

Edge universality in interacting *2d* topological insulators

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Joint with: G. Antinucci (UZH) and V. Mastropietro (Milan)

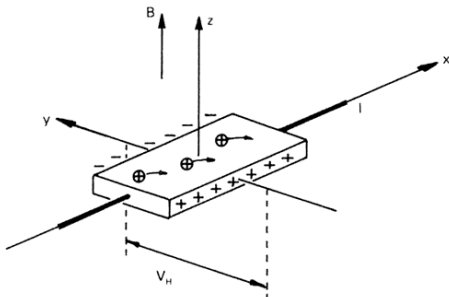
Summary

- Introduction: **edge transport** in noninteracting quantum Hall systems and time-reversal invariant systems. **Bulk-edge duality**.
- **Many-body quantum systems**. Results:
 - Edge transport coefficients for quantum Hall and TRI systems.
 - Interacting bulk-edge correspondence, Haldane relations.
- Sketch of the proof: **Renormalization group** and **Ward identities**.
- Conclusions.

Introduction: noninteracting systems

Integer quantum Hall effect

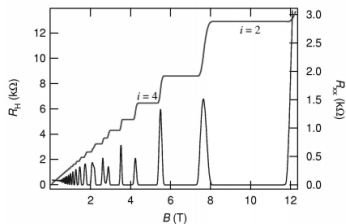
- **Bulk** topological order in condensed matter systems is deeply related to the emergence of **gapless** edge modes.
- **Example.** Integer quantum Hall effect [von Klitzing *et al.* '80]
 $2d$ insulators exposed to strong magnetic field and in-plane electric field.



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- **Example.** Integer quantum Hall effect [von Klitzing *et al.* '80]
 $2d$ insulators exposed to strong magnetic field and in-plane electric field.
Linear response: $J = \sigma E + o(E)$ with $\sigma =$ **conductivity matrix:**

$$\sigma = \begin{pmatrix} 0 & \frac{n}{2\pi} \\ -\frac{n}{2\pi} & 0 \end{pmatrix}, \quad n \in \mathbb{Z}.$$



Integer quantum Hall effect: theory

- **Noninteracting fermions.** $H = 1$ -particle Hamiltonian, on $\ell^2(\mathbb{Z}^2; \mathbb{C}^M)$. Suppose that $\sigma(H)$ is **gapped**, $\mu = \text{Fermi level} \in \text{gap}(H)$.



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- For simplicity, $H(x; y) \equiv H(x - y)$. **Bloch decomp.:** $H = \int_{\mathbb{T}^2}^{\oplus} dk \hat{H}(k)$
Let $\hat{P}_\mu(k) = \chi(\hat{H}(k) \leq \mu) = \text{Fermi projector}$. **Thouless et al. '82:**

$$\sigma_{12} = i \int_{\mathbb{T}^2} \frac{dk}{(2\pi)^2} \text{Tr}_{\mathbb{C}^M} \hat{P}_\mu(k) [\partial_{k_1} \hat{P}_\mu(k), \partial_{k_2} \hat{P}_\mu(k)] \in \frac{1}{2\pi} \mathbb{Z}$$

$\sigma_{12} = \text{Chern number}$ of Bloch bundle:

$$\mathcal{E}_B = \{(k, u) \in \mathbb{T}^2 \times \mathbb{C}^M \mid u \in \text{Ran} \hat{P}_\mu(k)\}$$

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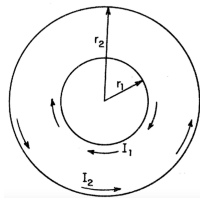
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- IQHE for general (**disordered**) systems:
 - **Bellissard *et al.* '94.** $\sigma_{12} =$ **Noncommutative** Chern number.
 - **Avron-Seiler-Simon '94.** $\sigma_{12} =$ **index** of a pair of projections.
 - **Aizenman-Graf '98.** Strong disorder \Rightarrow **Hall plateaux.**

Edge states in quantum Hall systems

- Halperin '82. Hall phases must come with **robust edge currents**.



- **Field theoretic approach.** For a weak, slowly varying vector potential A ,

$$\begin{aligned} \frac{\mathcal{Z}(A)}{\mathcal{Z}(0)} &= e^{i\sigma_{12} \int_{\Omega} A \wedge dA + \text{irr.}} && \text{(gap assumption)} \\ &= e^{i\sigma_{12} \int_{\Omega} (A + d\alpha) \wedge d(A + d\alpha) + \text{irr.}} && \text{(gauge inv.)} \\ &= \frac{\mathcal{Z}(A)}{\mathcal{Z}(0)} e^{i\sigma_{12} \int_{\partial\Omega} d\alpha \wedge A + \text{irr.}} && \text{(Stokes)} \end{aligned}$$

$\sigma_{12} \neq 0 \Rightarrow$ The gap assumption **cannot be true!** [Fröhlich '90+].

Edge states in quantum Hall systems: more precise

- Let H be a lattice Schrödinger operator on the **cylinder**:

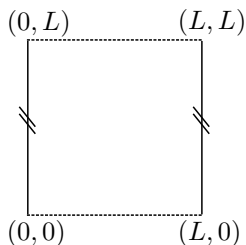


Figure: Dotted lines: **Dirichlet** boundary conditions. Identify vertical sides.

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- Let H be a lattice Schrödinger operator on the **cylinder**:
- Let H_p the counterpart of H with **periodic** b.c.. **Hyp.:** H_p is **gapped**.

Edge states in quantum Hall systems: more precise

- Let H be a lattice Schrödinger operator on the **cylinder**:
- Let H_p the counterpart of H with **periodic** b.c.. **Hyp.:** H_p is **gapped**.
 $\sigma(H)$ might differ from $\sigma(H_p)$ by the presence of **edge states**.

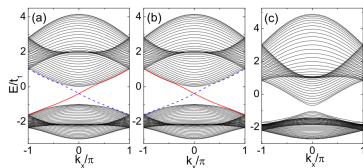


Figure: $H = \int_{\mathbb{T}^1}^{\oplus} dk_1 \hat{H}(k_1)$, $\hat{H}(k_1) = 1d$ Hamiltonian. **Spectrum** of $\hat{H}(k_1)$.

- **Red curve:** eigenvalue branch $\varepsilon(k_1)$, with eigenstates (**edge modes**)

$$\varphi_x(k_1) = e^{ik_1 x_1} \xi_{x_2}(k_1), \quad \text{with } \xi_{x_2}(k_1) \sim e^{-cx_2}.$$

The bulk-edge correspondence

- Bulk-edge duality:** relation between σ_{12} of H_p and the edge states of H .

$$\sigma_{12} = \sum_e \frac{\omega_e}{2\pi}$$

with $\omega_e = \pm 1$ (**chirality** of the edge state.)

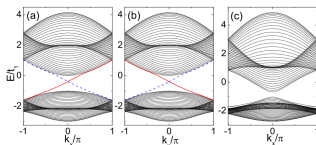


Figure: (a) : $\sigma_{12} = \frac{1}{2\pi}$, (b) : $\sigma_{12} = -\frac{1}{2\pi}$, (c) : $\sigma_{12} = 0$.

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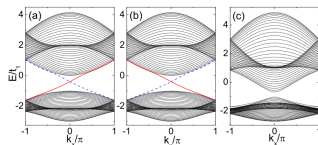
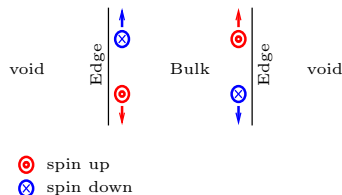


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- Rigorous results for **noninteracting systems:**
 - Hatsugai, '93: Translation invariant systems.
 - Schulz-Baldes *et al.* '00: Disordered systems (with bulk gap).
 - Graf *et al.* '02: Anderson localization regime.

Time-reversal invariant systems

- Quantum Hall systems are an example of **topological insulators**.
Necessary condition for $\sigma_{12} \neq 0$: **breaking** of TRS (magnetic field).
- Unbroken** TRS: charge transport is trivial but **spin transport** is possible.



- Spin Hall effect**: Murakami-Nagaosa-Zhang '03, ... (Fröhlich et al. '93.) **Model**: Kane-Mele '05. **Discovery**: Bernevig-Hughes-Zhang '06 (theory), König *et al.* '07.

Edge spin transport

- Gapped **TRI** model on a cylinder, Hamiltonian $H = \int_{\mathbb{T}^1}^{\oplus} dk_1 \hat{H}(k_1)$.
 TRS: $\hat{H}(k_1) = \Theta^{-1} \hat{H}(-k_1) \Theta$, with $\Theta^2 = -1$, Θ antiunitary.

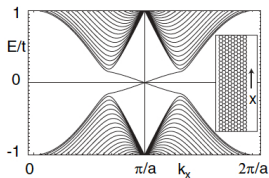
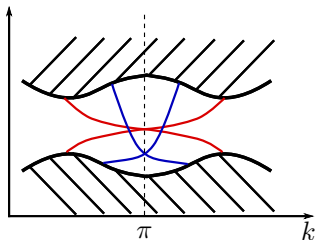


Figure: $\sigma(\hat{H}(k_1))$ for the **Kane-Mele** model. $\sigma(\hat{H}(k_1)) = \sigma(\hat{H}(-k_1))$.

- Eigenvalues at $k_1 = -k_1$ are **even degenerate** (Kramers degeneracy).
 \Rightarrow (**edge**) \mathbb{Z}_2 classification of H : **parity** of # of pairs of edge modes at μ .

Edge spin transport

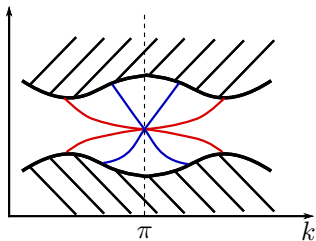
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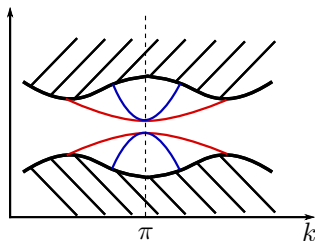
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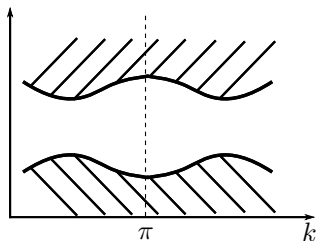
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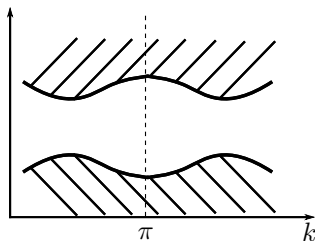
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- Eigenvalues at $k_1 = -k_1$ are **even degenerate** (Kramers degeneracy).
 \Rightarrow (**edge**) \mathbb{Z}_2 classification of H : **parity** of # of pairs of edge modes at μ .
- **Bulk** \mathbb{Z}_2 classif. is also possible (no direct connection with transport).
- **Graf-P. '13**: **bulk-edge duality** for TRI systems.

Many-body quantum systems

Many-body quantum systems

- **Interacting** many-body Fermi system on $\Lambda_L \subset \mathbb{Z}^2$.
- **Fock space Hamiltonian:** $\mathcal{H} = \mathcal{H}_0 + \lambda\mathcal{V}$ with

$$\mathcal{H}_0 = \sum_{x,y} \sum_{\sigma,\sigma'} a_{x,\sigma}^+ H(x,y) a_{y,\sigma}^- , \quad \mathcal{V} = \sum_{x,y} \sum_{\sigma,\sigma'} v(x-y) a_{x,\sigma}^+ a_{y,\sigma'}^+ a_{y,\sigma'}^- a_{x,\sigma}^-$$

with $\{a_{x,\sigma}^+, a_{y,\sigma'}^-\} = \delta_{x,y} \delta_{\sigma,\sigma'}$, $\{a_{x,\sigma}^+, a_{y,\sigma'}^+\} = \{a_{x,\sigma}^-, a_{y,\sigma'}^-\} = 0$,
and H, v short ranged.

- Finite volume, finite temperature **Gibbs state:**

$$\langle \cdot \rangle_{\beta,L} = \frac{\text{Tr} \cdot e^{-\beta(\mathcal{H}-\mu\mathcal{N})}}{\mathcal{Z}_{\beta,L}} , \quad \mathcal{Z}_{\beta,L} = \text{Tr} e^{-\beta(\mathcal{H}-\mu\mathcal{N})} , \quad \beta = 1/T$$

with μ = chemical potential and:

$$\mathcal{N} = \sum_x \sum_{\sigma} a_{x,\sigma}^+ a_{x,\sigma}^- \equiv \sum_x n_x .$$

Interacting bulk transport

- Periodic boundary conditions. **Many-body Kubo formula:**

$$\sigma_{ij} = \lim_{\eta \rightarrow 0^+} \lim_{\beta, L \rightarrow \infty} \frac{i}{\eta L^2} \left(\int_{-\infty}^0 dt e^{\eta t} \langle [\mathcal{J}_i(t), \mathcal{J}_j] \rangle_{\beta, L} - \langle [\mathcal{J}_i, \mathcal{X}_j] \rangle_{\beta, L} \right)$$

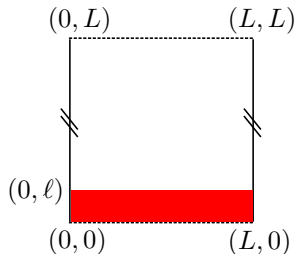
$\mathcal{X} = \sum_x x n_x$, $\mathcal{J} = i[\mathcal{H}, \mathcal{X}] =$ **current** operator and $\mathcal{J}_i(t) = e^{i\mathcal{H}t} \mathcal{J}_i e^{-i\mathcal{H}t}$.

- **Hastings-Michalakis '15. Quantization** of σ_{12} . (Quasi-adiabatic methods)
Hyp.: the ground state of \mathcal{H} is **gapped**.
- **Giuliani-Mastropietro-P. '16. Universality** of σ_{ij} . (RG & Ward identities)
Hyp.: fast enough **algebraic** decay of corr.. E.g.: gapped ground states; graphene-like models (+**Jauslin '16:** critical Haldane model).
- **Bachmann-de Roeck-Fraas '17. Validity** of Kubo formula.
Hyp.: the ground state of $\mathcal{H}(t)$ is gapped for all times.

Interacting edge transport

- **Edge transport.** Localize observables at distance $\leq \ell$ from $x_2 = 0$.

(Cylindric boundary conditions
& transl. inv. in direction x_1 .)



- Interesting quantities: **charge density** n_x and **current density** \vec{j}_x ,

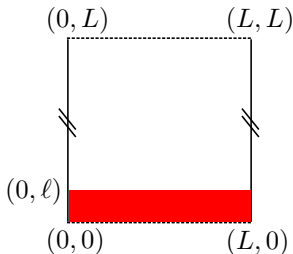
$$n_x = a_x^+ a_x^- , \quad \partial_t n_x(t) + \text{div}_x \vec{j}_x(t) = 0 .$$

$$\text{Let: } \hat{n}_{p_1} = \sum_{x=(x_1, x_2)} e^{ip_1 x_1} n_x \quad \text{and} \quad \hat{n}_{p_1}^\ell = \sum_{x_1} e^{ip_1 x_1} \sum_{x_2 \leq \ell} n_x .$$

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- **Spin** transport (current well defined if $[\mathcal{H}, \mathcal{S}_3] = 0$, with $\mathcal{S}_3 = \sum_\sigma \sigma n_{x, \sigma}$):

$$n_x \rightarrow n_{x, \uparrow} - n_{x, \downarrow} , \quad \dot{j}_{1, x} \rightarrow \dot{j}_{1, x, \uparrow} - \dot{j}_{1, x, \downarrow}$$

Edge transport coefficients

- Edge charge susceptibility:

$$\kappa^\ell(\eta, p_1) := - \lim_{\beta, L \rightarrow \infty} \frac{i}{L} \int_{-\infty}^0 dt e^{t\eta} \langle [\hat{n}_{p_1}(t), \hat{n}_{-p_1}^\ell] \rangle_{\beta, L}$$

(response of the edge density to a density perturbation)

- Edge charge conductance:

$$G^\ell(\eta, p_1) := - \lim_{\beta, L \rightarrow \infty} \frac{i}{L} \int_{-\infty}^0 dt e^{t\eta} \langle [\hat{n}_{p_1}(t), \hat{j}_{1, -p_1}^\ell] \rangle_{\beta, L}$$

(response of the edge current to a density perturbation)

- Edge Drude weight:

$$D^\ell(\eta, p_1) := \lim_{\beta, L \rightarrow \infty} \frac{i}{L} \int_{-\infty}^0 dt e^{t\eta} \langle [\hat{j}_{1, p_1}(t), \hat{j}_{1, -p_1}^\ell] \rangle_{\beta, L}$$

(response of the edge current to an electric field)

Effective description of the edge modes

- Effective $1d$ theory for a **single** edge mode: **chiral Luttinger model**.

$$\mathcal{H}_{\chi L} = \sum_{\sigma=\uparrow\downarrow} \int dk v_e k \hat{a}_{k,\sigma}^+ \hat{a}_{k,\sigma}^- + \lambda \int dp dk dk' \hat{a}_{k+p,\uparrow}^+ \hat{a}_{k'-p,\downarrow}^+ \hat{a}_{k,\downarrow}^- \hat{a}_{k',\uparrow}^-$$

- Wen '90**. Theory of interacting Hall edge currents based on χL .
Advantage: χL exactly solvable by **bosonization** [Mattis-Lieb '65.]
- Effective $1d$ theory for TRI systems: **helical Luttinger model**.

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- Remark**. Integrability is **nongeneric**: broken by, e.g., nonlinearities of the dispersion relation or by the bulk degrees of freedom.

Interacting edge transport: single mode edge currents

Theorem (Antinucci-Mastropietro-P., Comm. Math. Phys. '18)

Suppose that H has *one edge mode* per edge. Then, $\exists \lambda_0 > 0$ s.t. for $|\lambda| < \lambda_0$ the Gibbs state is analytic in λ . Moreover, the edge transport coefficients are:

$$\begin{aligned}\kappa^\ell(\eta, p_1) &= \frac{1}{\pi|v|} \frac{vp_1}{-i\eta + vp_1} + R_\kappa^\ell(\eta, p_1) \\ G^\ell(\eta, p_1) &= \frac{\omega}{\pi} \frac{vp_1}{-i\eta + vp_1} + R_G^\ell(\eta, p_1), \quad (\omega = \text{sgn}(v)) \\ D^\ell(\eta, p_1) &= \frac{|v|}{\pi} \frac{-i\eta}{-i\eta + vp_1} + R_D^\ell(\eta, p_1)\end{aligned}$$

$v \equiv v(\lambda) =$ *dressed Fermi velocity*, $\lim_{\ell \rightarrow \infty} \lim_{\eta, p_1 \rightarrow 0} R_\#^\ell(\eta, p_1) = 0$.

- The results agrees with the predictions based on **bosonization**:
“edge states \simeq noninteracting $1d$ **Bose** gas”.
- Proof based on **renormalization group methods** and on a rigorous comparison with χL [Benfatto-Falco-Mastropietro '10+].

Bulk-edge correspondence and Haldane relations

- Interacting **bulk-edge duality**:

$$G = \lim_{\ell \rightarrow \infty} \lim_{p_1, \eta \rightarrow 0^+} G^\ell(\eta, p_1) = \frac{\omega}{\pi}$$

$$\underbrace{\hspace{10em}}_{\text{bulk-edge corresp.}} = \sigma_{12}(\lambda = 0)$$

The bulk-edge duality follows from bulk universality: $\sigma_{12}(0) = \sigma_{12}(\lambda)$.

- In contrast, the Drude weight and the susceptibility are **nonuniversal**:

$$\kappa = \lim_{\ell \rightarrow \infty} \lim_{p_1, \eta \rightarrow 0^+} \kappa^\ell(\eta, p_1) = \frac{1}{\pi|v|}$$

$$D = \lim_{\ell \rightarrow \infty} \lim_{\eta, p_1 \rightarrow 0^+} D^\ell(\eta, p_1) = \frac{|v|}{\pi}. \quad (v \equiv v(\lambda))$$

Nevertheless, they satisfy the **Haldane relation**:

$$\frac{D}{\kappa} = v^2$$

first predicted to hold for 1d systems by [Haldane '80].

Interacting edge transport: TRI systems

Theorem (Mastropietro-P., Phys. Rev. B '17)

Suppose that \mathcal{H} is TRS, and that H has one *pair of edge states* per edge. Also, suppose that $[\mathcal{H}, \mathcal{S}_3] = 0$. Then, $\exists \lambda_0 > 0$ s.t. for $|\lambda| < \lambda_0$:

$$G^s = \frac{\omega}{\pi}, \quad \omega = \text{sgn}(v).$$

Moreover the charge and spin edge Drude weights and susceptibilities are:

$$\kappa^c = \frac{K}{\pi v}, \quad D^c = \frac{vK}{\pi}, \quad \kappa^s = \frac{1}{\pi v K}, \quad D^s = \frac{v}{\pi K}$$

with $K \equiv K(\lambda) = 1 + O(\lambda) \neq 1$, $v \equiv v(\lambda) = v_\uparrow + O(\lambda)$. Finally, the 2-point function decays with *anomalous exponent* $\eta = (K + K^{-1} - 2)/2$.

Remark. In the single edge mode case, $K = 1$ (no anomalous exponents.)

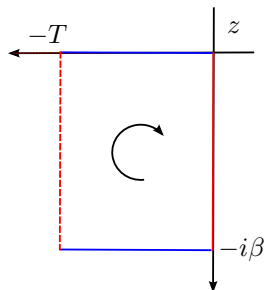
Sketch of the proof

(one edge mode)

Wick rotation

- Analytic continuation to **imaginary times**. Let $\eta_\beta \in \frac{2\pi}{\beta} \mathbb{N}_+$. We have:

$$\int_{-\infty}^0 dt e^{t\eta_\beta} \langle [A(t), B] \rangle_{\beta, L} = i \int_0^\beta dt e^{-it\eta_\beta} \langle A(-it)B \rangle_{\beta, L}$$



- Errors (dotted **red**) estimated via bounds on **Euclidean** correlations:

$$|\langle A(T - it)B \rangle_{\beta, L}| \leq \langle A(-it)A(-it)^* \rangle_{\beta, L}^{1/2} \langle B^* B \rangle_{\beta, L}^{1/2}$$

Perturbation theory

- Transport coefficients can be expressed via **imaginary time** correlations:

$$G^\ell(\eta, p_1) = \lim_{\beta, L \rightarrow \infty} \int_{-\beta/2}^{\beta/2} e^{-i\eta t} \frac{1}{L} \langle \mathbf{T} \hat{n}_{p_1}(-it); \hat{j}_{-p_1}^\ell \rangle_{\beta, L}.$$

- Let $A_t \equiv A(-it)$. Perturbative expansion of Euclidean correlations:

$$\langle \mathbf{T} A_t; B \rangle_{\beta, L} = \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{[0, \beta]^n} dt_1 \dots dt_n \langle \mathbf{T} A_t; B; \mathcal{V}_{t_1}; \dots; \mathcal{V}_{t_n} \rangle_{\beta, L} \Big|_{\lambda=0}$$

\rightsquigarrow Expansion in terms of **Feynman diagrams**. Covariance ($\beta, L \rightarrow \infty$):

$$g(t_1, x; t_2, y) = \langle \mathbf{T} a_{(t_1, x)}^- a_{(t_2, y)}^+ \rangle \Big|_{\lambda=0} = \\ \theta(t_1 - t_2) e^{(t_2 - t_1)(H - \mu)} P_\mu^\perp(H) - \theta(t_2 - t_1) e^{(t_2 - t_1)(H - \mu)} P_\mu(H)$$

- Problems.** 1) $(2n)!$ diagrams; 2) **gapless** modes: slow space-time decay.

Grassmann integral formulation

$$\frac{\mathrm{Tr} e^{-\beta\mathcal{H}}}{\mathrm{Tr} e^{-\beta\mathcal{H}_0}} = \int \mu(d\psi) e^{V(\psi)}$$

- $\psi_{\mathbf{x}}^{\pm}$ = Grassmann field, $V(\psi) = “\lambda\psi^4”$, $\mu(d\psi) = N^{-1} e^{-(\psi^+, g^{-1}\psi^-)} d\psi$
- $\psi = \psi_e + \psi_b$, where ψ_b has **gapped** covariance $g^{(\mathrm{bulk})} \equiv g\chi(|H - \mu| > \delta)$.

$$\int \mu(d\psi) e^{V(\psi)} = \int \mu_e(d\psi_e) \mu_b(d\psi_b) e^{V(\psi_e + \psi_b)} \equiv e^{F_{\beta, L}^{(b)}} \int \mu_e(d\psi_e) e^{V^{(e)}(\psi_e)}$$

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Brydges-Battle-Federbush formula. Solution of $2n!$ problem.

$$\langle \psi_b^{P_1}; \dots; \psi_b^{P_n} \rangle_{\mu_b} = \sum_{T \in \mathbf{T}} \alpha_T \prod_{\ell \in T} g_{\ell}^{(\mathrm{bulk})} \int \nu_T(d\mathbf{t}) \det G_T^{(\mathrm{bulk})}(\mathbf{t})$$

$$\begin{aligned} T &= \text{spanning tree of } \{P_i\}, & \#\{T\} &\leq Cn!, \\ \|\det G_T^{(\mathrm{bulk})}(\mathbf{t})\|_{\infty} &\leq C^n, & |g_{\ell}^{(\mathrm{bulk})}| &\leq (C/\delta) e^{-c\delta\|\ell\|}. \end{aligned}$$

Effective 1d model: RG analysis

$$\int \mu_e(d\psi_e) e^{V^{(e)}(\psi_e)} = \int \mu_{1d}(d\varphi) e^{V^{(e)}(\varphi * \xi^{(e)})} \equiv \int \mu_{1d}(d\varphi) e^{V^{(1d)}(\varphi)}$$

- $(\widehat{\varphi * \xi^{(e)}})(\omega, k_1, x_2) = \hat{\xi}_{x_2}^e(k_1) \hat{\varphi}_{(\omega, k_1)}$ with $\xi^{(e)} = \text{edge state}$ and:

$$\langle \hat{\varphi}_{(\omega, k_1)}^- \hat{\varphi}_{(\omega, k_1)}^+ \rangle = \frac{\chi_e(k_1)}{-i\omega + \varepsilon(k_1) - \mu} \simeq \frac{\chi_e(k_1)}{-i\omega + v(k_1 - k_F)} \quad \chi L \text{ model}$$

- **Multiscale evaluation** of the Grassmann integral:

[Gawedzki, Kupiainen, Feldman, Magnen, Rivasseau, Sénéor, Lesniewski, Benfatto, Gallavotti, Mastropietro, Balaban, Knörrer, Salmhofer, Trubowitz, Brydges, Slade...]

Write $\varphi = \sum_{h=h_\beta}^0 \varphi^{(0)}$ and integrate $\varphi^{(h)}$ progressively:

$$\int \mu_{1d}(d\varphi) e^{V^{(1d)}(\varphi)} = \int \mu_{h_\beta}(d\varphi^{(h_\beta)}) \cdots \mu_h(d\varphi^{(h)}) e^{V^{(h)}(\varphi^{(h_\beta)} + \dots + \varphi^{(h)})}$$

$\hat{\varphi}_{(\omega, k_1)}^{(h)}$ supported for $|\omega|^2 + |k_1 - k_F|^2 \sim 2^{2h}$, covariance $\hat{g}^{(h)}$.

The flow of the beta function

- Goal: control the map $(\mu_h, V^{(h)}) \rightarrow (\mu_{h-1}, V^{(h-1)})$. For instance,

$$V^{(h)}(\varphi^{(h)}) = \int dt \sum_{x_1} \lambda_h \varphi_{\underline{x}, \uparrow}^{(h)+} \varphi_{\underline{x}, \uparrow}^{(h)-} \varphi_{\underline{x}, \downarrow}^{(h)+} \varphi_{\underline{x}, \downarrow}^{(h)-} + \text{irrelevant terms}$$

$$\lambda_h = \lambda_{h+1} + \beta_{h+1}(\lambda_{h+1}, \dots, \lambda_0), \quad \lambda_0 \equiv \lambda.$$

- Marginal coupling: $|\beta_{h+1}| \leq C \max_{k \geq h} |\lambda_k|^2$. Not summable.

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- Marginal coupling: $|\beta_{h+1}| \leq C \max_{k \geq h} |\lambda_k|^2$. Not summable.
- Crucial remark: $\beta_{h+1} = \beta_{h+1}^{\chi L} + \delta\beta_{h+1}$, with [Falco-Mastropietro '08]

$$\beta_{h+1}^{\chi L} = 0, \quad |\delta\beta_{h+1}| \leq C 2^h \max_{k \geq h} |\lambda_k|^2.$$

Summable iteration! Analyticity of $V^{(h_\beta)}$, unif. in β, L , follows.

Comparison with the effective 1d theory

- RG allows to express the lattice correlations via the χL :

$$\langle \mathbf{T} j_{\mu,(t,x)} ; j_{\nu,y} \rangle = Z_{\mu}(x_2) Z_{\nu}(y_2) \langle \mathbf{T} n_{(t,x_1)} ; n_{y_1} \rangle_{\chi L} + \text{“small errors”}$$

where $|Z_{\mu}(x_2)| \leq C e^{-c|x_2|}$ (from the decay of edge modes), and:

$$(FT \langle \mathbf{T} n_{(t,x_1)} ; n_{y_1} \rangle_{\chi L})(\omega, p_1) = -\frac{1}{2\pi v} \frac{1}{Z^2(1-\tau)} \frac{-i\omega - vp_1}{-i\omega + \tilde{v}p_1},$$

$$\tilde{v} = \left(\frac{1-\tau}{1+\tau}\right)v, \quad \tau = \frac{\lambda}{2\pi v} = \text{anomaly}, \quad v = v_e + O(\lambda), \quad Z = 1 + O(\lambda^2).$$

Comparison with the effective 1d theory

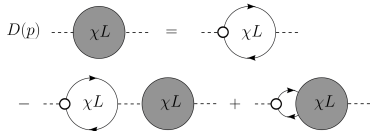
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$$(\text{empty bubble}) = \frac{\tau}{\lambda}(i\omega + vp_1)$$

- (i) $D(p) = -i\omega + vp_1$; (ii) the circle localizes the lines on the UV cutoff scale; (iii) the last term vanishes as the UV cutoff is removed.

Comparison with the effective 1d theory

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- To prove universality, need to find a **cancellation** between Z, τ, v and the **vertex renormalization**:

$$Z_{\mu} = \sum_{x_2} Z_{\mu}(x_2)$$

The cancellation follows from **Ward identities**: consequences of the continuity equation $\partial_{\mu} j_{\mu, \mathbf{x}} = 0$ for the correlations.

Ward identities

- Both $\langle \cdot \rangle_{\beta, L}$ and $\langle \cdot \rangle_{\chi L}$ satisfy **vertex WIs**: $(\mathbf{x} = (t, x_1, x_2) = (\underline{x}, x_2))$

$$\begin{aligned} \partial_\mu \langle \mathbf{T} j_{\mu, \mathbf{z}}; a_{\underline{y}}^- a_{\underline{x}}^+ \rangle_{\beta, L} &= [\delta_{\mathbf{x}, \mathbf{z}} \langle \mathbf{T} a_{\underline{y}}^- a_{\underline{x}}^+ \rangle_{\beta, L} - \delta_{\mathbf{y}, \mathbf{z}} \langle \mathbf{T} a_{\underline{y}}^- a_{\underline{x}}^+ \rangle_{\beta, L}] \\ (\partial_0 + v\partial_1) \langle \mathbf{T} n_{\underline{z}}; \varphi_{\underline{y}}^- \varphi_{\underline{x}}^+ \rangle_{\chi L} &= \frac{1}{Z(1-\tau)} [\delta_{\underline{x}, \underline{z}} \langle \mathbf{T} \varphi_{\underline{y}}^- \varphi_{\underline{x}}^+ \rangle_{\chi L} - \delta_{\underline{y}, \underline{z}} \langle \mathbf{T} \varphi_{\underline{y}}^- \varphi_{\underline{x}}^+ \rangle_{\chi L}] \end{aligned}$$

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- For large space-time distances:

$$\begin{aligned} \langle \mathbf{T} j_{\mu, (t_1, x)}; a_{(t_2, y)}^- a_z^+ \rangle_{\beta, L} &\simeq Z_\mu(x_2) \xi_{y_2} \bar{\xi}_{z_2} \langle \mathbf{T} n_{(t_1, x_1)}; \varphi_{(t_2, y_1)}^- \varphi_{z_1}^+ \rangle_{\chi L} \\ \langle \mathbf{T} a_{(t, x)}^-; a_y^+ \rangle_{\beta, L} &\simeq \xi_{x_2} \bar{\xi}_{y_2} \langle \mathbf{T} \varphi_{(t, x_1)}^-; \varphi_{y_1}^+ \rangle_{\chi L} \quad (*) \end{aligned}$$

- Plugging (*) in the WIs, we get **relations** between Z_μ , Z , τ and v :

$$Z_0 = Z(1 - \tau), \quad Z_1 = Zv(1 - \tau).$$

These identities imply the **universality** of the edge conductance.

Conclusions

- **Today:** Edge transport coefficients for $2d$ topological insulators with:
 - (i) single-mode edge currents, or
 - (ii) one pair of counterpropagating edge modes.

Consequences: bulk-edge duality, Haldane relation.

- Based on RG, and on Ward identities for relativistic & lattice model.
- Open problems:
 - (i) **Multi-edge states** topological insulators? (edge states scattering?)
 - (ii) Validity of edge **linear response theory**? (already for $\lambda = 0!$)
 - (iii) **Fractional** Quantum Hall effect...?

Thank you!