

Automatic Asymptotics in Isabelle/HOL

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$$\text{qs_cost} \in O[\text{length } \mathbf{going_to} \text{ at_top}](\lambda xs. (\text{length } xs)^2)$$

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Things like $(\lambda x. 2 * x + x * \ln \ln x) \in O(\lambda x. x * \ln x)$ get proven automatically.

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- ▶ Average-case complexity of QuickSort
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- ▶ Average number of integer divisors/co-primes/square-free integers

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Example: Lemma required for Akra–Bazzi

$$\lim_{x \rightarrow \infty} \left(1 - \frac{1}{b \log^{1+\varepsilon} x}\right)^p \left(1 + \frac{1}{\log^{\varepsilon/2} \left(bx + \frac{x}{\log^{1+\varepsilon} x}\right)}\right) - \left(1 + \frac{1}{\log^{\varepsilon/2} x}\right) = 0^+$$

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In Isabelle: 700 lines of messy proofs

lemma akra_bazzi_aux:

filterlim

$$\begin{aligned} & (\lambda x. (1 - 1/(b * \ln x^{(1 + \varepsilon))})^p) * \\ & (1 + \ln (b * x + x/\ln x^{(1 + \varepsilon)})^{-\varepsilon/2}) - \\ & (1 + \ln x^{-\varepsilon/2})) \\ & (\text{at_right } 0) \text{ at_top} \end{aligned}$$

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This is what we would like to have.

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This is what we would like to have.

Computer Algebra Systems can do this (sort of)

So why can't we?

Asymptotic Expansions

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- ▶ I will not show actual Isabelle code for better readability.

Related Work

- ▶ *Asymptotic Expansions of exp-log Functions* by Richardson, Salvy, Shackell, van der Hoeven
- ▶ *On Computing Limits in a Symbolic Manipulation System* by Gruntz

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Given the limits of $f(x)$ and $g(x)$, what is the limit of $f(x) \square g(x)$? (for $\square \in \{+, -, \cdot, /\}$)

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If the limits are $\in \mathbb{R}$: **Obvious**.

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If the limits are $\in \mathbb{R}$: **Obvious**.

But: $\infty - \infty = ?$ $0 \cdot \infty = ?$

We need to track more information than just the limit!

We need the *full asymptotic information*

Asymptotic Expansions

For $x \rightarrow 0$, we have:

$$e^x \sim 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

$$\frac{1}{1+x} \sim 1 - x + x^2 - x^3 + \dots$$

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This means: Cutting off $f(x) \sim a_0(x) + a_1(x) + \dots$ at term a_n yields error $O(a_{n+1}(x))$.

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Expansions contain the *full* asymptotic information.

They can be added/subtracted/multiplied/divided.

Limits can simply be 'read off'

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Not all functions have such easy expansions!
e. g. \exp (at $\pm\infty$) and \ln (at $\infty, 0$)

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For now, we only consider expansions of the form

$$f(x) \sim c_0 x^{e_0} + c_1 x^{e_1} + \dots$$

where $e_0 > e_1 > \dots$

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$xs + [] = xs$

$((c_1, e_1) :: xs) + ((c_2, e_2) :: ys)$

| $e_1 == e_2 = (c_1 + c_2, e_1) :: xs + ys$

| $e_1 < e_2 = (c_1, e_1) :: xs + ((c_2, e_2) :: ys)$

| $e_1 > e_2 = (c_2, e_2) :: ((c_1, e_1) :: xs) + ys$

Asymptotic Expansions – Multiplication

Multiplication with 'atomic' factor $c'x^{e'}$:

scale : $\mathbb{R} \rightarrow \mathbb{R} \rightarrow \text{Exp} \rightarrow \text{Exp}$

scale $c' e' xs = [(c * c', e + e') \mid (c, e) \leftarrow xs]$

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$$((c_1, e_1) :: xs) * ((c_2, e_2) :: ys) =$$

$$(c_1 * c_2, e_1 + e_2) :: \text{scale } c_1 \ e_1 \ ys + xs * ((c_2, e_2) :: ys)$$

Asymptotic Expansions – Power Series

Many functions have local power series expansions:

$$g(x) = \sum_{n=0}^{\infty} c_n x^n \quad (\text{for } |x| < R)$$

e. g. $c_n := 1/n!$ for the exponential function.

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power : \mathbb{R} list \rightarrow Exp \rightarrow Exp

power (c :: cs) xs = (c, 0) :: xs * power cs xs

Asymptotic Expansions – Reciprocal

Consider $f(x)$ with expansion $(c, e) :: xs$ where $c \neq 0$, i. e.

$$f(x) = c x^e + g(x)$$

where $g(x)$ expands to xs .

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Therefore:

$$\begin{aligned} \text{inv } ((c, e) :: xs) &= \text{scale } (1/c) \ (-e) \\ &\quad (\text{powser } (\text{cycle } [1, -1]) \ (\text{scale } (1/c) \ (-e) \ xs)) \end{aligned}$$

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Example: $(e^x, x, \ln x)$ is an asymptotic basis and generates monomials $e^{ax} x^b \ln^c x$

$$e^{4x} + 2x^3 \ln x \hat{=} [1 \cdot (4, 0, 0), 2 \cdot (0, 3, 1)]$$

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$$e^{4x} + 2x^3 \ln x \hat{=} [(4, (0, (0, 1))), (0, (3, (1, 2)))]$$

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- ▶ Limit is $\text{sgn}(c) \cdot \infty$ if $e_i > 0$
- ▶ Limit is c if all $e_i = 0$

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Before:

type Exp = $(\mathbb{R} \times \mathbb{R})$ llist

negate : Exp \rightarrow Exp

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datatype Exp : Basis  $\rightarrow$  Type where
```

```
  Const :  $\mathbb{R} \rightarrow$  Exp []
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  Exp   : (Exp bs  $\times$   $\mathbb{R}$ ) llist  $\rightarrow$  Exp (b :: bs)
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Exp : (Exp bs \times \mathbb{R}) llist \rightarrow Exp (*b* :: *bs*)

negate : Exp *bs* \rightarrow Exp *bs*

negate (Const *c*) = -*c*

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negate : Exp *bs* \rightarrow Exp *bs*

negate (Const *c*) = -*c*

negate (Exp xs) = Exp [(negate *c*, *e*) | (*c*, *e*) \leftarrow xs]

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Same trick as before: Given

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We might have to add $\ln x$ to our basis!

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- ▶ Lots of case distinctions
- ▶ Can introduce ugly new basis elements like $\exp(x + 1/x)$
- ▶ Lots of opportunities for implementation bugs
- ▶ Luckily, the Isabelle kernel caught them, of course. :)

What this looks like in Isabelle

Type α ms for multiserie with coefficients of type α

```
datatype  $\alpha$  ms = MS "( $\alpha \times \text{real}$ ) llist" "real  $\Rightarrow$  real"
```

E.g. expansion of order 3 would be 'real ms ms ms'.

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How do we turn this into a proof method?

Computing Expansions in Isabelle

Evaluation

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I had to implement lazy evaluation of HOL terms.

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Lazy evaluation framework:

- ▶ set of supported (co-)datatypes and their constructors

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- ▶ determine whether a pattern matches a term (modulo rewriting)
- ▶ bring a term into head-normal form
- ▶ produce an Isabelle theorem of the form
original term = reduced term

Does not support sharing

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- ▶ Addition of expansions involves comparisons of real numbers
- ▶ 'Trimming' expansions involves zeroness tests of real functions
- ▶ Both of these are difficult or even undecidable

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Solution: Heuristic approach using Isabelle's automation

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Additional backends (user input/Mathematica/Maple) possible

Proof method

At this point, we have all the ingredients:

- ▶ Parse and pre-process input expression

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- ▶ Use evaluation framework to trim expansions/determine signs whenever necessary
- ▶ **In the end:** Trim expansion to determine leading term

Proof method

With some pre-processing, we can automatically prove statements of the form

- ▶ $f(x) \longrightarrow c$
- ▶ $f(x) \sim g(x)$
- ▶ $f(x) \in L(g(x))$ for any Landau symbol L

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\sin, \cos, \tan at finite points also possible.

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lemma ( $\lambda n. (1 + 1/n)^n \longrightarrow \exp 1$ )  
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Example

```
lemma (( $\lambda x. (1 + y/x)^x \longrightarrow \exp y$ ) at_top  
proof (cases y = 0)  
  case False  
  thus ?thesis by exp_log_asymptotics  
qed simp_all
```

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lemma

assumes $c > 1$ and $k > 0$

shows $(\lambda n. n^k) \in o(\lambda n. c^n)$

using **assms** **by** `exp_log_asymptotics`

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lemma `akra_bazzi_aux`:

assumes $b \in \{0 < .. < 1\}$ and $\varepsilon > 0$

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$$(1 - H/(b * \ln x^{(1 + \varepsilon)}))^{p *}$$

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- ▶ Case distinctions have to be done manually
- ▶ Somewhat 'ad-hoc' formalisation

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- ▶ Implementation was tricky to get right

Comparison to CASs

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Maxima and Sage take very long for some of them
and give wrong result for this:

$$\exp\left(\frac{\log \log (x + e^{\log x \log \log x})}{\log \log \log (e^x + x + \ln x)}\right) \longrightarrow e$$

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Isabelle still isn't a CAS – but we're getting there.

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- ▶ Cannot handle oscillating functions or complex-valued asymptotics
- ▶ User interaction for zeroness tests could be improved

Questions? Demo?