Functional Integrals for Bose-Fermi Systems

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Workshop on Quantum Field Theory, Renormalisation and Stochastic PDE

Isaac-Newton Institute, Cambridge, October 22, 2018

joint work with Maximilian A. Nitzschner

Physical motivation

Bosons and fermions exist in nature as fundamental and composite particles

Spin-statistics theorem: in 3+1 dimensions, under very general, physically meaningful axioms, Bose and Fermi statistics are the only possibilities for identical particles (integer spin $\to$ bosons; half-integer spin $\to$ fermions).

Quantum-mechanical n-particle wavefunctions $\xi_j = (x_j, \alpha_j)$ satisfy, for all permutations π on $\{1, \ldots, n\}$,

$$\psi(\xi_{\pi(1)}, \ldots \xi_{\pi(n)}) = (\pm 1)^{\pi} \, \psi(\xi_1, \ldots \xi_n)$$

with $(+1)^{\pi} = 1$ for bosons and $(-1)^{\pi} = \mathrm{sign}(\pi)$ for fermions.

Fermi statistics is crucial for stability of matter, existence of metals, etc.

Bose statistics plays a crucial role in many types of collective phenomena.

Specific systems

(Formal) Hamiltonian of a Bose-Fermi mixture of cold atomic gases

$$H_{CAG}^{(M,N)} = -\sum_{i=1}^{M} \frac{\hbar^2}{2M}\Delta_{X_i} - \sum_{j=1}^{N} \frac{\hbar^2}{2m}\Delta_{x_j} + g\sum_{i=1}^{M}\sum_{j=1}^{N} W(X_i - x_j) + \dots$$

in second-quantized form ($\mathtt{b}$ - bosons; $\mathtt{f}$ - fermions)

$$\mathtt{H}_{CAG} = -\int \mathrm{d}k\, \frac{\hbar^2 k^2}{2M} \hat{\mathtt{b}}^{\dagger}{}_k \hat{\mathtt{b}}_k - \int \mathrm{d}k\, \frac{\hbar^2 k^2}{2m} \hat{\mathtt{f}}^{\dagger}{}_k \hat{\mathtt{f}}_k + g \int \mathrm{d}x \int \mathrm{d}y\, W(x-y) \mathtt{b}_x^{\dagger} \mathtt{b}_x\, \mathtt{f}_y^{\dagger} \mathtt{f}_y + \dots$$

e.g. mixtures of ^{6}Li and ^{133}Cs.

(Formal) Hamiltonian for the electron-phonon system

$$\mathtt{H}_{\mathrm{el-ph}} = \sum_k \omega(k) \hat{\mathtt{b}}^{\dagger}{}_k \hat{\mathtt{b}}_k + \sum_k \varepsilon(k) \hat{\mathtt{f}}^{\dagger}{}_k \hat{\mathtt{f}}_k + g \sum_{q,k} \sqrt{\omega(k)} \left(\mathtt{b}_k^{\dagger} \mathtt{f}_q^{\dagger} \mathtt{f}_{q-k} + \mathtt{b}_k \mathtt{f}_q^{\dagger} \mathtt{f}_{q+k} \right)$$

Perspective

Interesting phenomena

Electrons and phonons – basic model of Fröhlich and Bardeen-Cooper-Schrieffer for superconductivity; astonishingly accurate

Atomic Bose-Fermi mixtures – bosons may condense, Fermions then interact with sound waves of the condensate. Under certain conditions, the Fröhlich Hamiltonian (with appropriate $\omega(k)$ gives a good description. But not always.

BEC-BCS crossover, etc.

Major mathematical goals

rigorously prove the existence of superconductivity at low temperatures

give a non-perturbative proof of Migdal's theorem

study gapped systems under the influence of small interactions

Role of functional integrals

Using functional integral representations for these systems allows to deal only with commutative (sub)algebras and hence to apply standard expansion methods of statistical mechanics.

The functional integral for Bose particles is complex and oscillatory.

Bałaban, Feldman, Knörrer, Trubowitz have proven that the time continuum limit exists in an analytically useful form, and use it in their renormalization group study of the Bose condensation problem.

C. Blois, in her PhD thesis with Joel Feldman, studied existence of functional integral representations, using Duhamel formulas and time-ordered expansions.

Here, we give a simple derivation of the functional integral that illustrates some important points, and discuss what it can be applied to.

Mathematical Setup

. . . the standard C^*-algebraic framework of quantum statistical mechanics.

start with a model that is regularized both at short and long distances ('ultraviolet and infrared')

infrared – consider particles in a box of finite sidelength L ($\rightarrow$ discrete Fourier spectrum)

ultraviolet – in the continuum, impose a cutoff restricting all energies to lie below a maximal one, or use a lattice.

The effect is in all cases to make the one-particle Hilbert space finite-dimensional (10^{25}, say).

We choose the lattice option. For most of what is done, no translation group or regularity of the lattice is needed; it can be any finite graph for each species, with vertex set Λ_B for the bosons and Λ_F for fermions.

Notation: for a set X and functions f and g defined on X, we define the bilinear form

$$(f \mid g)_X = \sum_{x \in X} f_x g_x$$

Spaces

One-particle Hilbert spaces: $\mathscr{H}_B = \mathbb{C}^{\Lambda_B}$ for bosons, $\mathscr{H}_F = \mathbb{C}^{\Lambda_F}$ for fermions.

n-particle Hilbert spaces:

$$\mathscr{H}_B^{(n)} = \mathscr{H}_B^{\otimes_s n} = \mathscr{H}_B \otimes_s \ldots \otimes_s \mathscr{H}_B \,. \qquad \dim_{\mathbb{C}} \mathscr{H}_B^{(n)} = \binom{n + |\Lambda_B| - 1}{n}$$

$$\mathscr{H}_F^{(n)} = \bigwedge^n \mathscr{H}_F = \mathscr{H}_F \wedge \ldots \wedge \mathscr{H}_F \qquad \dim_{\mathbb{C}} \mathscr{H}_F^{(n)} = \binom{|\Lambda_F|}{n}$$

The bosonic and fermionic Fock spaces are

$$\mathscr{F}_B = \bigoplus_{n=0}^{\infty} \mathscr{H}_B^{(n)} \text{ and } \mathscr{F}_F = \bigoplus_{n=0}^{\infty} \mathscr{H}_F^{(n)} \,.$$

The elements of $\mathscr{F}_B$ and $\mathscr{F}_F$ can be identified with sequences $(\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots)$.

Vacuum vectors $\Omega_B = (1, 0, 0, \ldots) \in \mathscr{F}_B$ and $\Omega_F = (1, 0, 0, \ldots) \in \mathscr{F}_F$

$\mathscr{F}_F$ is a finite-dimensional Grassmann algebra: dim $\mathscr{F}_F = 2^{|\Lambda_F|}$.

$\mathscr{F}_B$ is infinite-dimensional. The standard subspace of 'finite' vectors $\mathscr{F}_B^{(0)}$ is dense in $\mathscr{F}_B$.

The Fock space of our models of bosons and fermions is $\mathscr{F} = \mathscr{F}_B \otimes \mathscr{F}_F$.

$\mathscr{F}^{(0)} = \mathscr{F}_B^{(0)} \otimes \mathscr{F}_F$ is dense in $\mathscr{F}$. $\quad \Omega = \Omega_B \otimes \Omega_F$ is the vacuum vector in $\mathscr{F}$.

Operators

$\mathscr{L}_B{}^{(0)} = \{\mathtt{A} : \mathscr{F}_B{}^{(0)} \to \mathscr{F}_B : \mathtt{A} \text{ linear}\}$, $\mathscr{L}_F = \{\mathtt{A} : \mathscr{F}_F \to \mathscr{F}_F : \mathtt{A} \text{ linear}\}$. Every $\mathtt{A} \in \mathscr{L}_F$ is continuous.

For $\mathtt{A} \in \mathscr{L}_B^{(0)}$ and $\mathtt{B} \in \mathscr{L}_F$, the tensor product of $\mathtt{A}$ and $\mathtt{B}$ acts on $\mathscr{F}^{(0)}$ as

$$\mathtt{A} \otimes \mathtt{B} : \mathscr{F}^{(0)} \to \mathscr{F}, \qquad f_1 \otimes f_2 \mapsto (\mathtt{A} f_1) \otimes (\mathtt{B} f_2).$$

If $\mathtt{A}$ is bounded, this operator extends uniquely to all of $\mathscr{F}$, and we denote it the same way. We call the operators

$$\mathtt{A} \otimes \mathtt{1}_{\mathscr{F}_F} : \mathscr{F}^{(0)} \to \mathscr{F}, \qquad f_1 \otimes f_2 \mapsto (\mathtt{A} f_1) \otimes f_2,$$
$$\mathtt{1}_{\mathscr{F}_B} \otimes \mathtt{B} : \mathscr{F} \to \mathscr{F}, \qquad f_1 \otimes f_2 \mapsto f_1 \otimes (\mathtt{B} f_2).$$

purely bosonic and purely fermionic, respectively.

By this definition, a purely fermionic operator commutes with all purely bosonic operators and vice versa.

$\mathscr{L}^{(0)} = \{\mathtt{A} : \mathscr{F}^{(0)} \to \mathscr{F} : \mathtt{A} \text{ linear}\}$ is a $*$-algebra over $\mathbb{C}$ with unit element $\mathtt{1}$.

Raising and lowering operators

The bosonic creation and annihilation operators are defined on $\mathscr{F}_B^{(0)}$ and satisfy the canonical commutation relations (CCR)

$$\mathtt{b}_x \mathtt{b}^\dagger_{x'} - \mathtt{b}^\dagger_{x'} \mathtt{b}_x = \delta_{x,x'} \qquad \mathtt{b}_x \mathtt{b}_{x'} - \mathtt{b}_{x'} \mathtt{b}_x = 0 \qquad \text{for all } x, x' \in \Lambda_B \ .$$

They are unbounded, purely bosonic, operators on $\mathscr{F}^{(0)}$.

The fermionic creation and annihilation operators are defined on $\mathscr{F}_F$ and satisfy the canonical anticommutation relations (CAR)

$$\mathtt{f}_y \mathtt{f}^\dagger_{y'} + \mathtt{f}^\dagger_{y'} \mathtt{f}_y = \delta_{y,y'} \qquad \mathtt{f}_y \mathtt{f}_{y'} + \mathtt{f}_{y'} \mathtt{f}_y = 0 \qquad \text{for all } y, y' \in \Lambda_F \ .$$

Their norm is bounded by 1. They act as purely fermionic operators on the entire Fock space $\mathscr{F}$.

Note that these are unit-lattice normalizations.

Local and total number operators:

$$\mathtt{n}_B(x) = \mathtt{b}^\dagger_x \mathtt{b}_x \quad \mathtt{n}_F(y) = \mathtt{f}^\dagger_y \mathtt{f}_y \quad \mathtt{N}_B = \sum_{x \in \Lambda_B} \mathtt{n}_B(x) \quad \mathtt{N}_F = \sum_{y \in \Lambda_F} \mathtt{n}_F(y) \ .$$

Bose-Fermi Hamiltonians

The kinetic terms in the Hamiltonian are given by hermitian matrices $\Theta : \Lambda_B \times \Lambda_B \to \mathbb{C}$, $(x, x') \mapsto \Theta_{x,x'}$, and $Q : \Lambda_F \times \Lambda_F \to \mathbb{C}$, $(y, y') \mapsto Q_{y,y'}$.

Let $\mathtt{G} : \Lambda_B \to \mathscr{L}_F$, $x \mapsto \mathtt{G}_x$, be a map associating a purely fermionic operator to each $x \in \Lambda_B$, e.g.

$$\mathtt{G}_x = \sum_{y \in \Lambda_F} G_{x,y} \mathtt{f}_y^\dagger \mathtt{f}_y \ ,$$

with suitable choice of $G_{x,y}$. Let $g \in \mathbb{C}$ (the 'coupling constant'). The Hamiltonian associated to Θ, Q and $\mathtt{G}$ is

$$\mathtt{H} = \mathtt{H}_B(\Theta) + \mathtt{H}_F(Q)$$

where

$$\mathtt{H}_B(\Theta) = (\mathtt{b}^\dagger \mid \Theta \mathtt{b} \ )_{\Lambda_B} + g \ (\mathtt{b}^\dagger \mid \mathtt{G})_{\Lambda_B} + \bar{g} \ (\mathtt{b} \ \mid \mathtt{G}^\dagger)_{\Lambda_B}$$

and

$$\mathtt{H}_F(Q) = (\mathtt{f}^\dagger \mid Q \mathtt{f} \ )_{\Lambda_F} \ .$$

$\mathtt{H}$ is a symmetric linear operator from $\mathscr{F}^{(0)}$ to $\mathscr{F}^{(0)}$.

$\mathtt{H}$ extends uniquely to a self-adjoint operator on $\mathscr{F}$.

Grand canonical ensemble

Let $\beta > 0$, and μ_B and μ_F be real. Set

$$\mathtt{K} = \mathtt{K}(\mu_B, \mu_F) = \mathtt{H} - \mu_B \mathtt{N}_B - \mu_F \mathtt{N}_F$$

It is natural to split $\mathtt{K}$ into a purely fermionic part $\mathtt{H}_F$ and the rest by setting

$$\mathtt{H}_B(\Theta) - \mu_B \mathtt{N}_B = \mathtt{H}_B(\Theta'), \qquad \mathtt{H}_F(Q) - \mu_F \mathtt{N}_F = \mathtt{H}_B(Q')$$

with

$$\Theta' = \Theta - \mu_B, \qquad Q' = Q - \mu_F.$$

Thus

$$\mathtt{K}(\mu_B, \mu_F) = \mathtt{H}_B(\Theta') + \mathtt{H}_F(Q') \; .$$

Let $\mathtt{A}$ be an operator defined on $\mathscr{F}^{(0)}$. If $\exp(-\beta \mathtt{K}(\mu_B, \mu_F))\ \mathtt{A}$ extends to a trace class operator on $\mathscr{F}$, the unnormalized grand canonical expectation value of $\mathtt{A}$ is defined as

$$[\mathtt{A}] = \mathrm{Tr}_{\mathscr{F}} \left(\mathrm{e}^{-\beta \mathtt{K}(\mu_B, \mu_F)} \mathtt{A} \right)$$

The grand canonical partition function is $Z_g(\beta, \mu) = [\mathtt{1}_{\mathscr{F}}]$. If it converges, it is nonzero (shown below), and we define the normalized expectation of $\mathtt{A}$ as

$$\langle \mathtt{A} \rangle = \frac{1}{Z_g}\ [\mathtt{A}] = \frac{1}{Z_g}\ \mathrm{Tr}_{\mathscr{F}} \left(\mathrm{e}^{-\beta(\mathtt{H} - \mu_B \mathtt{N}_B - \mu_F \mathtt{N}_F)}\ \mathtt{A} \right) \; .$$

Iteration of traces

The trace over $\mathscr{F}$ can be taken as subsequent traces over $\mathscr{F}_B$ and $\mathscr{F}_F$, as follows. Let $D = \dim \mathscr{F}_F$ and $\{\eta_n : n \in \{1, \dots D\}\}$ be an ONB of $\mathscr{F}_F$. The *bosonic trace* of any trace-class operator $\mathtt{B}$ on $\mathscr{F}$, defined as

$$\mathrm{Tr}_{\mathscr{F}_B}(\mathtt{B}) = \sum_{n,n'=1}^{D} \mathrm{Tr}_{\mathscr{F}} \left[\mathtt{B}(\mathtt{1}_{\mathscr{F}_B} \otimes |\eta_n\rangle\langle\eta_{n'}|)\right] \quad |\eta_{n'}\rangle\langle\eta_n| \; ,$$

is an operator on the finite-dimensional space $\mathscr{F}_F$ (which is independent of the choice of ONB on $\mathscr{F}_F$ used in the definition), and

$$\mathrm{Tr}_{\mathscr{F}}(\mathtt{B}) = \mathrm{Tr}_{\mathscr{F}_F}\Big(\mathrm{Tr}_{\mathscr{F}_B}(\mathtt{B})\Big) \; .$$

For proper choice of μ_B, the sum for the bosonic trace of $\mathrm{e}^{-\beta \mathtt{K}(\mu_B,\mu_F)}\ \mathtt{A}$ is norm-convergent for $\mathtt{A} = \mathtt{1}_{\mathscr{F}}$ and for the choices of $\mathtt{A}$ defining the usual correlation functions (m-particle reduced density matrices).

Existence of the grand canonical trace

Theorem Under the above assumptions

a. There is $\mu_B \in \mathbb{R}$ such that $\mathtt{H}_B(\Theta) - \mu_B \mathtt{N}_B$ is bounded below.

b. $\mathtt{H}_B(\Theta)$ extends uniquely to a self-adjoint operator, densely defined on $\mathscr{F}_B$.

c. Let $\mu_* > 0$. Then, for all $\mu_B \in \mathbb{R}$ for which $\Theta - \mu_B \geq \mu_*$,

$$\mathtt{K}(\mu_B, \mu_F) \geq \frac{\mu_*}{2}\,\mathtt{N}_B - 2\tfrac{|g|^2}{\mu_*}\,(\mathtt{G}^\dagger, \mathtt{G})_{\Lambda_B} + H_F(Q - \mu_F)$$

d. For all $\mu_B \in \mathbb{R}$ for which $\Theta - \mu_B \geq \mu_*$ and for any polynomial $\mathtt{P}$ in the creation and annihilation operators and any $\beta > 0$, $\mathrm{e}^{-\beta \mathtt{K}(\mu_B, \mu_F)}\,\mathtt{P} : \mathscr{F}^{(0)} \to \mathscr{F}$ exists and is a bounded operator. Its unique extension to an operator on $\mathscr{F}$ is trace class.

Essential points of the proof

For sufficiently negative μ_B, $\Theta' \geq \mu_*$, so $(\mathtt{b}^\dagger \mid \Theta' \mathtt{b}\ )_{\Lambda_B} \geq \mu_* (\mathtt{b}^\dagger \mid \mathtt{b}\ )$ and

$$\mathtt{H}_B(\Theta) - \mu_B \mathtt{N}_B = \mathtt{H}_B(\Theta') \geq \mathtt{H}_B(\mu_*)$$

and

$$\mathtt{K}(\mu_B, \mu_F) = \mathtt{H} - \mu_B \mathtt{N}_B - \mu_F \mathtt{N}_F \geq \mathtt{H}_B(\mu_*) + \mathtt{H}_F(Q')\ .$$

Let $\gamma = \frac{g}{\mu_*}$. Because purely bosonic and purely fermionic operators commute,

$$\begin{aligned}
\mathtt{H}_B(\mu_*) &= \mu_* (\mathtt{b}^\dagger, \mathtt{b}\ )_{\Lambda_B} + g(\mathtt{b}^\dagger, \mathtt{G})_{\Lambda_B} + \bar{g}(\mathtt{b}\ , \mathtt{G}^\dagger)\Lambda_B \\
&= \mu_* \left((\mathtt{b}\ + \gamma \mathtt{G})^\dagger, \mathtt{b}\ + \gamma \mathtt{G} \right)_{\Lambda_B} - \mu_* |\gamma|^2\, (\mathtt{G}^\dagger, \mathtt{G})_{\Lambda_B} \\
&\geq -\frac{|g|^2}{\mu_*}\, (\mathtt{G}^\dagger, \mathtt{G})_{\Lambda_B}
\end{aligned}$$

The traces converge because

$$\mathrm{Tr}_{\mathscr{F}_B} \left(\mathrm{e}^{-\frac{\tilde{\beta}\mu_*}{2} \mathtt{N}_B} \right) = \sum_{n=0}^{\infty} \binom{n + |\Lambda_B| - 1}{n}\, \mathrm{e}^{-\frac{\tilde{\beta}\mu_*}{2} n}$$

and $\binom{n+k}{n} \leq \frac{(n+k)^k}{k!}$ grows only powerlike in n.

Bosonic coherent states

The (unnormalized) coherent state labelled by $\phi \in \mathbb{C}^{\Lambda_B}$ is

$$v_\phi = \mathrm{e}^{(\phi|\mathsf{b}^\dagger)_{\Lambda_B}}\,\Omega = \prod_{x\in\Lambda_B} \mathrm{e}^{\phi_x \mathsf{b}_x^\dagger}\,\Omega\,.$$

The exponential series converges because $\left\|(\mathsf{b}_x^\dagger)^n\Omega\right\| = \sqrt{n!}$.

Thus v_ϕ is entire analytic in $\phi \in \mathbb{C}^{\Lambda_B}$. Using this analyticity, one then shows that

$$\mathsf{b}_x v_\phi = \phi_x v_\phi$$

and

$$\langle v_\phi | v_{\phi'}\rangle_{\mathscr{F}} = \mathrm{e}^{(\bar\phi|\phi')_{\Lambda_B}}\,.$$

Resolution of unity and integral representation for the trace

These results are taken from T. Bałaban, J. Feldman, H. Knörrer, E. Trubowitz, Annales Henri Poincaré **9** 1229-1273, 2008.

For $r > 0$ let

$$\mathrm{d}\kappa^{(r)}(\bar\phi, \phi) = \prod_{x \in \Lambda_B} \tfrac{\mathrm{d}\bar\phi_x \wedge \mathrm{d}\phi_x}{2\pi \mathrm{i}} \; \mathbb{1}\left(|\phi_x| \le r\right)$$

and

$$\mathtt{1}^{(r)}_{\mathscr{F}_B} w = \int \mathrm{d}\kappa^{(r)}(\bar\phi, \phi) \mathrm{e}^{-(\bar\phi|\phi)_{\Lambda_B}} \; v_\phi \; \langle v_\phi | w \rangle_{\mathscr{F}_B} \; .$$

Then

(1) $\|\mathtt{1}^{(r)}_{\mathscr{F}_B}\| \le 1$ for all $r > 0$ and $\mathtt{1}^{(r)}_{\mathscr{F}_B} \longrightarrow \mathtt{1}_{\mathscr{F}_B}$ strongly as $r \to \infty$.

(2) If $\mathtt{B}$ is trace class on $\mathscr{F}_B$, then $\mathtt{B}\mathtt{1}^{(r)}_{\mathscr{F}_B}$ is trace class for any $r > 0$,

$$\mathrm{Tr}_{\mathscr{F}_B}\left(\mathtt{B}\,\mathtt{1}^{(r)}_{\mathscr{F}_B}\right) = \int \mathrm{d}\kappa^{(r)}(\bar\phi, \phi)\; \mathrm{e}^{-(\bar\phi|\phi)_{\Lambda_B}} \; \langle v_\phi \mid \mathtt{B}\; v_\phi \rangle_{\mathscr{F}_B} \; ,$$

and $\mathrm{Tr}_{\mathscr{F}_B}\left(\mathtt{B}\,\mathtt{1}^{(r)}_{\mathscr{F}_B}\right) \to \mathrm{Tr}_{\mathscr{F}_B}\left(\mathtt{B}\right)$ as $r \to \infty$.

Formula for the bosonic trace

For $w, w' \in \mathscr{F}_B$, we define the operator $\langle\!\langle w \| w' \rangle\!\rangle$ on $\mathscr{F}_F$ as

$$\langle\!\langle w \| w' \rangle\!\rangle = \sum_{n,n'=1}^{D} \langle w \otimes \eta_n \mid w' \otimes \eta_{n'} \rangle \quad |\eta_{n'}\rangle\langle\eta_n| \ .$$

Then by the above, the bosonic trace of a trace class operator $\mathtt{A}$ on $\mathscr{F}$ is given by the limit

$$\mathrm{Tr}_{\mathscr{F}_B}\left(\mathtt{A}\right) = \lim_{r\to\infty} \int \mathrm{d}\kappa^{(r)}(\bar{\phi}, \phi)\ \mathrm{e}^{-(\bar{\phi}|\phi)_{\Lambda_B}}\ \langle\!\langle v_\phi \| \mathtt{A} v_\phi \rangle\!\rangle \ .$$

This is still an operator on $\mathscr{F}_F$.

Trotterization

Using a Trotter product formula

$$\mathrm{e}^{-\beta K(\mu_B,\mu_F)} = \operatorname*{s-lim}_{p\to\infty} \left(\mathrm{e}^{-\frac{\beta}{p}\mathtt{H}_B(\Theta') + \frac{1}{p^2}\mathtt{R}}\ \mathrm{e}^{-\frac{\beta}{p}\mathtt{H}_F(Q')} \right)^p$$

where $\varepsilon = \frac{\beta}{p}$ and $\mathtt{R}$ is purely fermionic,

$$\mathrm{Tr}_{\mathscr{F}_B}\left(\mathrm{e}^{-\beta K}\ \mathtt{P}\right) = \lim_{p\to\infty}\lim_{r\to\infty}\mathcal{P}_{p,r}$$

with

$$\mathscr{P}_{p,r} = \int \prod_{q=0}^{p} \mathrm{d}\kappa^{(r)}(\bar{\phi}_q, \phi_q)\ \mathrm{e}^{-(\bar{\phi}_q|\phi_q)}$$
$$\prod_{q=0}^{p-1}\left(\langle\!\langle v_{\phi_q} \| \mathrm{e}^{-\varepsilon \mathtt{H}_B(\Theta')} v_{\phi_{q+1}} \rangle\!\rangle\ \mathrm{e}^{-\frac{\mathtt{R}}{p^2}}\ \mathrm{e}^{-\varepsilon\mathtt{H}_F(Q')} \right)\ \langle\!\langle v_{\phi_p} \| \mathtt{P}\, v_{\phi_0} \rangle\!\rangle$$

Matrix elements

$$\langle v_\phi \mid \mathrm{e}^{-\varepsilon(\mathtt{b}^\dagger|\Theta'\mathtt{b}\ )} v_{\phi'} \rangle = \mathrm{e}^{(\bar{\phi}|(\mathrm{e}^{-\varepsilon\Theta'}-1)\phi')} \langle v_\phi \mid v_{\phi'} \rangle = \mathrm{e}^{(\bar{\phi}|\mathrm{e}^{-\varepsilon\Theta'}\phi')} \ .$$

The purely fermionic operator $\mathtt{G}_x = \sum_{y\in\Lambda_F} G_{x,y}\mathtt{f}_y^\dagger \mathtt{f}_y$ commutes with all bosonic operators and $[\mathtt{G}_x, \mathtt{G}_{x'}^\dagger] = 0$. This implies that

$$[\mathtt{b}_x^\dagger + \mathtt{G}_x^\dagger, \mathtt{b}_{x'} + \mathtt{G}_{x'}] = \delta_{x,x'}$$

and therefore

$$\langle\!\langle v_\phi \| \mathrm{e}^{-\varepsilon \mathtt{H}_B(\Theta')} v_{\phi'} \rangle\!\rangle \ = \mathrm{e}^{(\bar{\phi}'|\mathrm{e}^{-\varepsilon\Theta'}\phi)_{\Lambda_B}} \ \mathrm{e}^{-\varepsilon\bar{g}(\mathtt{1}_\varepsilon\phi|\mathtt{G}^\dagger)_{\Lambda_B} - \varepsilon g(\bar{\phi}'|\mathtt{1}_\varepsilon \mathtt{G})_{\Lambda_B}} \ \mathrm{e}^{-\varepsilon^2|g|^2(\mathtt{G}^\dagger|\mathtt{1}_{\varepsilon^2}\mathtt{G})_{\Lambda_B}}$$

Choose $\mathtt{R}$ so that it cancels the last term.

The quadratic form in the bosonic fields

Collecting all the terms that are proportional to $\mathbb{1}_{\mathscr{F}_F}$, and with $\mathbb{T}_B = \{0, \ldots, p\} \times \Lambda_B$,

$$\begin{aligned}(\bar{\phi} \mid \mathscr{K}^{(p)}\, \phi)_{\mathbb{T}_B} &= \sum_{q,q'=0}^{p} (\bar{\phi}_q \mid \mathscr{K}^{(p)}_{q,q'}\, \phi_{q'})_{\Lambda_B} \\ &= \sum_{q=0}^{p} (\bar{\phi}_q \mid \phi_q)_{\Lambda_B} - \sum_{q=0}^{p-1} (\bar{\phi}_q \mid \mathrm{e}^{-\varepsilon\Theta'}\, \phi_{q+1})_{\Lambda_B} - (\bar{\phi}_p \mid \mathrm{e}^{-\varepsilon\Theta'}\, \phi_0)_{\Lambda_B}\end{aligned}$$

The inverse $\mathscr{D}^{(p)}$ of $\mathscr{K}^{(p)}$ is the covariance of the Gaussian measure that arises in the limit $r \to \infty$. It turns out to be the standard continuous-time two-point function of free bosons with kinetic energy operator Θ'

$$\mathscr{D}^{(p)}_{q,q'} = \mathbb{B}(\tau_{q'} - \tau_q, \Theta')$$

where

$$\mathbb{B}(\tau, E) = \begin{cases} -b_\beta(-E)\exp(-\tau E), & \text{for } \tau \geq 0 \\ b_\beta(E)\exp(-\tau E), & \text{for } \tau < 0, \end{cases}$$

and $b_\beta(E) = (\exp(\beta E) - 1)^{-1}$ is the Bose distribution function. The effect of the discretization is only that it gets evaluated at the discrete times τ_q.

The cutoff Gaussian measure

We introduce the notation

$$\mathrm{d}\mu^{(r)}_{\mathscr{D}^{(p)}}(\overline{\phi},\phi) = \frac{1}{Z_B^{(0,r)}} \int \prod_{q=0}^{p} \mathrm{d}\kappa^{(r)}(\bar{\phi}_q,\phi_q)\ \mathrm{e}^{-(\bar{\phi}|\mathscr{K}^{(p)}\ \phi)_{\mathbb{T}_B}}$$

where $Z_B^{(0,r)} > 0$ normalizes the measure, so that $\int \mathrm{d}\mu^{(r)}_{\mathscr{D}}(\overline{\phi},\phi) = 1$. Recall that $\kappa^{(r)}$ cuts off each $|\phi_{q,x}|$ at r. Only in the limit $r \to \infty$ does μ become a Gaussian measure. In this limit, $Z_B^{(0,r)} \to Z_B^{(0)}$, where

$$Z_B^{(0)} = \det \mathscr{D}^{(p)} = \det\left(1 - \mathrm{e}^{-\beta\Theta'}\right)$$

is the partition function for free bosons. We also use the notation

$$\left\langle f(\overline{\phi},\phi)\right\rangle_{\mathscr{D}^{(p)},r} = \int \mathrm{d}\mu^{(r)}_{\mathscr{D}^{(p)}}(\overline{\phi},\phi)\ f(\overline{\phi},\phi)\ .$$

Summary: integral formula for the bosonic trace

Let μ_B be such that $\Theta' > 0$, and let $\mathtt{P} = (\mathtt{b}^\dagger)^{\tilde{X}} \mathtt{b}^X \mathtt{F}$ where $\mathtt{F}$ is purely fermionic.

$$\mathrm{Tr}_{\mathscr{F}_B} \left(\mathrm{e}^{-\beta K} \, \mathtt{P} \right) = \lim_{p \to \infty} \lim_{r \to \infty} Z_B^{(0,r)} \; \left\langle \overline{\phi}_p{}^{\tilde{X}} \; \phi_0^X \; \Pi_{p,r}^{(F)}(\overline{\phi}, \phi) \right\rangle_{\mathscr{D}^{(p)},r}$$

with $\Pi_{p,r}^{(F)}(\overline{\phi}, \phi) \in \mathscr{L}_F$ given by

$$\Pi_{p,r}^{(F)}(\overline{\phi}, \phi) = \prod_{q=0}^{p-1} \mathrm{e}^{-\varepsilon \bar{g}(\mathtt{1}_\varepsilon \phi_{q+1} | \mathtt{G}^\dagger)_{\Lambda_B} - \varepsilon g(\bar{\phi}_q | \mathtt{1}_\varepsilon \mathtt{G})_{\Lambda_B}} \; \mathrm{e}^{-\varepsilon \mathtt{H}_F(Q')} \; \mathtt{F} \; .$$

$\Pi_{p,r}^{(F)}(\overline{\phi}, \phi)$ is still a product of noncommuting operators on $\mathscr{F}_F$.

The Grassmann integral for the fermionic trace

The coherent states have fermionic analogues, with Grassmann variables $\bar{\psi}, \psi$ instead of the complex variables $\overline{\phi}, \phi$.

The Grassmann integral for the fermionic trace

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Everything looks formally similar to the bosonic case.[1] In particular

$$\langle v_{\bar{\psi}} \mid \mathrm{e}^{-\varepsilon(\mathtt{f}^\dagger | Q \mathtt{f}\ )_{\Lambda_F}} v_{\psi'} \rangle = \mathrm{e}^{(\bar{\psi} | \mathrm{e}^{-\varepsilon Q} \psi')_{\Lambda_F}}$$

leading to a Grassmann Gaussian measure with covariance $\mathscr{C}^{(p)}$, which is given by the the fermionic time-ordered two-point function:

$$\mathscr{C}^{(p)}_{q,q'} = \mathbb{C}(\tau - \tau', Q')$$

where

$$\mathbb{C}(\tau, E) = \begin{cases} -f_\beta(-E) \exp(-\tau E) & \text{for } \tau > 0 \\ f_\beta(E) \exp(-\tau E) & \text{for } \tau \le 0 \ , \end{cases}$$

and $f_\beta(E) = (1 + \exp(\beta E))^{-1}$ is the Fermi distribution function.

[1] although it has quite different algebraic properties

But...

in the term coupling bosons and fermions, one also gets

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Still, we have the

Theorem. Let the bosonic chemical potential μ_B be chosen such that $\Theta' > 0$. Then for all $X, \tilde X, Y$, and $\tilde Y$, the unnormalized grand canonical expectation value of $\mathtt{P} = (\mathtt{b}^\dagger)^{\tilde X}\mathtt{b}^X(\mathtt{f}^\dagger)^{\tilde Y}\mathtt{f}^Y$ is

$$[\mathtt{P}] = Z_B^{(0)}\ Z_F^{(0)}\ \lim_{p\to\infty}\left\langle \mathrm{e}^{-\varepsilon\bar g(\phi_{q+1}|\bar G\bar\psi_q\psi_q)_{\Lambda_B}-\varepsilon g(\bar\phi_q|G\bar\psi_q\psi_q)_{\Lambda_B}}\ \overline{\phi}_p^{\tilde X}\,\phi_0^X\,\overline{\psi}_p^{\tilde Y}\,\psi_0^Y\right\rangle_{\mathscr{D}^{(p)},\mathscr{C}^{(p)}} \ .$$

Pauli rides to the rescue

The nilpotency of the Grassmann variables helps at this point. Because the coupling is to local fermionic bilinears, hence

$$
\begin{aligned}
&\mathrm{e}^{\sum_{q=0}^{p}(\bar\psi_q|(\mathrm{e}^{-\varepsilon g(G\overline{\phi}_q)}-1)\;\psi_q)_{\Lambda_F}} \\
&= \prod_{(q,y)\in\mathbb{T}_F} \mathrm{e}^{(\mathrm{e}^{-\varepsilon g(G\overline{\phi}_q)_y}-1)\;\bar\psi_q(y)\,\psi_q(y)} \\
&= \prod_{(q,y)\in\mathbb{T}_F} \Big(1+(\mathrm{e}^{-\varepsilon g(G\overline{\phi}_q)_y}-1)\;\bar\psi_q(y)\,\psi_q(y)\Big)
\end{aligned}
$$

by nilpotency of the Grassmann variables. Thus we can arrange things to get only exponents linear in ϕ and $\bar\phi$. It is, however, not trivial to implement this observation: expanding the product as

$$
\sum_{T\subset\mathbb{T}_F}\prod_{(q,y)\in T}(\mathrm{e}^{-\varepsilon g(G\overline{\phi}_q)_y}-1)\;\bar\psi_q(y)\,\psi_q(y)\;,
$$

makes clear that care has to be taken controlling the sum over T, since $|\mathbb{T}_F| = 2^{p|\Lambda_F|}$, hence there potentially is a serious growth in p in estimates.

Strategy for estimates

The quadratic form defining the Gaussian integral is complex-valued. At fixed p, $|\mathrm{e}^{-(\overline{\phi}|\mathscr{K}^{(p)}\phi)}| = \mathrm{e}^{-\mathrm{Re}\,(\overline{\phi}|\mathscr{K}^{(p)}\phi)}$ decays, so one can take absolute values and take $r \to \infty$. Estimates obtained by putting in absolute values at this stage are non-uniform in p.

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For the bosonic integrals we use

$$\int \mathrm{d}\mu_{\mathscr{D}^{(p)}}(\overline{\phi}, \phi) \prod_{i=1}^{m} \overline{\phi}_{q_i', x_i'} \prod_{j=1}^{n} \overline{\phi}_{q_j, x_j} = \delta_{m,n} \mathrm{perm} \left(\mathscr{D}^{(p)}_{(q_i', x_i'), (q_j, x_j)} \right)_{i,j} .$$

where perm(...) denotes the *permanent* of the matrix. The permanent is then estimated by

$$n!\, \kappa^n, \qquad \kappa = \sup_{q', x', q' x} \left| \mathscr{D}^{(p)}_{(q', x'), (q, x)} \right|$$

The factorials are controllable because the coupling of ϕ to the fermionic bilinears is only linear. κ is finite because $\Theta' > 0$.

The effective fermionic theory

Now, finally, the action is quadratic in the ϕ's. Integrating over the bosons gives an interaction (in continuum notation $\varepsilon \sum_q f(\tau_q) = \int \mathrm{d}\tau f(\tau)$)

$$W(\bar{\psi}, \psi) = -|g|^2 \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' \, (\bar{G}\bar{\psi}\psi)(\tau, x) \, \mathbb{B}(\tau' - \tau, \Theta') \, (G\bar{\psi}\psi)(\tau', x') \, .$$

Persistence of exponential decay of truncated correlations when both Q and Θ have spectrum away from zero holds in the limit $\beta \to \infty$. This generalizes the statement proven for a general class of purely fermionic Hamiltonians in [W. De Roeck & MS, arXiv:1712.00977].

The proof is based on fermionic tree expansion techniques (for an exposition, see, e.g. [MS & C. Wieczerkowski, J. Stat. Phys. **99** (2000) 557–586]).

This does not require any assumptions on translation invariance: the determinant bound is finite for any self-adjoint Q.

The gapless case

The case where $\Theta \geq 0$ is of special interest for the models discussed in the beginning (acoustic phonons have $\omega(k) = c|k|$ and $\mu_B = 0$; c is the speed of sound).

The fact that the coupling to the fermion density also vanishes at $k = 0$ in this model allows to remove the zero mode and prove convergence of the time-continuum limit.

Because G is real in this case, the boson propagator gets symmetrized and the resulting phonon interaction is the standard one,

$$\hat{D}(k_0, k) = \frac{\omega(k)^2}{k_0^2 + \omega(k)^2} \qquad \omega(k) = c|k| \; .$$

For this interaction, we have shown to all orders in perturbation theory using Wilson renormalization group iterations that vertex corrections are uniformly bounded by const c (Migdal's theorem). The representation derived here can be useful to prove it beyond perturbation theory (above T_c).

Conclusion and Outlook

Even in the case of a linear coupling, showing the convergence of the fermionic action is not completely obvious, but possible.

The result is the expected interaction that includes retardation effects in the form of the boson propagator.

The derivation did not assume that the interactions are weak.

We are also working on the case of a quartic Bose-Fermi interaction. This is a borderline case for the expansion used in the presented proof (quadratic perturbation).