Contagion in Financial Systems: A Bayesian Network Approach

Carsten Chong
Technical University of Munich

Cambridge, August 2016

Based on a paper with Claudia Klüppelberg (TUM)
(available at SSRN ID 2804291)
Systemic risk refers to the risk of a cascading failure in the financial sector, caused by interlinkages within the financial system, resulting in a severe economic downturn.

SRC Systemic Risk Centre
Literature review

Clearing mechanism:
Eisenberg & Noe 01, Gouriéroux et al. 12, Rogers & Veraart 13

Connectedness and stability:
Gai & Kapadia 10, Elliott et al. 14, Acemoglu et al. 15

Empirical work:
Elsinger et al. 06, Cont et al. 13, Craig and von Peter 14

Stochastic modeling:
Fouque & Sun 13, Chong & Klüppelberg 15, Amini et al. 16

Measures of systemic risk:
Battiston et al. 12, Cont et al. 13, Brownlees & Engle 15, Adrian & Brunnermeier 16
Systemic Risk model with 5 key features:

- Structural model with stochastic dynamics
- Identification of channels of financial contagion
- Explicit computation of default probabilities
- Quantification of systemic importance
- Applicable to all network structures
## Model at time $t = 0$

### Financial system with firms $i = 1, \ldots, N$

<table>
<thead>
<tr>
<th>Operating assets $X_i$</th>
<th>Loans $\sum_{j=1}^{N} L_{ij}$</th>
<th>Liabilities $F_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>- to bank account</td>
</tr>
<tr>
<td></td>
<td></td>
<td>- to other firms $\sum_{j=1}^{N} L_{ji}$</td>
</tr>
<tr>
<td>Cash $K_i$</td>
<td></td>
<td>Equity $E_i(0)$</td>
</tr>
</tbody>
</table>

| Assets at $t = 0$      | Liabilities and equity at $t = 0$ |

**Balance sheet of firm $i$**
**Model at time** \( t = T \)

**Financial system with firms** \( i = 1, \ldots, N \)

<table>
<thead>
<tr>
<th>Operating assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_i(T) )</td>
<td>( F_i )</td>
</tr>
<tr>
<td>Loans ( \sum_{j=1}^{N} L_{ij} \mathbf{1}_{S(j)} )</td>
<td>- to bank account ( F_i )</td>
</tr>
<tr>
<td>Cash ( K_i )</td>
<td>- to other firms ( \sum_{j=1}^{N} L_{ji} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Assets at ( t = T )</th>
<th>Liabilities and equity at ( t = T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balance sheet of firm ( i )</td>
<td></td>
</tr>
</tbody>
</table>
Model specifications

Stochastic dynamics:

\[ X_i(T) = X_i \exp((\mu_i - \sigma_i^2/2)T + \sigma_i B_i(T)) \]

- \( B_i \) i.i.d. standard Brownian motions
- \( \mu_i \in \mathbb{R} \) drift and \( \sigma_i > 0 \) volatility

Solvency equations:

1. \( i \in S : \iff E_i(T) \geq 0 \)

2. \( E_i(T) := X_i(T) + \sum_{j=1}^{N} L_{ij} \mathbf{1}_{S(j)} + K_i - F_i - \sum_{j=1}^{N} L_{ji} \)
Questions

- Can we solve the solvency equations?
- Are solutions unique?
- What is the joint distribution of $D_1, \ldots, D_N$?

**Notation:**

$$D_i := \begin{cases} 
1 & \text{if } i \text{ defaults} \\
0 & \text{if } i \text{ survives.}
\end{cases}$$
We associate to the presented financial network the redemption graph \( G = (\mathcal{V}, \mathcal{E}) \) where

\[
\mathcal{V} = \{1, \ldots, N\} = [N]
\]

and

\[
(i, j) \in \mathcal{E} : \iff L_{ji} > 0 \iff i \text{ has to repay } j \text{ at time } T.
\]

Example:
Existence and Uniqueness

Theorem (CK 16)

There exists a unique solution with probability 1

$\iff$

$G$ has no directed cycles.
Existence and Uniqueness

Theorem (CK 16)

There exists a unique solution with probability 1 ⇐⇒ $G$ has no directed cycles.

In this case: $D_1, \ldots, D_N$ form a Bayesian network over $G$.

Example:

![Graph diagram with nodes and edges labeled with monetary values]
Problem with cycles

Solution 1: Both firms default.
Problem with cycles

Solution 1: Both firms default.

Solution 2: Both firms survive.
Problem with cycles

Solution 1: Both firms **default**.  →  Strict default rule

Solution 2: Both firms **survive**.  →  Mild default rule
Mild default rule

1. **Assume:** All firms repay their debts.
   **Question:** Are there still firms with negative equity?
   \[ \rightarrow \mathcal{D}_1 \]

2. **Assume:** All firms repay their debts except those in \( \bigcup_{m=1}^{n-1} \mathcal{D}_m \).
   **Question:** Is there now a new firm with negative equity?
   \[ \rightarrow \mathcal{D}_n \]

3. We set \( \mathcal{D} = \bigcup_{m=1}^{N} \mathcal{D}_m \) and \( S = [N] \setminus \mathcal{D} \).
Problem: No Bayesian network structure due to directed cycles!

Idea:

- Incorporate mild default rule algorithm into graphical structure!
- Get back Bayesian network structure on an augmented graph!
Acyclic transformation

Original graph $G$  

Acyclic augmentation $\tilde{G}$
Acyclic transformation

Original graph $G$

Acyclic augmentation $\tilde{G}$
Acyclic transformation

Original graph $G$

Acyclic augmentation $\tilde{G}$
Acyclic transformation

Original graph $G$

Acyclic augmentation $\tilde{G}$
Acyclic transformation

Original graph $G$

Acyclic augmentation $\bar{G}$
Acyclic transformation

Original graph $G$

Acyclic augmentation $\tilde{G}$
Define: Auxiliary variables

\( D_{in} = \) Has \( i \) defaulted up to the \( n^{th} \) step of the mild default algorithm?

Main Theorem (CK 16)

Under the mild default rule the variables \( \{D_{in}: i \in [N], \ n \in [N_i]\} \) form a Bayesian network on \( \bar{G} \) with explicit conditional probabilities.
What is the Bayesian network structure good for?

- Detection of contagion channels
  \[ \rightarrow \text{d-separation} \]

- Measures of systemic importance
  \[ \rightarrow \text{total variation distance, Kullback-Leibler divergence etc.} \]

- Computation of default probabilities
  \[ \rightarrow \text{junction tree algorithm, Monte-Carlo methods etc.} \]
Example: Core–periphery network

\[
\begin{align*}
\mu_C &= 0.1, & \sigma_C &= 0.2, \\
\mu_P &= 0.05, & \sigma_P &= 0.1, \\
V_C &= 2000, & F_C &= 500, \\
V_P &= 80, & F_P &= 90, \\
L_{CC} &= 1600/(N - 1) = 400, \\
L_{PC} &= 35 \text{ (if } C \rightarrow P) .
\end{align*}
\]
Distribution of number of defaults at time $T = 1$

Solid line: Mild default rule
Dashed line: Strict default rule
Summary

**Economic Interest:** Systemic risk model with
- Structural model with stochastic dynamics
- Identification of channels of financial contagion
- Explicit computation of default probabilities
- Quantification of systemic importance
- Applicable to all network structures

**Mathematical interest:**
- Transformation of directed cyclic models into Bayesian networks
- Interpretation of directed cycles as cascade paths with indeterminate starting point
Thank you very much!