come from chains of elementary submodels.
Theorem (Paul Erdős and Richard Rado, 1956)

The Continuum Hypothesis implies $\left( \mathbb{N}_1 \right) \nrightarrow \left( \mathbb{N}_1 \right)_{2}^{1,1}$

Here $\left( \begin{array}{c} \theta \\ \omega \end{array} \right) \rightarrow \left( \begin{array}{c} \theta \\ \omega \end{array} \right)_{2}^{1,1}$ means for every function $c : \theta \times \omega \rightarrow 2$ there are $A \subseteq \theta$ and $B \subseteq \omega$ such that $|A| = \theta$, $B$ is infinite and $c \upharpoonright (A \times B)$ is constant.
Theorem (Shimon Garti and Saharon Shelah, 2014)

It is consistent with ZFC, that all cardinals $\theta$ with $\aleph_1 \leq \theta \leq 2^\omega$,

\[
\left( \begin{array}{c} \theta \\ \omega \end{array} \right) \rightarrow \left( \begin{array}{c} \theta \\ \omega \end{array} \right)^{1,1}_2
\]

Earlier they had shown the partition relation was true for $\theta < s$ (splitting number) and here show it is true for $\tau < \theta$ (reaping number).

Their model has $\mathfrak{N}_1 = \tau =< s = \mathfrak{N}_2 = c$.

Question: is it consistent that $\tau < s$ and $c > \mathfrak{N}_2$. 


Claribet Piña has investigated for given countable ordinals $\alpha$ and $\beta$, the minimum number of colors $m$ needed, so that if pairs from $\alpha$ are colored with finitely many colors (at least 2), then there is a topological copy of $\beta$ whose pairs realize at most $m$ colors.

This may be expressed compactly as $(\forall \ell > 1)(\alpha \rightarrow_{top} (\beta)^2_{\ell,m})$.

**Theorem (Piña, 2013, 2015)**

- for all $n > 1$, $(\forall \ell > 1)(\omega^{\omega \cdot n} \rightarrow_{top} (\omega^2 + 1)^2_{\ell,6})$, but $(\omega^{\omega \cdot n} \not\rightarrow_{top} (\omega^2 + 1)^2_{9,5})$.

- for all $k > 1$, $(\forall \ell > 1)(\omega^{\omega^k} \rightarrow_{top} (\omega^1)^2_{\ell,6})$, but $\omega^{\omega^k} \not\rightarrow_{top} (\omega^2 + 1)^2_{3(k+1),5}$.
Topological Pigeonhole Principle

Building on the ordinal pigeonhold principle of Eric Milner and Richard Rado which calculates $P^{\text{ord}}(\alpha_i)$, and work of Bill Weiss, of Prikry and Solovay, and of Shelah, Jacob Hilton constructed a pigeonhole principle for ordinal topological spaces. Here is the theorem he describes as his main breakthrough.

**Theorem (Jacob Hilton, JSL, to appear)**

Suppose $0 < \alpha_1, \alpha_2, \ldots, \alpha_k < \omega_1$.

1. $P^{\text{top}}(\omega^{\alpha_1} + 1, \omega^{\alpha_2} + 1, \ldots, \omega^{\alpha_k} + 1) = \omega^{\alpha_1 \# \alpha_2 \# \ldots \# \alpha_k + 1}$.
2. $P^{\text{top}}(\omega^{\alpha_1}, \omega^{\alpha_2}, \ldots, \omega^{\alpha_k}) = \omega^{P^{\text{ord}}(\alpha, \alpha_2, \ldots, \alpha_k)}$.

If $\alpha = \omega^{a_0} \cdot m_0 + \cdots + \omega^{a_p} \cdot m_p$ and $\beta = \omega^{a_0} \cdot n_0 + \cdots + \omega^{a_p} \cdot n_p$ where $a_0 > a_1 > \cdots > a_p$ and $m_0, m_1, \ldots, m_p, n_0, n_1, \ldots, n_p < \omega$, then the Hessenberg sum (natural sum) is defined by $\alpha \# \beta = \omega^{a_0} \cdot (m_0 + n_0) + \cdots + \omega^{a_p} \cdot (m_p + n_p)$.
Topological Pigeonhold Principle

There is a reason why powers of $\omega$ and their successors appear so often.
We say $X$ and $Y$ are biembeddable and write $X \equiv Y$ if $X$ is homeomorphic to a subspace of $Y$ and $Y$ is homeomorphic to a subspace of $X$.

**Lemma (Hilton)**

If $Y \equiv V$ and $X_i \equiv U_i$ for all $i < \kappa$, then $Y \rightarrow_{top} (X_i)_{i<\kappa}$ if and only if $V \rightarrow_{top} (U_i)_{i<\kappa}$. 
In their search for Ramsey numbers of ordinal topological spaces, Caicedo and Hilton looked at two kinds of goals: $R^{top}$ when the goals were homeomorphic to given ordinal spaces and $R^{cl}$ when the goals that were both order-isomomorphic to given ordinal spaces and closed in their own supremum.

As noted above, Hilton proved a pigeonhole principle $P^{top}(\alpha_i)_{i<\kappa}$ for goals that were topological spaces and Caicedo and Hilton found many cases of Hilton’s proof carried over to the closed pigeonhole principle, $P^{cl}(\alpha_i)_{i<\kappa}$, and they computed the values in the remaining cases quite handily.
Topological Ramsey numbers

They write $R^{top}(\alpha_i)_{i<\kappa}$ for the topological Ramsey number, namely the least ordinal $\beta$ such that

$$\beta \rightarrow_{top} (\alpha_i)^2_{i<\kappa}$$

Similarly, they write $R^{cl}(\alpha_i)_{i<\kappa}$ for the closed Ramsey number, namely the least ordinal $\beta$ such that

$$\beta \rightarrow_{cl} (\alpha_i)^2_{i<\kappa}$$
Topological Ramsey numbers

Paul Erdős and Eric Milner proved the following theorem for ordinal partition relations:

**Theorem (Erdős, Milner)**

If $\omega^\alpha \to (\omega^{1+\beta}, k)^2$, then $\omega^{\alpha+\beta} \to (\omega^{1+\beta}, 2k)^2$.

Andres Caicedo and Jacob Hilton found a counterpart which they call the Weak topological Erdős-Milner Theorem.

**Theorem (Caicedo, Hilton)**

Let $\alpha$ and $\beta$ be countable non-zero ordinals, and let $k > 1$ be a positive integer. If

$$\omega^{\omega^\alpha} \to_{top} (\omega^\beta, k)^2,$$

then

$$\omega^{\omega^{\alpha+\beta}} \to_{top} (\omega^\beta, k + 1)^2,$$
Topological Ramsey numbers

In a corollary to their main theorem, Caicedo and Hilton prove these Ramsey numbers coincide in certain cases and give upper bounds for them.

Theorem (Caicedo, Hilton)
Suppose $0 < \alpha < \omega_1$ and $k$, $m$ and $n$ are positive integers.

1. $R^{top}(\alpha, k)$ is countable.
2. $R^{top}(\omega^\alpha, k + 1) = R^{cl}(\omega^\alpha, k + 1) \leq \omega^{\omega^\alpha \cdot k}$.
3. $R^{top}(\omega^\alpha + 1, k + 1) = R^{cl}(\omega^\alpha + 1, k + 1)$ and the common value is bounded by
   $\omega^{\omega^\alpha \cdot k} + 1$ if $\alpha$ is infinite, and bounded by
   $\omega^{\omega^{(n+1) \cdot k - 1}} + 1$ if $\alpha = n$ is finite.
4. $R^{top}(\omega^n \cdot m, k + 2) = R^{cl}(\omega^n \cdot m, k + 2) \leq \omega^{\omega^k \cdot n \cdot R(m, k + 2) + 1}$. 
Question: is there a topological partition ordinal $\alpha > \omega$?

That is, is there $\alpha > \omega$ for which $\alpha \rightarrow_{top} (\alpha, 3)^2$?

Question: is it possible to reduce the computation of $R^{top}(\alpha, k)$ to finite combinatorial problems, even for $\alpha < \omega^2$?
Halpern-Läuchli Theorem

Copies of illustrations of a $k$-dense subset and a $k$-$x$-dense subset from *Ramsey Theory* by Todorcevic:

![Figure 3.2 A k-dense set and a k-x-dense set.](image)

A $k$-dense matrix ($k$-$\vec{x}$-dense matrix) for $\prod_{i<d} T_i$ is $\prod_{i<d} X_i$ where each $X_i$ is a $k$-dense matrix ($k$-$x_i$-dense matrix).
Halpern-Läuchli Theorem

(Asymmetric Version of the Halpern-Läuchli Theorem)

For every coloring of the finite product of finitely branching trees, \( \prod_{i<d} T_i = K_0 \cup K_1 \), either

- \( K_0 \) contains a \( k \)-dense matrix for every integer \( k \), or
- there is some \( \vec{x} \in \prod_{i<d} T_i \) so that \( K_1 \) contains a \( k-\vec{x} \)-dense matrix for every integer \( k \).
Halpern-Läuchli Theorem

Laver’s Conjecture:

For every (level set) coloring of the infinite product of downwards closed perfect $\prod_{i<\omega} T_i = K_0 \cup K_1$, there is some $j < 2$ and some $\vec{x} \in \prod_{i<\omega} T_i$ such that for each $k < \omega$, $K_j$ contains a $k$-$\vec{x}$-dense (level set) matrix.
Theorem (Denis Devlin, 1979)

\[ \mathbb{Q} \to (\mathbb{Q})_{<\omega,t_n}^n \quad \text{and} \quad \mathbb{Q} \to (\mathbb{Q})_{<\omega,t_n-1}^n, \]

where \( t_n \) is the \( n \)th tangent number.

For regular cardinals \( \kappa \), there is a natural order \( \leq_Q \) extending lexicographic order on \( \kappa > 2 \) such that \( \mathbb{Q}_\kappa = \langle \kappa > 2, \leq_Q \rangle \) is a \( \kappa \)-dense linear order.
Joint work with Džamonja and Mitchell

Theorem (Džamonja, Larson, Mitchell, 2009)

For every positive integer $m$ there is $t_m^+ < \omega$ such that for any cardinal $\kappa$ which is measurable after generically adding $\lambda$ many Cohen subsets of $\kappa$ where $\lambda \rightarrow (\kappa)_{2\kappa}^{2m}$, the $\kappa$-dense linear order $Q_\kappa$ satisfies

$$Q_\kappa \rightarrow (Q_\kappa)^m_{<\omega,t_m^+} \text{ and } Q_\kappa \nrightarrow (Q_\kappa)^m_{<\omega,t_m^+-1},$$

where $t_m^+$ is the cardinality of a finite set of trees.

Furthermore, for $m \geq 3$, $t_m^+ > t_m$: $t_3 = 16 < 20 = t_3^+$; $t_4 = 272 < 776 = t_4^+$; $t_5 = 7936 < 151,184 = t_5^+$. 
Joint work with Džamonja and Mitchell

A key tool:

Theorem (Saharon Shelah, 1988)

Suppose that $m < \omega$ and $\kappa$ is a cardinal which is measurable in the generic extension obtained by adding $\lambda$ Cohen subsets of $\kappa$ where $\lambda \rightarrow (\kappa)^{2m}_{2\kappa}$. Then for any coloring $d$ of the $m$-element antichains of $\kappa^>$ onto $\sigma < \kappa$ colors, and any well-ordering $\prec$ of the levels of $\kappa^>$, there is a strong embedding $e : \kappa^> \rightarrow \kappa^>$ and a dense set of elements $w$ such that

- $e(s) \prec e(t)$ for all $s \prec t$ from $\text{Cone}(w)$, and
- $d(e[a]) = d(e[b])$ for all $\prec$-similar $m$-element antichains $a$ and $b$ of $\text{Cone}(w)$. 

Based on work by Hajnal and Komjáth, the positive partition relation for $\mathcal{Q}_\kappa$ does not follow from any large cardinal hypothesis on $\kappa$.

**Theorem (András Hajnal and Péter Komjáth)**

*There is a forcing of size $\aleph_1$ which adds an order type $\theta$ of size $\aleph_1$ with the property that $\psi \nrightarrow [\theta]^{2}_{\omega_1}$ for every order type $\psi$, regardless of its size.*
Work with Džamonja and Mitchell

Our proof uses the theorem of Shelah quoted above to show that all the types we say occur do occur in every large set.

Does $Q_\kappa \rightarrow (Q)^m_{<\kappa, t^+_{m-1}}$ hold for sufficiently large $m < \omega$ for some regular cardinal which is not measureable after generically adding $\lambda$ Cohen subsets of $\kappa$ where $\lambda \rightarrow (\kappa)_{2^\kappa}^{2m}$?
Remembering Mary Ellen Rudin


Bibliography


Bibliography

Thank you!