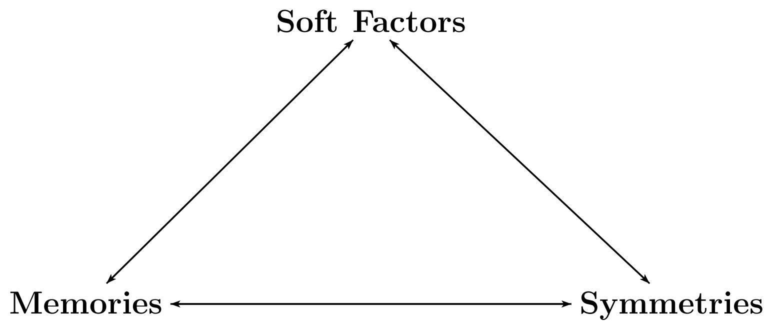
2 Day 2: Memory Effects

i) Weinberg – photon $\mathcal{O}(\frac{1}{\omega})$
ii) Weinberg – graviton $\mathcal{O}(\frac{1}{\omega})$
iii) Cachazo & Strominger – graviton $\mathcal{O}(1)$



i) Liénard-Wiechert / Bieri & Garfinkle
ii) Zeldovich & Polnarev / Christodoulou
iii) Pasterski, Strominger, & Zhiboedov

(global)	(asymptotic)
i) e-charge	large U(1)
ii) p^μ	supertranslations
iii) $J^{\mu\nu}$	superrotations

∘ $$z = e^{i\phi}\tan\frac{\theta}{2}, \quad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}, \quad p_k^\mu = \frac{E_k}{1+z_k\bar{z}_k}\left(1+z_k\bar{z}_k, \bar{z}_k+z_k, i(\bar{z}_k-z_k), 1-z_k\bar{z}_k\right) \tag{2.1}$$

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + 2\tfrac{m_B}{r}du^2 \\ &+\big(rC_{zz}dz^2 + D^zC_{zz}dudz + \tfrac{1}{r}(\tfrac{4}{3}N_z - \tfrac{1}{4}\partial_z(C_{zz}C^{zz}))dudz + c.c.\big) + ... \end{aligned} \tag{2.2}$$

$$8\pi G Q[Y] = \int du \int d^2z\sqrt{\gamma}\partial_u[-uY^AD_Am_B + Y^AN_A + ...] \tag{2.3}$$

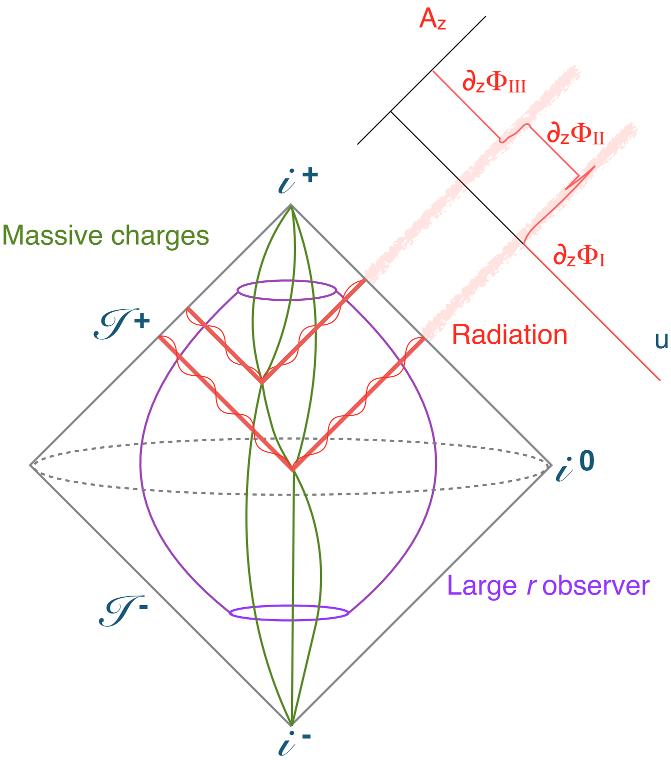
$$\begin{aligned} \partial_u m_B &= \frac{1}{4}\left[D_z^2N^{zz} + D_{\bar{z}}^2N^{\bar{z}\bar{z}}\right] - T_{uu}, \\ T_{uu} &\equiv \frac{1}{4}N_{zz}N^{zz} + 4\pi G\lim_{r\to\infty}[r^2T^M_{uu}], \end{aligned} \tag{2.4}$$

$$\begin{aligned} \partial_u N_z &= \frac{1}{4}\partial_z\left[D_z^2C^{zz} - D_{\bar{z}}^2C^{\bar{z}\bar{z}}\right] + \partial_z m_B - T_{uz}, \\ T_{uz} &\equiv 8\pi G\lim_{r\to\infty}[r^2T^M_{uz}] - \frac{1}{4}D_z[C_{zz}N^{zz}] - \frac{1}{2}C_{zz}D_zN^{zz}, \end{aligned} \tag{2.5}$$

$$\partial_u A_u = \partial_u(D^zA_z + D^{\bar{z}}A_{\bar{z}}) + e^2j_u, \tag{2.6}$$

Yesterday, background to gain familiarity with the gravitational case so that we can treat $U(1)$ and gravity soft theorem memory effect together.

With some intuition about null infinity, we want to consider the connection between position and momentum space for massless scattering (with an interlude on soft factors for massive charges)



The essence is

- The saddle point in the plane wave phase picks out the same position space direction as the photon or graviton momentum.
- Long-time-integrated observables pick up low frequency modes.

Note: when we see soft factors appear later we want to think of the soft momentum direction as the location of an observer in the radiation zone.

Let us look at the gauge field for a photon:

$$A_{\bar{z}} = e \lim_{r\to\infty} \partial_{\bar{z}} x^\mu \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \left(\varepsilon_\mu^{\alpha *}(\vec{q}) a_\alpha^{out}(\vec{q}) e^{-i\omega_q u - i\omega_q r(1-\cos\theta)} + h.c. \right) \tag{2.7}$$

$$= -\frac{ie\hat{\varepsilon}_{\bar{z}}^{+}}{8\pi^2} \int_0^\infty d\omega_q (a_-^{out}(\omega_q \hat{x}) e^{-i\omega_q u} - a_+^{out}(\omega_q \hat{x})^\dagger e^{i\omega_q u}). \tag{2.8}$$

And for a graviton:

$$C_{\bar{z}\bar{z}} = 2 \lim_{r\to\infty} \frac{1}{r} \partial_{\bar{z}} x^\mu \partial_{\bar{z}} x^\nu \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \left[\varepsilon_{\mu\nu}^{\alpha *}(\vec{q}) a_\alpha^{\text{out}}(\vec{q}) e^{-i\omega_q u - i\omega_q r(1-\cos\theta)} + h.c. \right], \tag{2.9}$$

$$= -\frac{i}{4\pi^2} \hat{\varepsilon}_{\bar{z}\bar{z}}^{+} \int_0^\infty d\omega_q [a_-^{\text{out}}(\omega_q \hat{x}) e^{-i\omega_q u} - a_+^{\text{out}}(\omega_q \hat{x})^\dagger e^{i\omega_q u}]. \tag{2.10}$$

Note: ∂_u will pull down powers of ω_q and $\int du$ will give delta functions supported on $\omega_q = 0$.

At this point a bit of clarification on conventions for polarization vectors is due:

$$\begin{aligned} \varepsilon^{+\mu}(\vec{q}) &= \tfrac{1}{\sqrt{2}} \left(\bar{w}, 1, -i, -\bar{w} \right), \\ \varepsilon^{-\mu}(\vec{q}) &= \tfrac{1}{\sqrt{2}} \left(w, 1, i, -w \right). \end{aligned} \tag{2.11}$$

for q like p_k above with $E_k \to \omega_q$ and $z_k \to w$

$$\varepsilon_{\bar{z}}^{+}(\vec{q}) = \partial_{\bar{z}} x^\mu \varepsilon_\mu^{+}(\vec{q}) = \frac{\sqrt{2} r \left(1 + z\bar{w}\right)}{\left(1 + z\bar{z}\right)^2}, \qquad \varepsilon_{\bar{z}}^{-}(\vec{q}) = \partial_{\bar{z}} x^\mu \varepsilon_\mu^{-}(\vec{q}) = \frac{\sqrt{2} r z \left(w - z\right)}{\left(1 + z\bar{z}\right)^2}. \tag{2.12}$$

$$\hat{\varepsilon}_{\bar{z}}^{+} = \frac{\partial_{\bar{z}} x^\mu}{r} \varepsilon_\mu^{+} = \frac{\sqrt{2}}{1 + z\bar{z}}. \tag{2.13}$$

on saddle point.

If we plug in these mode expansions into the constraint equations above, we see the Ward Identity

$$\langle out | Q^+(\xi) \mathcal{S} - \mathcal{S} Q^-(\xi) | in \rangle = 0, \tag{2.14}$$

arising from an expectation-value-interpretation of the soft factor. In particular the soft factors give the following expansion:

$$\langle z_{n+1}, z_{n+2}, ...|a_-(q)\mathcal{S}|z_1, z_2, ...\rangle = \left(S^{(0)-} + S^{(1)-}\right)\langle z_{n+1}, z_{n+2}, ...|\mathcal{S}|z_1, z_2, ...\rangle + \mathcal{O}(\omega). \quad (2.15)$$

Where for gravity:

$$S^{(0)-} = \sum_k \frac{\kappa}{2}\frac{(p_k \cdot \varepsilon^-)^2}{p_k \cdot q} \quad (2.16)$$

$$S^{(1)-} = -i\sum_k \frac{p_{k\mu}\varepsilon^{-\mu\nu}q^\lambda J_{k\lambda\nu}}{p_k \cdot q}. \quad (2.17)$$

and for the photon:

$$S^{(0)-} = \sum_k eQ_k\frac{p_k \cdot \varepsilon^-}{p_k \cdot q}, \quad (2.18)$$

Now we want to look at the expressions for the charges corresponding keeping in mind the role of the constraint equations, the mode expansions for the radiative fields and the soft factors.

$$Q^+_\varepsilon = \frac{1}{e^2}\int_{\mathcal{I}^+_-} d^2z\gamma_{z\bar{z}}\varepsilon F_{ru} = \frac{1}{e^2}\int_{\mathcal{I}^+} dud^2z\varepsilon[\partial_u(\partial_{\bar{z}}A_z + \partial_z A_{\bar{z}}) + e^2 j_u] \quad (2.19)$$

Where the second equality follows from the constraint equation.

Note: $A_\mu(u, z, \bar{z})$ are the leading coefficients of the $\frac{1}{r}$ expansion of $\mathcal{A}_\mu(r, u, z, \bar{z})$ with $\mathcal{A}_u = \mathcal{O}(r^{-1})$, $\mathcal{A}_z = \mathcal{O}(1)$. The gauge choice $A_r = 0$ gives the following relations to the large r limit of the field strength tensor:

$$\begin{aligned} F_{ur} &= A_u \\ F_{z\bar{z}} &= \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z \\ F_{uz} &= \partial_u A_z, \end{aligned} \quad (2.20)$$

where $\mathcal{F}_{ur} = \mathcal{O}(r^{-2})$, $\mathcal{F}_{z\bar{z}} = \mathcal{O}(1)$, $\mathcal{F}_{uz} = \mathcal{O}(1)$, and $F_{\mu\nu}$ are the corresponding leading coefficients in the radial expansion of $\mathcal{F}_{\mu\nu}$. In the case where all of the charged matter is massive, the current will be zero at the position of the detector. Note that F_{ur} corresponds to the radial electric field ($A_u = -e^2r^2E_r$), $F_{z\bar{z}}$ to the radial magnetic field, and F_{uz} to the radiative fields (tangent to the S^2).

Similarly for supertranslations

$$Q_f^+ = \frac{1}{4\pi G}\int_{\mathcal{I}^+_-} d^2z\gamma_{z\bar{z}} f m_B \tag{2.21}$$

while the superrotation charge was described yesterday and is written above.

Keeping in mind the prefactors that come with the mode expansion and the combination that appear in the constraint equations, we can plug in the soft factors.

At this point, it is instructive to look for a moment at the soft factor for a massive scatterer (its contribution to the above sum will be denoted with a p subscript)

$$p^\mu = m\gamma(1,\vec{\beta}) \tag{2.22}$$

from the mode expansion and expectation value interpretation

$$\Delta A_z = -\frac{e}{4\pi}\hat{\varepsilon}_z^{*+}\omega S^{(0)+}, \tag{2.23}$$

$$-\tfrac{e}{4\pi}\lim_{\omega\to 0}\omega[D^z\hat{\varepsilon}_z^{*+}S_p^{(0)+} + D^{\bar{z}}\hat{\varepsilon}_{\bar{z}}^{*-}S_p^{(0)-}] \;\; = -e^2\tfrac{Q}{4\pi}\tfrac{1}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^2}, \tag{2.24}$$

meanwhile

$$E_r = \frac{Q}{4\pi r^2}\frac{1}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^2}. \tag{2.25}$$

Similarly for a boosted mass

$$\Delta m_B = \frac{1}{4}\left[D^zD^z\Delta C_{zz} + D^{\bar{z}}D^{\bar{z}}\Delta C_{\bar{z}\bar{z}}\right], \quad \Delta C_{zz} = -\frac{\kappa}{4\pi}\hat{\varepsilon}_{zz}^{*+}S^{(0)+}. \tag{2.26}$$

$$-\tfrac{\kappa}{4\pi}\lim_{\omega\to 0}\omega[D^zD^z\hat{\varepsilon}_{zz}^{*+}S_p^{(0)+} + D^{\bar{z}}D^{\bar{z}}\hat{\varepsilon}_{\bar{z}\bar{z}}^{*-}S_p^{(0)-}] \;\; = \tfrac{4Gm}{\gamma^3(1-\vec{\beta}\cdot\hat{n})^3} = 4m_B(\vec{\beta}), \tag{2.27}$$

Similar identities appear in the massless charge limit where poles and delta functions appear. The ward identities follow from the matter fluxing through null infinity.

Note in the massless case we set empty initial and final m_B and matched near i^0 whereas one could read the constraint equations as giving a relation between the net change in the coulomb field compared to the radiation field.

One has a nice picture then of the memory effect being some net radiation observable required to be present as a result of the scattering matrix in and out states conditioning on fixed $|p, h, Q\rangle$

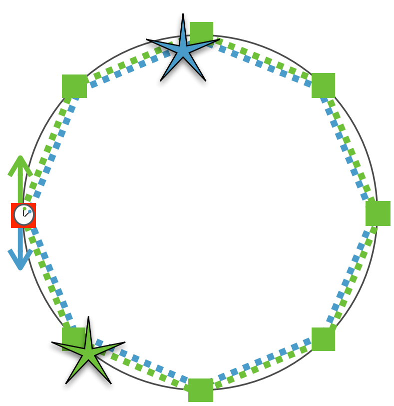
The observables correspond to steps and the additional "large gauge" symmetries span the range of allowable fluctuations. Example of test masses moving, reset with a supertranslation. Even though pure gauge, intuition of no unique vacuum prescription– how do you know nothing scattered before?

With the triangle in mind, one can go about looking for new memory effects where other vertices have been filled in.

A supertranslated vacuum has the following form:

$$C_{zz} = -2D_z^2 C \tag{2.28}$$

The time integral of the u derivative gives a finite result, but at first one might be worried that since the subleading soft factor is $\mathcal{O}(1)$ versus $\mathcal{O}(\frac{1}{\omega})$ that what we integrate over time could be divergent. This led to the isolation of a chiral observable that neatly projects out the supertranslation-transition part.



If one imagines a BMS-coordinate fixed ring with counterrotating lightlike beams upon a single orbit there is a time delay

$$\Delta P = \oint_{\mathcal{C}} \left(D^z C_{zz} dz + D^{\bar{z}} C_{\bar{z}\bar{z}} d\bar{z}\right). \tag{2.29}$$

coming from the *dudz* term in the metric. (To calculate this look at infinitesimal lightlike motions in this metric with cancellation in some terms from oppositely propagating beams moving through each segment in an orbit). Sampling such delays over each circuit of the loop, one ends up with a net time delay that is a u integral of this term:

$$\Delta^+ u = \frac{1}{2\pi L} \int du \oint_{\mathcal{C}} (D^z C_{zz} dz + D^{\bar{z}} C_{\bar{z}\bar{z}} d\bar{z}) \,. \tag{2.30}$$

With an appropriate greens function multiplying the constraint equations involving N_z one can extract such a contour integral from the curl term appearing and we find that the subleading soft graviton theorem corresponds to this Spin Memory Effect.

From the point of view of a ring in free fall, there is a net rotation with respect to a BMS ring, so that this analogous to the Sagnac effect.

So we have seen how different vertices of the triangle fit together and hint at filling in missing ones. The initial and final data specified for a given scattering process also restrict certain radiation observables (ex. ramping down energy not memory effect) and the soft factors encode this data.

A generic scattering process results in a net change in the gauge fields such that a canonical zero baseline is not physical, while a certain class of pure gauge transformations take on a special role.

As a final food for thought, it is curious to connect this story to what happens to the memory effect when there is a horizon. Note that this is where translating between data on an initial spacelike cauchy slice to data on null infinity goes awry. From the point of view of an observer near null infinity any acceleration behind a horizon would not source radiation that this observer can absorb. (Note from the memory effect perspective the horizon is naturally special precisely because it is the edge of what a null infinity observer can see.) So the memory effect due to in versus out states that would be expected from a hairless final state would naively be different from the radiation sourced by the trajectory truncated at the horizon...