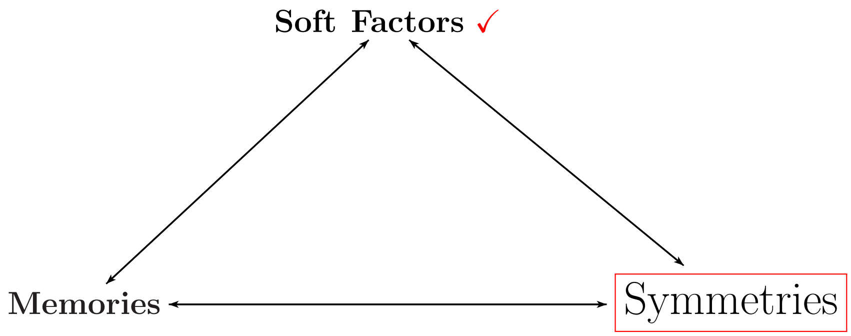
1 Day 1: Null Infinity



Since amplitudes workshop, the two talks will focus on the lower two vertices
day 1 Asymptotic Symmetries @ Null Infinity
day 2 Memory Effects

- We see multiple iterations of the above triangle in gauge theories with vertices filled in at different times 1950s to now
- Can use intution from simpler examples like $U(1)$ photon in flat space to predict new iterations ex:
  - Subleading Soft Graviton Theorem (CS 1404.4091)
  - Superrotations (KLPS 1406.3312)
  - Spin Memory (PSZ 1502.06120)

We will be studying 4 dimensional massless scattering so a good first step is to look at the Penrose diagram for minkowski space and set up coordinates convenient for massless trajectories.

If we define advanced and retarded time cordinates

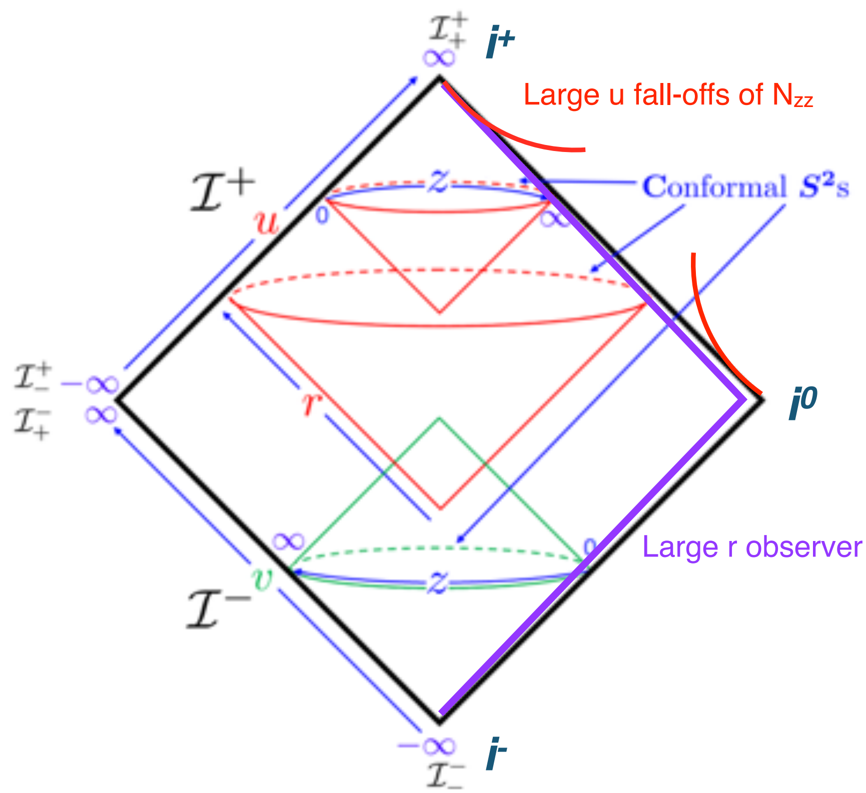
$$u = t - r, \quad v = t + r \tag{1.1}$$

Then we can define rescaled coordinates that allow us to plot minkowski space on a finite diagram that captures causal structure but distorts length scales by a conformal factor:

$$u = L \tan U, \quad v = L \tan V \tag{1.2}$$

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 = \frac{L^2}{\cos^2 U \cos^2 V}(-dUdV + \frac{\sin^2(U-V)}{4}d\Omega^2) \tag{1.3}$$

where $U, V \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$.



- massive trajectories enter at i^- and leave at i^+
- massless trajectories enter at $\mathcal{I}^-$ and leave at $\mathcal{I}^+$ which are **null boundary** components

Further conventions include stereographic coordinates on the S^2:

$$z = e^{i\phi}\tan\frac{\theta}{2}, \quad d\Omega^2 = 2\gamma_{z\bar{z}}dzd\bar{z}, \quad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2} \tag{1.4}$$

so that a massless four momentum is described by an energy and a direction on the S^2

$$p_k^\mu = \frac{E_k}{1+z_k\bar{z}_k}(1+z_k\bar{z}_k, \bar{z}_k + z_k, i(\bar{z}_k - z_k), 1 - z_k\bar{z}_k) \tag{1.5}$$

D_z will denote a covariant derivative with respect to the round metric.

- At this point we see that past and future null infinity provide a natural arena for describing the initial and final data of massless degrees of freedom
- In the 1980's Ashtekar and collaborators studied how to construct phase space in terms of connections on null infinity as compared to data on a spacelike cauchy slice and how to interpolate between the two.
- This takes advantage of the notion that we can still think of attaching a null infinity like boundary to spacetimes which are only asymptotically flat as opposed to Minkowski, so that we can study gravity a la BMS (1960s)

We will use coordinates for the physical spacetime adapted to the behavior near past and future null infinity rather than manipulating data defined strictly on null infinity. The metric near future null infinity takes the following form

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + 2\tfrac{m_B}{r}du^2 \\ &+\big(rC_{zz}dz^2 + D^zC_{zz}dudz + \tfrac{1}{r}(\tfrac{4}{3}N_z - \tfrac{1}{4}\partial_z(C_{zz}C^{zz}))dudz + c.c.\big) + ... \end{aligned} \tag{1.6}$$

while near past null infinity

$$\begin{aligned} ds^2 &= -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + 2\tfrac{m_B}{r}dv^2 \\ &+\big(-rD_{zz}dz^2 + D^zD_{zz}dvdz + \tfrac{1}{r}(\tfrac{4}{3}N_z + \tfrac{1}{4}\partial_z(D_{zz}D^{zz}))dvdz + c.c.\big) + ... \end{aligned} \tag{1.7}$$

This parameterization is based on the work of BMS in the 1960s. Their method was to

1. Set up coordinates that allowed a series expansion in $\frac{1}{r}$ near null infinity
   - fixed (u, θ, ϕ) for an outgoing null geodesic
   - luminosity distance for radial coordinate
2. Solve Einstein's equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ (in vacuum originally so $T_{\mu\nu} = 0$) to look at structure of solutions and isolate free data
   - 6 equations solved sliced by slice in u + 1 trivially satisfied + 3 constraint equations which need to hold at one value of r.

- integration constants introduced along the way as potential free data, some related by constraint equations
- complex news tensor $N_{zz}(u,z,\bar{z}) = \partial_u C_{zz}$ and $M_{zz} = \partial_v D_{zz}$ become the in and out descriptions of the two polarization states of the graviton

3. identify residual coordinate transformations that preserve the asymptotic form of the metric

Explicitly for changes at order:

$$\mathcal{L}_{\xi^+} g_{ur} = \mathcal{O}(r^{-2}), \quad \mathcal{L}_{\xi^+} g_{uz} = \mathcal{O}(1), \quad \mathcal{L}_{\xi^+} g_{zz} = \mathcal{O}(r), \quad \mathcal{L}_{\xi^+} g_{uu} = \mathcal{O}(r^{-1}). \tag{1.8}$$

these vector fields are given by:

$$\begin{aligned} \xi^+ =& (1+\frac{u}{2r})Y^{+z}\partial_z - \frac{u}{2r}D^{\bar{z}}D_z Y^{+z}\partial_{\bar{z}} - \frac{1}{2}(u+r)D_z Y^{+z}\partial_r + \frac{u}{2}D_z Y^{+z}\partial_u + c.c. \\ &+ f^+\partial_u - \frac{1}{r}(D^z f^+\partial_z + D^{\bar{z}} f^+\partial_{\bar{z}}) + D^z D_z f^+\partial_r, \end{aligned} \tag{1.9}$$

allowing infinitesimal superrotations as suggested by Barnich and Banks.

Here Y is a holomorphic CKV of the S^2 so up to quadratic order give the global transformations corresponding the the Lorentz subgroup of Poincare. One can similarly identify the ordinary translations within the supertranslations given by the arbitrary function $f(z,\bar{z})$ (ex change coordinates and write ∂_μ for Minkowski (t,x,y,z) in this form $(u,r,z,\bar{z})$)

Similar expressions hold near past null infinity, but the generators should be related so that the same transformation is done on both incoming and outgoing. This is at the heart of the Ward Identity interpretation of Strominger and collaborators:

$$\langle out|Q^+(\xi)\mathcal{S} - \mathcal{S}Q^-(\xi)|in\rangle = 0, \tag{1.10}$$

To get to this point, one uses the constraint equations (for flat space the null generator of $\mathcal{I}^+$ is $n = \partial_u - \frac{1}{2}\partial_r$) $G_{uu} = 8\pi G T^M_{uu}$

$$\begin{aligned} \partial_u m_B &= \frac{1}{4}\left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right] - T_{uu}, \\ T_{uu} &\equiv \frac{1}{4} N_{zz} N^{zz} + 4\pi G \lim_{r\to\infty}[r^2 T^M_{uu}], \end{aligned} \tag{1.11}$$

and $G_{uz} = 8\pi G T^M_{uz}$

$$\begin{aligned} \partial_u N_z &= \frac{1}{4}\partial_z\left[D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}\right] + \partial_z m_B - T_{uz}, \\ T_{uz} &\equiv 8\pi G \lim_{r\to\infty}[r^2 T^M_{uz}] - \frac{1}{4} D_z[C_{zz} N^{zz}] - \frac{1}{2} C_{zz} D_z N^{zz}, \end{aligned} \tag{1.12}$$

These equations have three parts

1. A $\partial_u\cdot$ term – if we integrate this along u then we end up with ex. $m_B|_{\mathcal{I}^+_+} - m_B|_{\mathcal{I}^+_-}$ so if we do the same thing along $\mathcal{I}^-$ we can hope to cancel near i^0 and use boundary conditions near $i^\pm$ to end up with an expression relating the right hand sides at future and past null infinity. We thus see that the following things are required

   - boundary conditions at $\mathcal{I}^\pm_\pm$ a la CK
   - matching conditions between $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$ near i^0 a la S

   Note: This left hand side convolved with an arbitrary angular function integrated along $\mathbb{R} \times S^2$ is essentially giving the S^2-integral-defined charge generating these large diffeomorphisms. For example, for superrotations from Barnich

   $$8\pi G Q[Y] = \int du \int d^2z \sqrt{\gamma}\,\partial_u[-uY^A D_A m_B + Y^A N_A + ...] \tag{1.13}$$

   supressing quadratic in C_{zz} terms. These are the charges one associates with the vector field acting on the manifold a la WZ 9911095 and Barnich

2. A linear term in the radiative metric component. This is where the mode expansion and soft factor comes in

    - a la Ashtekar one has brackets between the radiative data on null infinitity

$$[M_{\bar{z}\bar{z}}(v,z,\bar{z}), M_{ww}(v',w,\bar{w})] = 2i\gamma_{z\bar{z}}\delta^2(z-w)\partial_v\delta(v-v'), \tag{1.14}$$

    - in cases where one needs to extend it to boundary values of the metric D_{zz} vs $\partial_v D_{zz}$ one can compare the bracket induced by the charge to the change in the metric component expected from taking the lie derivative along ξ.

$$\delta_{Y^-} M_{zz}(v,z,\bar{z}) = D_z^3 Y^{-z}. \tag{1.15}$$

$$Q_S^- = \frac{1}{2}\int_{\mathcal{I}^-} dvd^2zD_z^3Y^{-z}vM^z{}_{\bar{z}}. \tag{1.16}$$

    - this is also where certain combinations of positive and negative frequency zero modes are picked out as associated to

   Note: This is the soft part of the charge, due to the linearized radiation.

3. A matter (combined with quadratic metric perturbations) term, which in the charge gives rise to a familiar stress tensor as generator of diffeomorphism along ξ:

$$Q_H^- = \lim_{\Sigma\to\mathcal{I}^-}\int_\Sigma d\Sigma\ \xi^\mu n_\Sigma^\nu T^M_{\mu\nu}. \tag{1.17}$$

   This is the hard part of the charge, due to the scatters.

The verification of the Ward Identity amounts then to showing that for the soft factor contribution matches a matter flux contribution when convolved against appropriate vector fields or after certain differential operators act on them. Tomorrow we see that a nice way of seeing such relations is through Memory Effects.