Symmetry Breaking in a Gas of Bosons

T.Balaban, J.Feldman, H.Knörrer, E.Trubowitz

Gas of Bosons in thermodynamic equilibrium at temperature T

- $X = X(s) = \mathbb{Z}^3 / L^s \mathbb{Z}^3$
- single particle operator $h = -\Delta$
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- $\beta = \frac{1}{kT}$, k the Boltzmann constant
- $H = H_0 + \lambda V$ where $H_0 = \sum_{\mathrm{x,y} \in X} a^\dagger(\mathrm{x}) h(\mathrm{x}, \mathrm{y}) a(\mathrm{y})$

$$V = \sum_{\mathrm{x,y} \in X} a^\dagger(\mathrm{x}) a^\dagger(\mathrm{y}) v(\mathrm{x}, \mathrm{y}) a(\mathrm{x}) a(\mathrm{y})$$

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Number Symmetry: H commutes with N

Correlation Functions:

$$C_p(x_1, \cdots, x_p) = \langle a^{(\dagger)}(x_p) \cdots a^{(\dagger)}(x_1) \rangle$$

- $x_j = (\tau_j, \mathrm{x}_j) \in [\frac{-\beta}{2}, \frac{\beta}{2}] \times X, \quad \tau_1 \leq \tau_2 \leq \cdots \leq \tau_p$
- $a^{(\dagger)}(\tau, \mathrm{x}) = e^{\tau(H-\mu N)} a^{(\dagger)}(\mathrm{x}) e^{-\tau(H-\mu N)}$

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Number Symmetry implies that $C_p(x_1, ..., x_p) = 0$, unless there are as many $a^\dagger$ factors as a factors.

Expected Result on Symmetry Breaking:

For all sufficiently small coupling constants $\lambda > 0$, the zero temperature – infinite volume correlation functions
$C_p(x_1, ..., x_p) = \lim_{T\to 0} \lim_{s\to\infty} C_p(x_1, \cdots, x_p)$
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exist and there is a (number) symmetry breaking phase transition:

- If $\mu < 0$, $C_1\big((0,x)\big) = \lim_{T\to 0} \lim_{s\to\infty} \langle a^\dagger(\mathrm{x})\rangle = 0$, and $C_p(0, x_2, ..., x_p)$ is exponentially decreasing (localized phase)
- If $\mu > 0$, $C_1\big((0,x)\big) \neq 0$, and $C_p(0, x_2, ..., x_p)$ is not even integrable.
  There are "extended collective excitations (Goldstone bosons)"

Coherent State Functional Integral Representation

Formally

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int \prod_{x\in[0,\beta]\times X} \tfrac{d\varphi^*(x)\wedge d\varphi(x)}{2\pi i}\, e^{-\mathrm{Action}(\varphi^*,\varphi)}$$

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where

$$\mathrm{Action}(\varphi^*,\varphi) \;=\; \langle\varphi^*, (-\partial_0 + h)\,\varphi\rangle \;+\; \lambda\langle\varphi^*\varphi,\; v\,\varphi^*\varphi\rangle \;-\; \mu\langle\varphi^*,\varphi\rangle$$

with $\langle\ ,\ \rangle$ the "natural" real inner product

The Temporal Ultraviolet Limit (2012)

$$Z = \int_{S_-} \prod_{x\in\mathcal{X}_-} \tfrac{d\phi^*(x)\wedge d\phi(x)}{2\pi i}\ e^{-Act_-(\phi^*,\phi)+\mathcal{E}_-(\phi^*,\phi)} \ +\ O(e^{-1/\lambda^\varepsilon})$$

- $\mathcal{X}_- = (\mathbb{Z}/L^t\mathbb{Z}) \times X$ where t an integer close to $\log_L \beta$.
- The "small field region" S_- is the space of complex valued functions ϕ on $(\mathbb{Z}/L^t\mathbb{Z}) \times X$ such that $|\phi(x)|, |\partial_\nu\phi(x)| < \lambda^{-1/3}$ for all x and all $\nu = 0, 1, 2, 3$

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- $Act_-(\phi^*,\phi) = \langle \phi^*, (-\partial_0 + h)\phi\rangle + \lambda\mathcal{V}(\phi^*,\phi) - \mu\langle\phi^*,\phi\rangle$ with $\mathcal{V}(\phi_*,\phi)$ an explicit quartic term
- $\mathcal{E}_-(\phi_*,\phi)$ a perturbative correction; a power series in the two independent complex fields ϕ_*,ϕ of lower degree at least six.

$$Z = \int_{S_-} \prod_{x \in \mathcal{X}_-} \frac{d\phi^*(x) \wedge d\phi(x)}{2\pi i} \; e^{-Act_-(\phi^*,\phi) + \mathcal{E}_-(\phi^*,\phi)} \; + \; O(e^{-1/\lambda^{\varepsilon}})$$

What have we achieved?

$$Z = \int_{S_-} \prod_{x \in \mathcal{X}_-} \tfrac{d\phi^*(x) \wedge d\phi(x)}{2\pi i} \; e^{-Act_-(\phi^*,\phi) + \mathcal{E}_-(\phi^*,\phi)} \; + \; O(e^{-1/\lambda^\varepsilon})$$

What have we achieved?

If $\psi = z$ is a constant field, then the leading term in Act_- is proportional to $(|z|^2 - \frac{\mu}{\lambda})^2 - (\frac{\mu}{\lambda})^2$

The Infrared Analysis

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- A "block spin renormalization group flow" with parabolic scaling that integrates out many of the fields and leads to actions with much better developped potential well, that justifies the Bogoliubov Ansatz.
- A "block spin renormalization group flow" with elliptic scaling in radial and tangential variables around a point of the bottom of the well that will will push the action into the symmetry broken, superfluid fixed point.

At the beginning:

$$Z = \int_{\mathcal{H}_-} d\mu(\phi^*, \phi)\ e^{-Act_-(\phi^*,\phi) + \mathcal{E}_-(\phi^*,\phi)}\ +\ O(e^{-1/\lambda^\varepsilon})$$

After a certain number of block spin transformations:

$$Z = \int_{\mathcal{H}} d\mu(\psi^*, \psi)\ e^{-\mathcal{A}(\psi^*,\psi;\phi_{*\mathrm{bg}},\phi_{\mathrm{bg}}) + \mathcal{E}(\psi^*,\psi)}\ +\ O(e^{-1/\lambda^\varepsilon})$$

$$\mathcal{A}(\psi_*, \psi; \phi_*, \phi) = \langle \psi_* - M_-\phi_*,\ \mathfrak{Q}(\psi - M_-\phi)\rangle + \mathfrak{A}(\phi_*, \phi)$$

$$\mathfrak{A}(\phi_*, \phi) = -\langle \phi_*, (\alpha\partial_0 + \Delta)\phi\rangle - \mu_r\langle \phi_*, \phi\rangle + \mathrm{quartic}$$

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Insert $1 = \int_{\mathcal{H}_+} d\mu(\theta^*, \theta)\, e^{-b\langle\theta^* - M\psi^*,\ \theta - M\psi\rangle}$ to get

$$Z = \int_{\mathcal{H}_+} d\mu(\theta^*, \theta) \int_{\mathcal{H}} d\mu(\psi^*, \psi)\ e^{-\mathcal{A}_{\mathrm{eff}}(\theta^*,\theta;\psi^*,\psi;\phi_{*\mathrm{bg}},\phi_{\mathrm{bg}})+\mathcal{E}(\psi^*,\psi)}\ +\ \ldots.$$

with

$$\mathcal{A}_{\mathrm{eff}}(\theta_*, \theta; \psi_*, \psi; \phi_*, \phi) = b\langle\theta_* - M\psi_*,\ \theta - M\psi\rangle + \mathcal{A}(\psi_*, \psi; \phi_*, \phi)$$

Critical point equations

For fixed (θ_*, θ), the critical point equations for the map

$$(\psi_*, \psi) \mapsto \mathcal{A}_{\text{eff}}\big(\theta_*, \theta; \psi_*, \psi; \phi_{*\text{bg}}(\psi_*, \psi), \phi_{\text{bg}}(\psi_*, \psi)\big)$$

are

$$\begin{aligned} (bM^*M + \mathfrak{Q})\psi_* &= bM^*\theta_* + \mathfrak{Q}M_-\phi_{*\text{bg}}(\psi_*, \psi) \\ (bM^*M + \mathfrak{Q})\psi \ &= bM^*\theta \ + \mathfrak{Q}M_- \phi_{\text{bg}}(\psi_*, \psi) \end{aligned}$$

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They have a "unique" solution $\psi_{(*)\text{cr}}(\theta_*, \theta)$.

⚠ $$\psi_{*\text{cr}}(\theta^*, \theta) \neq \psi_{\text{cr}}(\theta^*, \theta)^*$$

Block spin transformation

Theorem

Set $\check{\phi}_{(*)\mathrm{bg}} = \phi_{(*)\mathrm{bg}} \circ \psi_{(*)\mathrm{cr}}$ *and* $\check{M}_- = M \circ M_-$ *. Then*

$Z = F(\theta^*, \theta) \cdot \int d\mu(\theta^*,\theta)\, e^{-\check{\mathcal{A}}(\theta^*,\theta;\check{\phi}_{*\mathrm{bg}},\check{\phi}_{\mathrm{bg}})+\mathcal{E}(\psi_{*\mathrm{cr}},\psi_{\mathrm{cr}})}$ *with*

- $\check{\mathcal{A}}(\theta^*, \theta; \phi_*, \phi) = \langle \theta_* - \check{M}_-\phi_*, \check{\mathfrak{Q}}(\theta - \check{M}_-\phi)\rangle + \mathfrak{A}(\phi_*, \phi)$
- $\check{\mathfrak{Q}} = (\frac{1}{b}\mathbf{1} + M\mathfrak{Q}^{-1}M^*)^{-1}$
- *the fluctuation integral* $F(\theta_*, \theta) = \int_{D(\theta_*,\theta)} d\mu(\partial\psi_*,\partial\psi) e^{-\partial\mathcal{A}+\partial\mathcal{E}}$ *with*

$$\partial\mathcal{A}(\theta_*, \theta; \partial\psi_*, \partial\psi) = \mathcal{A}_{\mathrm{eff}}(\theta_*, \theta; \psi_*, \psi; \phi_{*\mathrm{bg}}, \phi_{\mathrm{bg}})\Big|^{\psi_{(*)}=\psi_{(*)\mathrm{cr}}+\partial\psi_{(*)}}_{\psi_{(*)}=\psi_{(*)\,\mathrm{cr}}}$$

$$D(\theta_*, \theta) = \{(\partial\psi_*, \partial\psi) \,|\, \psi_{*\mathrm{cr}}(\theta_*, \theta) + \partial\psi_* = (\psi_{\mathrm{cr}}(\theta_*, \theta) + \partial\psi)^*\}$$

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$$D(\theta_*, \theta) = \{(\partial\psi_*, \partial\psi) \,|\, \psi_{*\mathrm{cr}}(\theta_*, \theta) + \partial\psi_* = (\psi_{\mathrm{cr}}(\theta_*, \theta) + \partial\psi)^*\}$$

Also, $\check{\phi}_{(*)\mathrm{bg}}(\theta_*, \theta)$ *is critical for* $(\phi_*, \phi) \mapsto \check{\mathcal{A}}(\theta_*, \theta; \phi_*, \phi)$ *.*

Resulting Challenges

- The fluctuation integral: Move the integration region $D(\theta^*, \theta)$ to a region where $\partial\psi_* = \partial\psi^*$! This is done by a Cauchy–Stokes Theorem. Then estimate $\log F$ by a modification of polymer expansion!

Resulting Challenges

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- Renormalization: $Z = \int_{d\mu(\theta^*,\theta)} e^{\log F - \check{\mathcal{A}} + \mathcal{E}}$
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  $\log F$ contains terms that have to be subsumed into the explicit part $\check{\mathcal{A}}$ of the action.
- Study the critical field equations and the background field equations!

End of parabolic regime (small field analysis):

Assume that $\mu > O(\lambda)$.

$$Z = \int_{\mathcal{H}} d\mu(\psi^*, \psi)\ e^{-\mathcal{A}(\psi^*,\psi;\phi_{*\mathrm{bg}},\phi_{\mathrm{bg}})+\mathcal{E}(\psi^*,\psi)} \ +\ O(e^{-1/\lambda^{\varepsilon}})$$

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$$\mathfrak{A}(\phi_*, \phi) = -\langle \phi_*, (\alpha\partial_0 + \Delta)\phi\rangle - \mu_r\langle \phi_*, \phi\rangle + \mathrm{quartic}$$

with

- $\mu_r = O(1)\ ,\ \ \alpha = O(1)$,
- radius of potential of order $\frac{1}{\lambda^{3/4}}$,
- depth of potential well of order $\frac{1}{\lambda^{3/2}}$

Critical Fields and Background Fields

One can show

$$\psi_{(*)cr}(\theta_*, \theta) = (bM^* M + \mathfrak{Q})^{-1}(bM^* \theta_{(*)} + \mathfrak{Q} M_- \check{\phi}_{(*)\mathrm{bg}}(\theta_*, \theta))$$

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Therefore one can concentrate on the "background field equations" at each scale:

$$\begin{aligned}
(M_-^* M_- - \alpha \partial_0 - \Delta - \mu_r)\phi_* &= M_-^* \psi_* - \mathfrak{v}\, \phi_* \phi^2 \\
(M_-^* M_- - \alpha \partial_0 - \Delta - \mu_r)\phi\ &= M_-^* \psi\ - \mathfrak{v}\, \phi_*^2 \phi
\end{aligned}$$

(the scalar $\mathfrak{v}$ is the dominant part of the interaction)

Parabolic Regime – Background field equations

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The operator $(M_-^* M_- - \alpha\partial_0 - \Delta - \mu_r)$ is boundedly invertible, and one can apply a contraction mapping argument.

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Scaling and the form of the averaging operators are adjusted to the parabolic differential operator $\alpha\partial_0 + \Delta$.

A model for $(M_-^* M_- - \alpha\partial_0 - \Delta - \mu_r)^{-1}$ is the inverse Fourier transform of

$$\hat{s}(k_0, \mathbf{k}) = \frac{1}{-ik_0 + m^2 + k_1^2 + k_2^2 + k_3^2}$$

namely

$$s(\tau, \mathbf{x}) = \begin{cases} 0 & , \ \tau > 0 \\ \frac{1}{|t|^{3/2}} e^{-m|t|^2} e^{-|\mathbf{x}|^2/4|t|} & , \ \tau < 0 \end{cases}$$

- continuous for $(\tau, \mathbf{x}) \neq (0,0)$, integrable singularity there
- exponentially decaying for large $(\tau, \mathbf{x})$

Elliptic Regime – Background field equations

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μ_r is large, $\mathfrak{v}$ is small, potential well has minima on a circle of radius $r = \sqrt{\mu_r/\mathfrak{v}}$. Change of variables near bottom of the well

$$\psi = re^{R+i\vartheta} \quad \psi_* = re^{R-i\vartheta} \quad \phi = re^{X+iH} \quad \phi_* = re^{X-iH}$$

In these coordinates, the background field equations are

$$\Box \begin{bmatrix} X \\ H \end{bmatrix} = M_-^* \begin{bmatrix} R \\ \vartheta \end{bmatrix} \quad , \qquad \Box = \begin{bmatrix} 2\mu_r - \Delta & i\alpha\partial_0^* \\ i\alpha\partial_0 & -\Delta \end{bmatrix} + M_-^* M_-$$

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Observe

$$\det \begin{bmatrix} 2\mu_r - \Delta & i\alpha\partial_0^* \\ i\alpha\partial_0 & -\Delta \end{bmatrix} = \alpha^2 \Big(\partial_0^* \partial_0 + 2\tfrac{\mu_r}{\alpha^2}(-\Delta) + \tfrac{1}{\alpha^2}(-\Delta)^2 \Big)$$

In the elliptic regime, $\frac{\mu_r}{\alpha^2} = O(1)$ while $\frac{1}{\alpha^2}$ is small.

Elliptic Regime – Action near bottom of the well

$$\frac{1}{r^2}\mathrm{Act}(\psi_*,\psi) = \left\langle \begin{bmatrix} R \\ \vartheta \end{bmatrix}, (1 - M_- \Box^{-1} M_-^*)^{-1} \begin{bmatrix} R \\ \vartheta \end{bmatrix} \right\rangle + \text{higher order}$$

Elliptic Regime – Action near bottom of the well

$$\frac{1}{r^2}\mathrm{Act}(\psi_*, \psi) = \left\langle \begin{bmatrix} R \\ \vartheta \end{bmatrix}, (1 - M_- \Box^{-1} M_-^*)^{-1} \begin{bmatrix} R \\ \vartheta \end{bmatrix} \right\rangle + \text{higher order}$$

Deep in the elliptic regime the Fourier transform of $1 - M_- \Box^{-1} M_-^*$ is approximately

$$\begin{bmatrix} 1 & 0 \\ 0 & k_0^2 + \mathbf{k}^2 \end{bmatrix}$$

Elliptic operator in tangential direction, mass in radial direction.

Thank you