A scaling limit from Euler to Navier-Stokes equations with random perturbation

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The talk deals with a special scaling limit connecting 2D Euler and Navier-Stokes equations, both with noise, but of different type:

\begin{align*}
\partial_t \omega + u \cdot \nabla \omega &= \zeta \circ \nabla \omega \quad \text{Euler, multiplicative noise} \\
\partial_t \omega + u \cdot \nabla \omega &= \Delta \omega + \eta \quad \text{Navier-Stokes, additive noise}
\end{align*}

where

\begin{align*}
u &= \text{velocity, } \text{div} \ u = 0 \\
\omega &= \nabla^\perp u = \text{vorticity} \\
\zeta, \eta &= \text{space-dependent noise.}
\end{align*}
Overview of some directions in stochastic fluid dynamics

Due to the interdisciplinary character of the meeting, let me spend a few introductory words on stochastic fluid dynamics.

- The most studied equation is the stochastic Navier-Stokes equation with additive noise:

\[
\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \dot{W}
\]

- From the time of Kolmogorov it is indicated as a potential model to explain some feature of turbulence.

- For instance, the formal average energy balance

\[
\frac{1}{2} \mathbb{E} \int |u_t|^2 \, dx + \nu \int_0^t \mathbb{E} \int |\nabla u_s|^2 \, dx ds = \frac{1}{2} \mathbb{E} \int |u_0|^2 \, dx + t \cdot \text{Trace} (Q)
\]

put the basis for investigating dissipation for small \(\nu\): energy input is under control, and dissipation is constant.

- Much work has been devoted to the attempt to prove well posedness in 3D, outstanding open problem.
Overview of some directions in stochastic fluid dynamics

For Euler equation, a more natural noise is of transport type

\[ \partial_t u + u \cdot \nabla u + \nabla p = \zeta \circ \nabla u. \]

It preserves certain physical quantities (energy, in the form above, enstrophy in the 2D model \( \partial_t \omega + u \cdot \nabla \omega = \zeta \circ \nabla \omega \)).

Intuitively, it may correspond to the Lagrangian motion of small scales, acting on larger scales.

Similarly to additive noise for 3D Navier-Stokes equation, an open problem is whether this noise "regularizes", namely it produces better results of well posedness.

For instance, it prevents point vortex collision (F.-Gubinelli-Priola ’11).
Thus the two models

\[ \partial_t \omega + u \cdot \nabla \omega = \xi \circ \nabla \omega \quad \text{Euler, multiplicative noise} \]

\[ \partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \eta \quad \text{Navier-Stokes, additive noise} \]

have some physical and mathematical motivation. (Here they are written in vorticity form)
The purpose of this talk is to discuss a special scaling limit which connects the two, based on a computation similar in spirit to a renormalization.
Disclaimer

There is an obvious misunderstanding that we have to declare. In deterministic fluid mechanics, one of the most important open problems is the *vanishing viscosity limit*. Is it true that the solution $\omega_\epsilon$ of

$$
\partial_t \omega_\epsilon + u_\epsilon \cdot \nabla \omega_\epsilon = \epsilon \Delta \omega_\epsilon \quad \text{Navier-Stokes}
$$

converges to a solution $\omega$ of

$$
\partial_t \omega + u \cdot \nabla \omega = 0 \quad \text{Euler}
$$

as $\epsilon \to 0$? *This is not the limit investigated in this talk.*
Somewhat opposite, we aim to prove that solutions $\omega_\varepsilon$ of
\[
\partial_t \omega_N + u_N \cdot \nabla \omega_N = \xi_N \circ \nabla \omega_N \quad \text{Euler}
\]
converge to a solution $\omega$ of
\[
\partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \eta \quad \text{Navier-Stokes}
\]
as $N \to \infty$. In other words, in a suitable scaling limit, a transport type noise
\[
\xi_N \circ \nabla \omega_N
\]
gives rise to
\[
\Delta \omega + \eta.
\]
The stochastic Euler equations

The equations will be considered on the torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$. The noise $\xi_N$ in

$$\partial_t \omega_N + u_N \cdot \nabla \omega_N = \xi_N \circ \nabla \omega_N$$

has the form

$$\xi_N (x, t) = \epsilon_N \sum_{|k| \leq N} \sigma_k (x) \frac{d \beta_k (t)}{dt}$$

where

$$k \in \mathbb{Z}^2_* = \mathbb{Z}^2 \setminus \{0\}$$

$$\sigma_k (x) : \mathbb{T}^2 \to \mathbb{R}^2 \text{ are divergence free fields}$$

$$\beta_k \text{ are independent scalar BM's.}$$
Coefficients on the noise

\[ \partial_t \omega_N + u_N \cdot \nabla \omega_N = \zeta_N \circ \nabla \omega_N \]

\[ \zeta_N = \varepsilon_N \sum_{|k| \leq N} \sigma_k \frac{d\beta_k}{dt} \]

We assume \( \sigma_k (x) : \mathbb{T}^2 \to \mathbb{R}^2 \), divergence free fields, of the form

\[ \sigma_k (x) = \frac{k^\perp}{|k|^2} e_k (x) \]

where \((k_1, k_2)^\perp = (k_2, -k_1)\),

\[ e_k (x) = \begin{cases} 
\sin (2\pi k \cdot x) & \text{for } k \in \mathbb{Z}_+^2 \\
\cos (2\pi k \cdot x) & \text{for } k \in \mathbb{Z}_-^2 
\end{cases} \]
Consider the divergence free noise (infinite series)

\[ \xi^{(\gamma)}(x, t) = \sum_{k \in \mathbb{Z}_*^2} \frac{k \perp}{|k|^\gamma} e_k(x) \frac{d\beta_k(t)}{dt} \]

parametrized by \( \gamma \).

- The case \( \gamma = 1 \) is the divergence free analog of \textit{space-time white noise}. Completely untractable as a transport noise.
- The case \( \gamma = 2 \) is the divergence free analog of \textit{Gaussian free field noise}. It is our case but with truncated series. The infinite series (as a transport noise) is an open problem.
- Only the case \( \gamma > 2 \) (infinite series) can been treated by standard methods.
The threshold $\gamma = 2$ appears when rewriting Stratonovich into Itô:

$$
\xi(\gamma) = \sum_{k \in \mathbb{Z}^2_*} \frac{k^\perp}{|k|^\gamma} e_k \frac{d\beta_k}{dt}
$$

$$
\xi(\gamma) \circ \nabla \omega - \xi(\gamma) \cdot \nabla \omega = \frac{1}{2} \sum_{k \in \mathbb{Z}^2_*} \left( e_k \frac{k^\perp}{|k|^\gamma} \cdot \nabla \right) \left( e_k \frac{k^\perp}{|k|^\gamma} \cdot \nabla \omega \right).
$$

and it turns out that the second order differential operator is well defined only for $\gamma > 2$. Let us see the details.
The Itô-Stratonovich corrector

\[
\frac{1}{2} \sum_{k \in \mathbb{Z}^2_+} \left( e_k \frac{k^\perp}{|k|^\gamma} \cdot \nabla \right) \left( e_k \frac{k^\perp}{|k|^\gamma} \cdot \nabla \omega \right)
\]

\[
k^\perp \cdot \nabla e_k = 0 \quad \Rightarrow \quad \frac{1}{2} \sum_{k \in \mathbb{Z}^2_+} e_k^2 \left( \frac{k^\perp}{|k|^\gamma} \cdot \nabla \right) \left( \frac{k^\perp}{|k|^\gamma} \cdot \nabla \omega \right)
\]

\[
\sin^2 + \cos^2 = 1 \quad \Rightarrow \quad \frac{1}{2} \sum_{k \in \mathbb{Z}^2_+} \left( \frac{k^\perp}{|k|^\gamma} \cdot \nabla \right) \left( \frac{k^\perp}{|k|^\gamma} \cdot \nabla \omega \right)
\]

\[
= \frac{1}{2} \sum_{k \in \mathbb{Z}^2_+} \frac{1}{|k|^{2\gamma}} \sum_{i,j=1}^2 k_i k_j \partial_i \partial_j \omega
\]

\[
= \frac{1}{2} \left( \sum_{k \in \mathbb{Z}^2_+} \frac{1}{|k|^{2\gamma-2}} \right) \Delta \omega \quad \text{finite only for } \gamma > 2.
\]
The Itô-Stratonovich corrector

- The previous computation shows that, for $\gamma > 2$, the Itô-Stratonovich corrector in
  \[ \frac{1}{2} \left( \sum_{k \in \mathbb{Z}_+^2} \frac{1}{|k|^{2\gamma-2}} \right) \Delta \omega. \]

- For $\gamma = 2$ the coefficient diverges.

- But, for $\gamma = 2$, the Itô-Stratonovich corrector of the noise
  \[ \zeta_N(x, t) = \epsilon_N \sum_{|k| \leq N} \sigma_k(x) \frac{d\beta_k(t)}{dt} \]
  is
  \[ \frac{1}{2} \left( \epsilon_N^2 \sum_{|k| \leq N} \frac{1}{|k|^2} \right) \Delta \omega \]
  which converges if we take
  \[ \epsilon_N \sim 1/ \sqrt{\log N}. \]
For the Euler equation

\[ \partial_t \omega_N + u_N \cdot \nabla \omega_N = \zeta_N \circ \nabla \omega_N, \quad \zeta_N = \epsilon_N \sum_{|k| \leq N} \sigma_k \frac{d \beta_k}{dt} \]

with \( \sigma_k(x) = \frac{k^\perp}{|k|^2} e_k(x) \) and \( \epsilon_N \sim 1/\sqrt{\log N} \) we can write

\[ \partial_t \omega_N + u_N \cdot \nabla \omega_N = \underbrace{\zeta_N \cdot \nabla \omega_N}_{\text{Itô}} + \nu_N \Delta \omega_N \]

with \( \nu_N \to \nu \in \mathbb{R}^+ \).

In the case \( \sigma_k(x) = \frac{k^\perp}{|k|^\gamma} e_k(x) \) with \( \gamma > 2 \) we get the same result without rescaling with \( \epsilon_N \).

What about the martingale term?
The martingale term

The martingale

\[ M_N(t, x) := \epsilon_N \sum_{|k| \leq N} \int_0^t \sigma_k(x) \cdot \nabla \omega_N(s, x) \, d\beta_k(s) \]

in general depends strongly on the solution \( \omega_N(s, x) \) and preserves this dependence in the limit \( N \to \infty \).

This dependence spoils the dissipativity of \( \Delta \omega_N \): the sum of the martingale and corrector is the Stratonovich term, which is not dissipative. For \( \gamma > 2 \), not rescaled by \( \epsilon_N \), this remains true in the limit of the infinite dimensional noise: we get the stochastic Euler equation

\[ \partial_t \omega + u \cdot \nabla \omega = \xi \circ \nabla \omega, \quad \xi = \sum_{k \in \mathbb{Z}^2} \frac{k^\perp}{|k|^\gamma} e_k \frac{d\beta_k}{dt}. \]
The martingale term

In the case $\gamma = 2$, the martingale

$$M_N (t, x) := \frac{1}{\sqrt{\log N}} \sum_{|k| \leq N} \int_0^t \frac{k^\perp}{|k|^2} e_k (x) \cdot \nabla \omega_N (s, x) \, d\beta_k (s)$$

behaves in a special way for special solutions $\omega_N$ (described below) and converges to a Gaussian process, precisely of the form

$$\nabla^\perp \cdot W (t, x)$$

where $W' (t, x)$ is a divergence free space-time white noise. When true, the limit of the 2D Euler equation will be

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega + \nabla^\perp \cdot W' (t, x).$$

It remains to describe the special solutions when this happens.
Again

To avoid misunderstanding, let us insist on the fact that usually the Laplacian in the Itô-Stratonovich reformulation

\[ \partial_t \omega_N + u_N \cdot \nabla \omega_N = \underbrace{\tilde{\zeta}_N \cdot \nabla \omega_N + \nu_N \Delta \omega_N}_{\text{Itô}} \]

is a \textit{fake} Laplacian, it does not dissipate. The sum of Itô + corrector is just Stratonovich. Thus, in general, both the approximating and the limit equation are of hyperbolic type.
But we have identified a special regime where the Laplacian (the corrector) converges to a Laplacian, and the Itô term converges to a Gaussian process. In this case the limit equation changes nature, becomes parabolic.
Consider, on $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$, the equation

$$
\partial_t \omega + u \cdot \nabla \omega = \zeta^{(\gamma)}(\gamma) \circ \nabla \omega \\
\omega = \nabla \perp u, \quad \text{div } u = 0
$$

where $\zeta^{(\gamma)}(x, t) = \sum_{k \in \mathbb{Z}^2} \frac{k^\perp}{|k|^\gamma} \mathbf{e}_k(x) \frac{d\beta_k(t)}{dt}$.

Theorem (F.-Luo '18)

If $\gamma > 2$, there exists a solution, weak in the probabilistic and analytic sense, stationary process with trajectories of class $C([0, T]; H^{-1-})$, such $\omega(t)$ is a white noise in space, for every $t \geq 0$.

In spite of many attempts, we do not know how to extend this theorem to $\gamma = 2$. 
Remarks

- For $\gamma > 3$, solutions in a more classical sense, precisely of class $\omega \in L^\infty$, have been obtained by Brzezniak-F.-Maurelli, ARMA 2016. In the class $\omega \in L^\infty$, also uniqueness is known, as a generalization of the deterministic result of Yudovich.

- Below $\omega \in L^\infty$, deterministic 2D Euler equations lack uniqueness and our hope is that this kind of transport noise may restore it.

- Therefore we investigate all possible ranges of noise, since perhaps those which may regularize are very rough. The range $\gamma > 2$ is covered by the previous result. $\gamma = 2$ is the threshold we do not know how to approach (with infinite dimensional noise).
Consider the equation

$$\partial_t \omega_N + u_N \cdot \nabla \omega_N = \zeta_N \circ \nabla \omega_N$$

with

$$\zeta_N (x, t) = \epsilon_N \sum_{|k| \leq N} \frac{k^\perp}{|k|^2} e_k (x) \frac{d \beta_k (t)}{dt}.$$ 

Similarly to above,

**Lemma (F.-Luo ’18)**

*There exists a solution (called below White Noise solution), weak in the probabilistic and analytic sense, stationary process with trajectories of class \( C ([0, T] ; H^{-1 -}) \), such \( \omega_N (t) \) is a white noise in space, for every \( t \geq 0 \).*
Main result

**Theorem (F.-Luo ’18)**

For every $N \in \mathbb{N}$, let $\omega_N$ be a White Noise solution of

$$\partial_t \omega_N + u_N \cdot \nabla \omega_N = \zeta_N \circ \nabla \omega_N, \quad \zeta_N = \epsilon_N \sum_{|k| \leq N} \frac{k^\perp}{|k|^2} e_k \frac{d\beta_k}{dt}$$

given by the previous lemma. Assume

$$\epsilon_N = \frac{1}{\sqrt{\log N}}.$$

Then $\omega_N$ converges weakly to $\omega$, unique solution of the stochastic Navier-Stokes equation with space-time noise

$$\partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \nabla^\perp \cdot \dot{W}.$$
2D Navier-Stoke equations with space-time white noise

\[ \partial_t u + u \cdot \nabla u + \nabla p = \Delta u + \dot{W} \]

have been studied by Da Prato-Debussche, Albeverio-Ferrario and others. It is well posed also in the strong sense (for general initial conditions). The vorticity formulation, \( \omega = \nabla^\perp u \), in the equation above:

\[ \partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \nabla^\perp \cdot \dot{W} \]

It has a White Noise solution as a stationary solution. This is the regime considered by the theorem above.
Tightness of the laws of solutions of the approximating Euler equations can be proved in $C \left( [0, T] ; H^{-1-} \right)$, taking advantage of their stationarity in time and knowledge of the time-marginals. The identification of the limit requires standard work for the nonlinear term (based on recent works on White Noise solutions of 2D Euler) and for the Laplacian arising as a corrector. The difficult part is the convergence of the martingale term

$$M_N (t, x) := \frac{1}{\sqrt{\log N}} \sum_{|k| \leq N} \int_0^t \frac{k^\perp}{|k|^2} e_k (x) \cdot \nabla \omega_N (s, x) \, d\beta_k (s)$$

to the Gaussian process

$$\nabla^\perp \cdot W (t, x).$$
Quadratic variation

We have to prove, for

\[ M_N(t, x) := \frac{1}{\sqrt{\log N}} \sum_{|k| \leq N} \int_0^t \frac{k^\perp}{|k|^2} e_k(x) \cdot \nabla \omega_N(s, x) \, d\beta_k(s) \]

that the joint variation

\[ \left[ \langle M_N, e_l \rangle, \langle M_N, e_m \rangle \right]_t \]

converges to

\[ \left[ \langle \nabla^\perp \cdot W, e_l \rangle, \langle \nabla^\perp \cdot W, e_m \rangle \right]_t = \int_0^t \langle \nabla e_l, \nabla e_m \rangle \, ds = t |l|^2 \delta_{l,m}. \]
Quadratic variation

From

\[ M_N (t, x) = \frac{1}{\sqrt{\log N}} \tilde{M}_N (t, x) \]

\[ \tilde{M}_N (t, x) := \sum_{|k| \leq N} \int_0^t \frac{k \perp}{|k|^2} e_k (x) \cdot \nabla \omega_N (s, x) \, d\beta_k (s) \]

one can show, similarly to the corrector above, that

\[ \left[ \left\langle \tilde{M}_N, e_l \right\rangle, \left\langle \tilde{M}_N, e_m \right\rangle \right]_t \] diverges.

Precisely,

\[ \left[ \left\langle \tilde{M}_N, e_l \right\rangle, \left\langle \tilde{M}_N, e_m \right\rangle \right]_t = \sum_{|k| \leq N} \frac{1}{|k|^4} \left( k \perp \cdot l \right) \left( k \perp \cdot m \right) \int_0^t \left\langle \omega, e_k e_{-l} \right\rangle \left\langle \omega, e_k e_{-m} \right\rangle \, ds \]
Quadratic variation

\[
\left[ \langle \tilde{M}_N, e_l \rangle, \langle \tilde{M}_N, e_m \rangle \right]_t \\
= \sum_{|k| \leq N} \frac{1}{|k|^4} \left( k^\perp \cdot l \right) \left( k^\perp \cdot m \right) \int_0^t \langle \omega, e_k e_{-l} \rangle \langle \omega, e_k e_{-m} \rangle \, ds
\]

\[
\mathbb{E} \left[ \langle \tilde{M}_N, e_l \rangle, \langle \tilde{M}_N, e_m \rangle \right]_t \\
\omega(t) \overset{WN}{=} t \cdot \sum_{|k| \leq N} \frac{1}{|k|^4} \left| k^\perp \cdot l \right|^2 \delta_{l,m}
\]

\[
\sim \left( \sum_{|k| \leq N} \frac{1}{|k|^2} \right) t |l|^2 \delta_{l,m}
\]

and the coefficient \( \sum_{|k| \leq N} \frac{1}{|k|^2} \) diverges.
Analogy: renormalized energy

In the theory of 2D Euler with the enstrophy measure there is a well-known renormalization: the interaction kinetic energy. The average kinetic energy of the fluid is \((u_k = \text{Fourier components of } u)\)

\[
\mathbb{E} \left[ \frac{1}{2} \int |u(x)|^2 \, dx \right] = \frac{1}{2} \sum_{k \in \mathbb{Z}_2^*} \mathbb{E} \left[ |u_k|^2 \right]
\]

\(\omega = \nabla u = \text{WN} \quad \frac{1}{2} \sum_{k \in \mathbb{Z}_2^*} \frac{1}{|k|^2} = +\infty\)

but the renormalized partial sums (corresponding to interaction energy in the case of point vortices)

\[
\sum_{|k| \leq N} |u_k|^2 - \mathbb{E} \left[ \sum_{|k| \leq N} |u_k|^2 \right]
\]

converge in \(L^2_\mu \) \((\mu = \text{law of White Noise on } H^{-1-})\) to a well defined random variable

\[ : \mathcal{E} : \]

called renormalized kinetic energy.
Renormalizing quadratic variation

- For $[\langle M_N, e_l \rangle, \langle M_N, e_m \rangle]_t$ it is similar.
- *Without* the term $\frac{1}{\log N}$, we have seen above that
  $$\mathbb{E} \left[ \langle \tilde{M}_N, e_l \rangle, \langle \tilde{M}_N, e_m \rangle \right]_t$$
  diverges.
- Moreover, it happens that
  $$\left[ \langle \tilde{M}_N, e_l \rangle, \langle \tilde{M}_N, e_m \rangle \right]_t - \mathbb{E} \left[ \langle \tilde{M}_N, e_l \rangle, \langle \tilde{M}_N, e_m \rangle \right]_t$$
  converges in $L^2_{\mu}$ ($\mu$=law of White Noise on $H^{-1-}$) to a well defined random variable.
Renormalizing quadratic variation

As a consequence, including the term \( \frac{1}{\log N} \),

\[
L^2_\mu = \lim_{N \to \infty} \left( \langle M_N, e_l \rangle, \langle M_N, e_m \rangle \right)_t - \mathbb{E} \left[ \langle M_N, e_l \rangle, \langle M_N, e_m \rangle \right]_t = 0.
\]

And

\[
\mathbb{E} \left[ \langle M_N, e_l \rangle, \langle M_N, e_m \rangle \right]_t = \left( \frac{1}{\log N} \sum_{|k| \leq N} \frac{1}{|k|^2} \right) t |l|^2 \delta_{l,m}
\]

\[
\rightarrow t |l|^2 \delta_{l,m}
\]

\[
= \left[ \langle \nabla \cdot W, e_l \rangle, \langle \nabla \cdot W, e_m \rangle \right]_t.
\]
Discussion: extensions, interpretation, motivations

- There are heuristic reasons to believe that part of the scaling limit discussed above is a more general fact; in particular, the presence of the Laplacian in the limit equation, with a true parabolic character.
- Maybe precisely additive white noise is special, due to the "white noise" solution regime.
- [Private discussions years ago with Weber and Hauray.] A common space-dep noise with very short correlation range acting on particles is similar to independent BMs and should give rise to a Laplacian in the mean field limit.
- With Dejun Luo we spent much time trying to prove that multiplicative noise restores uniqueness in 2D Euler equations and had the heuristic impression that $\gamma = 2$ was relevant (but it is not admissible).
- In a sense we have studied here the case $\gamma = 2$, and the limit equation, stochastic Navier-Stokes, has uniqueness!