Computational Higher-dimensional Type Theory & RedPRL

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Computational Types

programs/
realizers

computation

Computational Types

programs/ realizers	<------	computational type theory
computation		theory of computation

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programs/
realizers

<-----

computational
type theory

computation

theory of
computation

meaning
explanation

<----

Martin-Löf
type theory

pre-mathematical
in M-L's work

Computational Types

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Ex: "bool type" because "bool" evals to "bool"
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Ex: "if(true;false,42) ∈ bool" because
"if(true;false,42)" evals to "false"
and "false" is in "bool".

Go Higher-Dimensional

The Book HoTT and the CCHM system have higher-dim. interpretations

Can we do the same?

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realizers/programs
higher-dim. (dims & Kan)
interesting spaces
universes + univalence

Potential Benefits

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1. $\doteq$ is closer to equational reasoning in standard math

   functional extensionality
   full universal properties
   (ex: eta for natural numbers)
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2. adding more theorems cannot break computation

   realizers & proof theory separated
   perfect for programming

Clarification #0

computational type theory

IS NOT

"extensional type theory"

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Realizers can be
a model of "ETT", **a model of** "ITT",
or potentially **a model of** HoTT.

You can collect your favorite **theorems**
as rules in your favorite theory

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operational sem. + canonicity

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most formal type theories (defined by rules)
want **decidable type-checking**

$$f \doteq g \in \texttt{nat -> nat}$$

not decidable in general
but can be proved by **induction** in CTT
(as proving a **theorem** about realizers)

Clarification #2

most formal type theories
want decidable type-checking

=> undecidable rules were ruled out

computational type theory
is fine with "undecidable rules"

=> ask users for guidance in practice
(can be interactive & tactic-based)

Cubical Realizers

higher-dimensional programming

[Angiuli, Harper & Wilson]

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higher-dimensional programming

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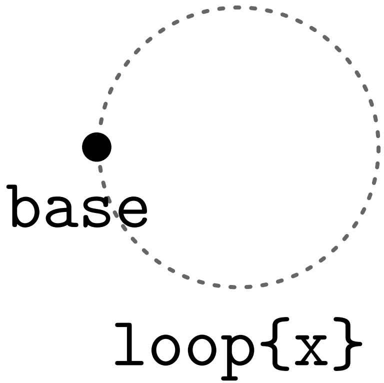
with dim expr `r := 0 | 1 | x`

Circle



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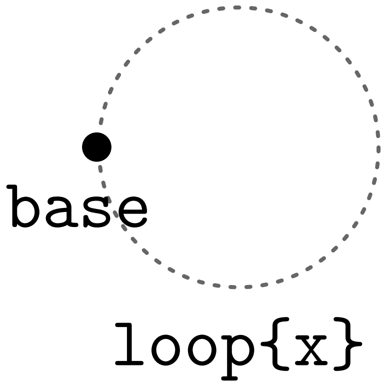
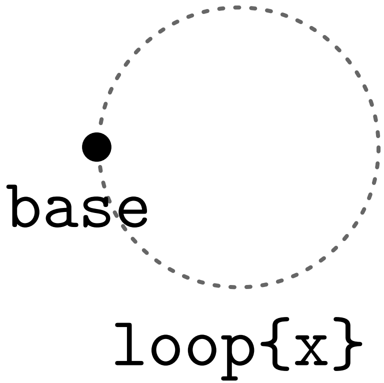


S1 value

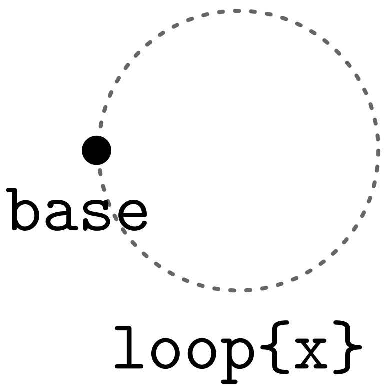
Circle



base value

S1 value

Circle



S1 value

base value

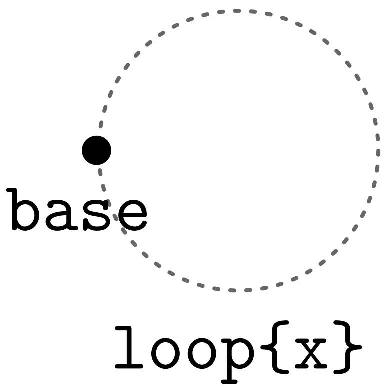
loop{r} → dim expr

loop{x} value

loop{0} ↦ base

loop{1} ↦ base

Circle



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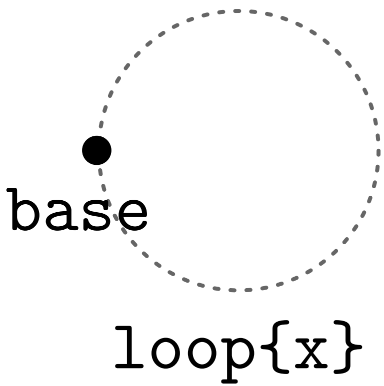
M ↦ M'
---------------------
S1elim(a.A, M, B, u.L)
 ↦ S1elim(a.A, M', B, w.L)
```

Circle



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  ↦ B
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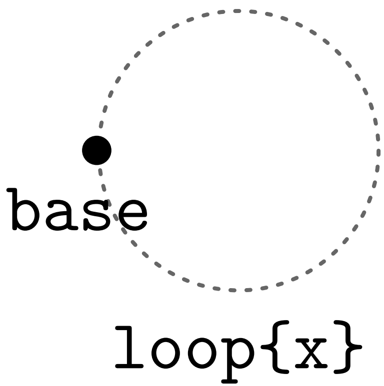
Circle



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```
S1elim(a.A, base, B, w._)
 ↦ B
```

```
S1elim(a.A, loop{x}, _, w.L)
 ↦ L<x/w>
```

Kan: Coercions

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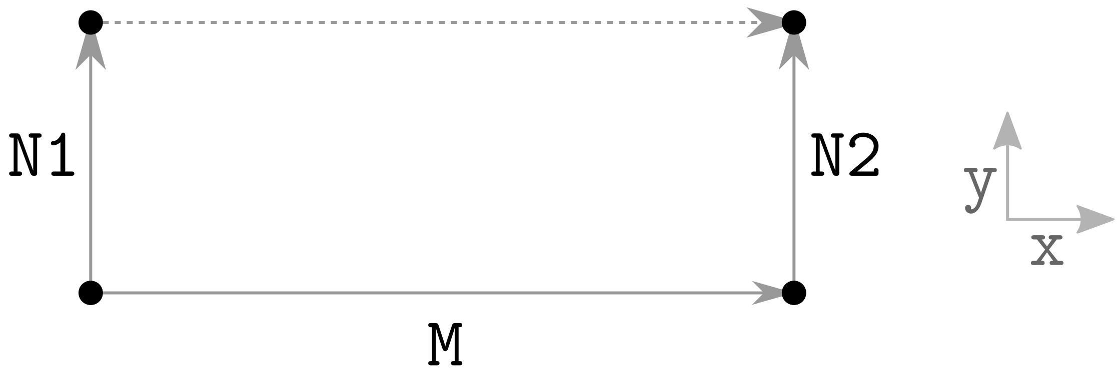
coe{r->r'}(x.A, M) ∈ A<r'/x>

⋔

A<r/x>

Kan: Homogeneous Comp.



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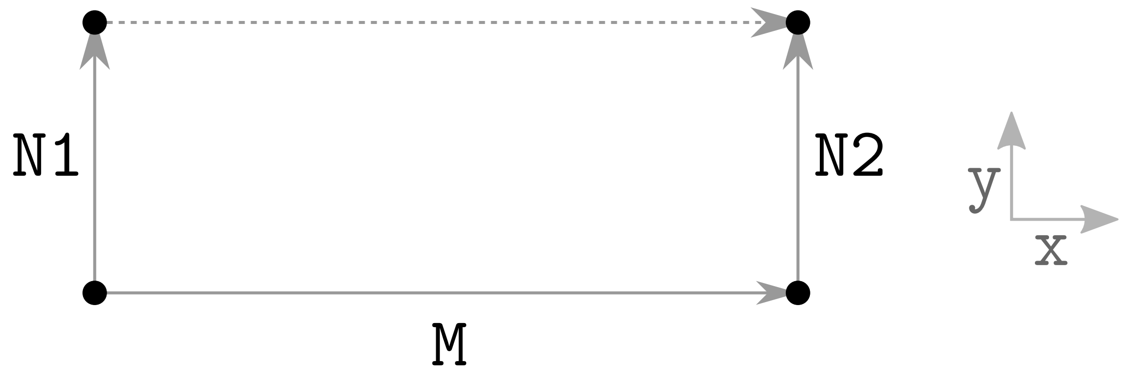


```
hcom{r->r'}(A, M)
  [x -> (y.N1, y.N2)]
```

adjacency: needs exact equality (≐)

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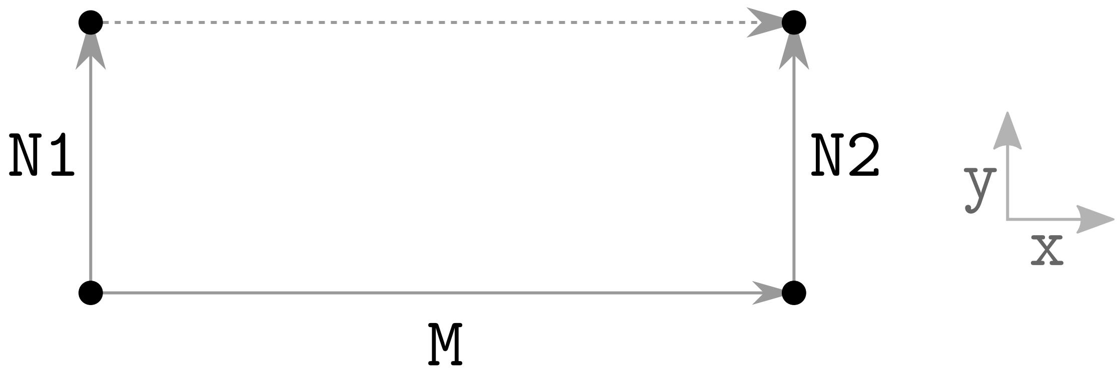


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note: we forbid empty systems

Kan Circle

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coe{r->r'}(_.S1, M) ↦ M
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r != r'  ri is 0 or 1 (the first)
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hcom{r->r'}(S1, M)...[ri -> (y.N_0, y.N_1)]...
 ↦ N_ri<r'/y>
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hcom{r->r'}(S1, M)... value
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formal composition

Kan Circle

S1elim needs to handle new values

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```
r != r'  ri != 0  ri != 1
-----------------------------------
S1elim(a.A, hcom{r->r'}(S1, M)..., B, w.L)
 ↦ com{r->r'}(z.A[hcom{r->z}(...).../a],
      S1elim(M, B, w.L))...
```

(using the definable heterogeneous comp.)

Cubical Stability

Dimension substs. do not commute with evaluation!

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```
S1elim(a.A,
 loop{x}, B, w.L)  |----------> L<x/w>
                                  |
                                  |  <0/x>
                                  v

                                L<0/w>
```

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                                    ¦
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        v                           v

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Judgments for cubically stable types, memberships, values, etc

Cubical Type Theory

stability: consider every substitution

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A ≐ B type =>
under any further substitution ψ...
Aψ and Bψ stably* eval to A' and B',
stably* recognizing the same stable* values
and having stably* equal Kan structures

(*see our arXiv & POPL papers for details)

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M ≐ N ∈ A =>
A ≐ A type and A evals to A',
M and N stably* eval to M' and N',
A' stably* views N' and M' as the same value

(*see our arXiv & POPL papers for details)

Comparison with CCHM

based on realizability

(closer to standard math equality)

no connections

no empty systems

(handled by coe)

Current Progress

Done:

dependent functions
dependent pairs
strict bools
"weak" bools
circle
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univalence for
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Still working:

Univalence & Kan universes

RedPRL

a proof assistant based on cubical realizability

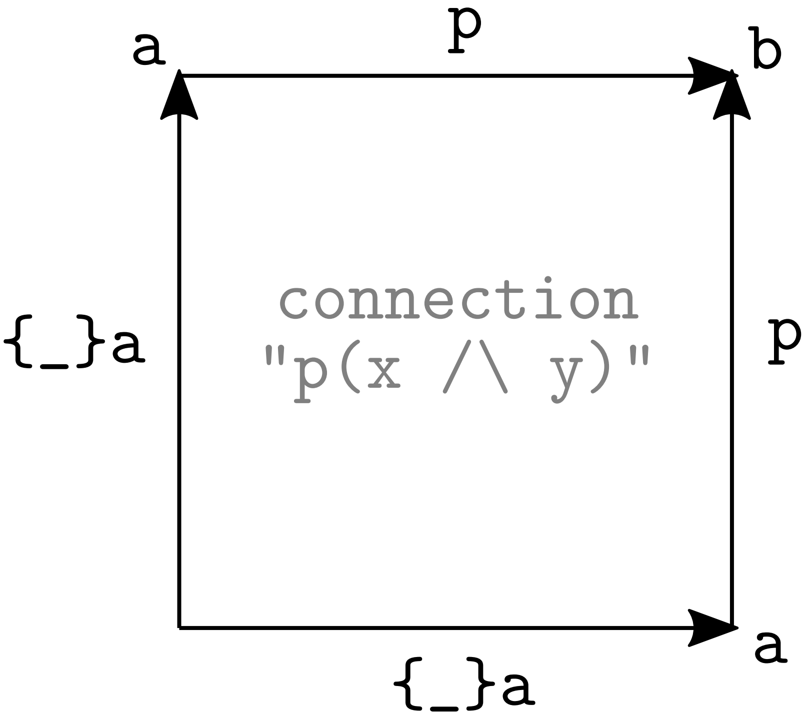
incorporates recent advances such as dependent subgoals

(inspired by Spiwack's work)

still nascent, changing everyday

https://github.com/RedPRL/sml-redprl

RedPRL Example



Conclusion

Cubical extension of Nuprl realizers

Canonicity on points (diff. from CCHM)

Still working on Kan universes

RedPRL (to be) an implementation

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Acknowledgements

Our work is strongly influenced by the BCH and CCHM papers, work by Licata and Brunerie, and many inspiring discussions within the HoTT community.