Automation in Isabelle's Analysis

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Bigproof 2017 / INI / Cambridge

What is in Isabelle/HOL's Analysis

- Real & complex numbers
- Topological & metric spaces, limits, continuity, ...
- Real vector spaces, Euclidean spaces, derivatives, ...
- Integration, measures & probabilities
- Convex analysis & complex function analysis
- ...

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  (also open & closed sets)
- Measurability
  (of functions & sets)
- Limits
  (given f compute $\lim f = x$)
- Differentiability
  (is a function differentiable, what's the derivative at x)

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- Limits
  (given f compute $\lim f = x$)
- Differentiability
  (is a function differentiable, what's the derivative at x)
- Very generic: not restricted to $\mathbb{R}^n$
- Tools: Combination of rewriting and classical reasoning

Generic Methods

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- Sledgehammer

Simplifier (`simp`, $\sim$ 1989)

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$$P\ x_1\ x_2 \implies Q\ x_1\ x_2 \implies t\ x_1\ x_2 = s\ x_1$$

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- Rule database (setup by packages & theories)

Classical reasoner (`auto`, `blast`, ∼ 1994)

- Introduction & elimination rules

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- blast: tableau prover using safe & unsafe rules

Adding Specific Automation

Explicit (Statement is in the right format), e.g.:

`algebra` Gröbner bases

`approximation` Interval arithmetic

`sos` Sums of squares: $q(x) = \sum_{i \in I} p(x)^2 \geq 0$

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(Semi-) Implicit: **Rule sets**

Measurability, Continuity & Limits

General Approach

- Use introduction rules to destruct terms:

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- Is there a *rule format*? How to handle *higher-order* functions?

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What is required:

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- Product spaces $A \times B$
- *No* function spaces $A \to B$
  (i.e. no topology of continuous functions)

Rule formats

How to show continuity of combinations of continuous functions?

- Just add $\ldots \implies (\lambda x.\ f\ (g\ x)) \in A \to C$ as intro-rule
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- Use general rule format:

$$\frac{?f \in ?A \to B \qquad ?g \in ?A \to C \qquad P\ ?p}{(\lambda x.\ c\ ?p\ (?f\ x)\ (?g\ x)) \in\ ?A \to D}$$

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Higher-order Example

$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ – extended real numbers form a complete lattice
Measurability of least/greatest fixed points in $\overline{\mathbb{R}}$:

$$\frac{(\forall f \in A \to \overline{\mathbb{R}}.\ F\ f \in A \to \overline{\mathbb{R}}) \qquad \sqcup\text{−continuous } F}{\text{lfp } F \in A \to \overline{\mathbb{R}}}$$

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Proof:

1. $\sqcup$−continuous $(\lambda f\ \omega.\ \mathsf{hd}\ \omega + f\ (\mathsf{tl}\ \omega))$
2. $\forall f \in \mathsf{stream}\ \overline{\mathbb{R}} \to \overline{\mathbb{R}}.\ (\lambda \omega.\ \mathsf{hd}\ \omega + f\ (\mathsf{tl}\ \omega)) \in \mathsf{stream}\ \overline{\mathbb{R}} \to \overline{\mathbb{R}}$

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$$(\lambda(x, y).\ f\ x\ y) \in A \times B \to C \quad \Longrightarrow$$
$$\left(\forall g, h, D.\ \frac{g \in D \to A \qquad h \in D \to B}{(\lambda x.\ f\ (g\ x)\ (h\ x)) \in D \to C} \right)$$

Specialities

- Cut of terms with *countable*, *discrete* range for measurable functions:

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- Measurable, open, closed sets are closed under predicates in set-comprehension. E.g.

$$\frac{f \in A \to \mathbb{R} \qquad g \in A \to \mathbb{R}}{\mathsf{open}_A\ \{x \mid f\ x < g\ x\}}$$

Also logical connectives, quantifiers, ...

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- Derivatives are similar

Discussion

- Measurability works quite good
- Continuity, Differentiability, Limits work but require more user interaction
- Works (mostly) by introduction and simp rules; some special cases necessary
- More spaces (i.e. discrete spaces $(\mathbb{N}, \mathbb{B}, \ldots)$ or measures $(\mathbb{P}(A))$) lead to more measurable functions, more fine grained rules