Renormalisation in Regularity Structures: Part II

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One main example

The Geometric KPZ is the most natural stochastic evolution on loop space. The system of equations in local coordinates is given by

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\partial_x u^\beta \partial_x u^\gamma + \sigma_i^\alpha(u)\xi_i\ .$$

where

- the ξ_i are independent space-time white noises.
- the $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols.
- the σ_i are a collection of smooth vector fields on the manifold.

Main issues

- Give a meaning to a singular SPDE: ill-defined distributional products.
- Are there notions of solution that are covariant under changes of coordinates?
- Are there notions of solution that, in law, depend only on $g^{\alpha\beta}(u) = \sigma_i^\alpha(u)\sigma_i^\beta(u)$ rather than on the arbitrary choice of vector fields σ_i?

Good algebraic structures are needed for answering these questions.

A perturbative expansion

$$\partial_t u = \partial_x^2 u + (\partial_x u)^2 + \xi, \quad \partial_t v = \partial_x^2 v + \xi$$

Then $v = K * \xi \in \mathcal{C}^{\frac{1}{2}-\kappa}$ and we look at $u = v + w$ where $w \in \mathcal{C}^\alpha$ with $\alpha > \frac{1}{2}$.

A perturbative expansion

$$\partial_t u = \partial_x^2 u + (\partial_x u)^2 + \xi, \quad \partial_t v = \partial_x^2 v + \xi$$

The ... $v \in \mathcal{C}^\alpha$

with ...

Decorated Trees $\mathcal{T}$

$\xi = \mathbf{\Pi}$ , $\quad K * \xi = \mathbf{\Pi}$ , $\quad \partial_x K * \xi = \mathbf{\Pi}$ 

$(\partial_x K * \xi)^2 = \mathbf{\Pi}$ , $\quad K * ((\partial_x K * \xi)^2) = \mathbf{\Pi}$ 

A perturbative expansion

$$\partial_t u = \partial_x^2 u + (\partial_x u)^2 + \xi, \quad \partial_t v = \partial_x^2 v + \xi$$

Th ... α

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Degree deg **on** $\mathcal{T}$

deg $(\circ) = -\frac{3}{2} - \kappa$, deg () $= \frac{1}{2} - \kappa$, deg () $= -\frac{1}{2} - \kappa$

deg () $= -1 - 2\kappa$, deg () $= 1 - 2\kappa$

A perturbative expansion

$$\partial_t u = \partial_x^2 u + (\partial_x u)^2 + \xi, \quad \partial_t v = \partial_x^2 v + \xi$$

Then $v = K * \xi \in \mathcal{C}^{\frac{1}{2}-\kappa}$ and we look at $u = v + w$ where $w \in \mathcal{C}^{\alpha}$ with $\alpha > \frac{1}{2}$.

$$\partial_t w = \partial_x^2 w + \mathbf{\Pi} \vee + 2\, \mathbf{\Pi} \backslash\, (\partial_x w) + (\partial_x w)^2.$$

We go on

$$u = \mathbf{\Pi} \backslash + \mathbf{\Pi}\, Y + \tilde{w}, \quad w = \mathbf{\Pi}\, Y + \tilde{w}.$$

This expansion is not sufficient: the equation for the reminder needs a special treatment.

Construction of the decorated trees T

We put constraints of each nodes depending on the non-linearities.
For example,

$$(\partial_x u)^2 + \xi \longrightarrow T = \{\circ, X^k, X^k \overset{\tau_1}{\big|}\,, \overset{\tau_1 \quad \tau_2}{\bigvee} : \tau_1, \tau_2 \in T, k \in \mathbb{N}^2\}$$

$$f(u)\xi \longrightarrow T = \{X^k \circ, X^k \overset{\tau_1 \cdots \tau_n}{\bigvee} : \tau_i \in T, k \in \mathbb{N}^2\}.$$

We set $\tilde{T} = \{X^k, \overset{\tau}{\big|} : \tau \in T, k \in \mathbb{N}^2\}$.

A new Taylor expansion

We obtain a local expansion of the solution u_ε by recentering these monomials around a point x

$$u_\varepsilon(y) = \sum_{\tau \in \tilde{T}} (\Upsilon \tau)\,(u_\varepsilon, \partial_x u_\varepsilon)\,(\Pi_x^{(\varepsilon)} \tau)(y) + r(x,y).$$

For the KPZ equation, we have

$u_\varepsilon(y) = u_\varepsilon(x) + (\Pi_x^{(\varepsilon)}$  $)(y) + (\Pi_x^{(\varepsilon)}$  $)(y) + r(x,y),$

with

$(\Pi_x^{(\varepsilon)}$  $)(y) = (K * \xi_\varepsilon)(y) - (K * \xi_\varepsilon)(x), \quad \xi_\varepsilon = \varrho_\varepsilon * \xi.$

Two renormalisations

Positive renormalisation

- Recentering procedure
- Smooth model (Π_x, Γ_{xy}), for every decorated tree τ, $(\Pi_x \tau)(y) \lesssim |y-x|^{\deg \tau}$.
- Structure group G, $\Gamma_{xy} \in G$ (Hopf algebra).

Negative renormalisation

- Renormalisation group $\mathcal{R}$ (Hopf algebra). Let $M : \mathcal{T} \to \mathcal{T}$, $M \in \mathcal{R}$.
- Action onto the model (Π_x^M, Γ_{xy}^M).
- Action onto the equation (Pre-Lie Structure).

Co-interaction

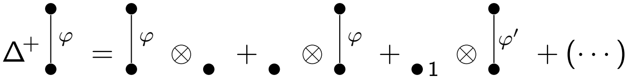
$\Pi_x^M = \Pi_x M$, on some $\mathcal{T}_{\text{ex}}$, $\mathcal{T} \subset \mathcal{T}_{\text{ex}}$.

Abstract Taylor expansion

Expansion of $\varphi \in \mathcal{C}^\infty$ around 0:

$$\varphi(y) - \varphi(0) - y\varphi'(0) = r(y), \quad |r(y)| \lesssim |y|^2.$$

Cut of one decorated edge:



Recentering monomials

Take $\mathcal{T}$ the linear span of the abstract polynomials $\{X^k, k \in \mathbb{N}\}$, $(\bullet_k = X^k)$. It is a Hopf algebra with $\mathbf{1} = X^0$ and:

- The multiplicative coproduct Δ^+ is given by

$$\Delta^+ X = X \otimes \mathbf{1} + \mathbf{1} \otimes X, \quad \Delta^+ X^n = \sum_{k=0}^{n} \binom{n}{k} X^k \otimes X^{n-k}$$

- Co-unit $\mathbf{1}^\star$: $\mathbf{1}^\star(X^k) = \mathbf{1}_{k=0}$.
- Antipode $\mathcal{A}$ is given by $\mathcal{A}\mathbf{1} = \mathbf{1}$, $\mathcal{A}X = -X$.
- Structure group is isomorphic to $\mathbb{R}$: $\Gamma_g X^k = (X + g(X))^k$.

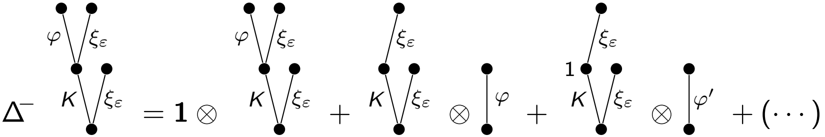
Negative renormalisation

We are looking at the renormalised integral with the heat kernel and a mollifier ϱ_ε:

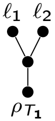
$$I_\varepsilon(y) = \int K(y-z)\varrho_\varepsilon^{(2)}(y-z)\left(\varphi(z)-\varphi(y)-(z-y)\varphi'(y)\right)dz,$$

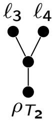
where $\varrho_\varepsilon^{(2)}(y-z) = (\varrho_\varepsilon * \varrho_\varepsilon)(y-z) = \mathbb{E}\left(\xi_\varepsilon(z)\xi_\varepsilon(y)\right).$

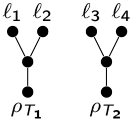
Extraction of one subtree:

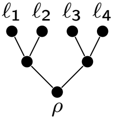


Notations

- Let $T_1 =$ 

and $T_2 =$ 

, then $T_1 \cdot T_2 =$ 

  and $T_1 \star T_2 =$ 

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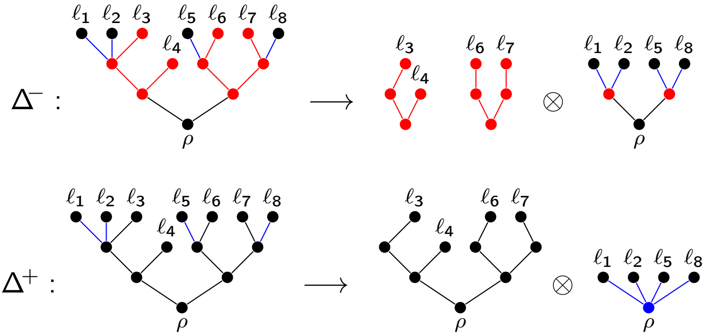
- $\mathcal{T}$ linear span of decorated trees.
- $\mathcal{T}_+$ linear span of decorated trees with positive degree.
- $\mathcal{T}_-$ linear span of negative decorated forests.

Co-actions



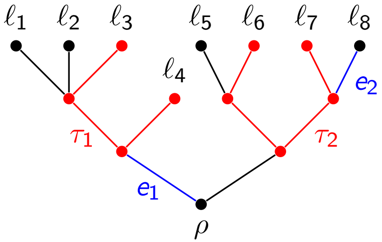
There are Taylor expansions on the blue edges.

Two Hopf algebras in co-interaction

Theorem (B., Hairer, Zambotti, 2016)

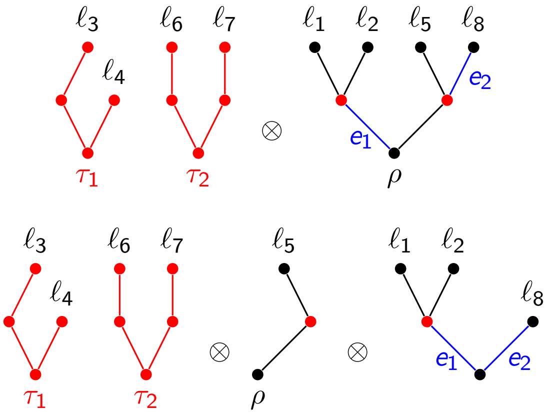
1. *The algebra $\mathcal{T}_+$ endowed with the product $\star$ and the coproduct Δ^+ is a Hopf algebra. Moreover Δ^+ turns $\mathcal{T}$ into a right comodule over $\mathcal{T}_+$.*
2. *The algebra $\mathcal{T}_-$ endowed with the product $\cdot$ and the coproduct Δ^- is a Hopf algebra. Moreover Δ^- turns $\mathcal{T}$ into a left comodule over $\mathcal{T}_-$.*
3. *They co-interact.*

Co-interaction

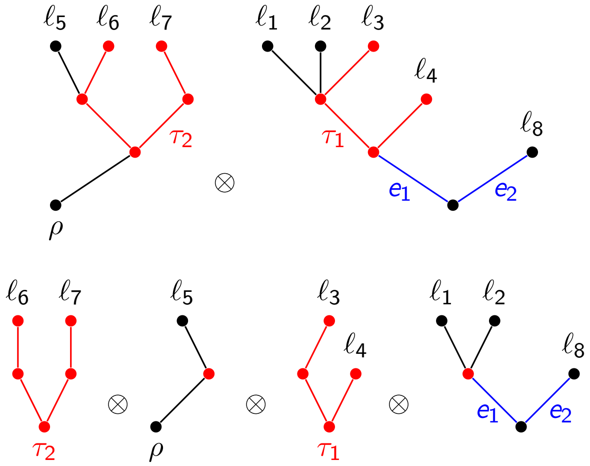


1. Let consider a subforest $A = \{\tau_1, \tau_2\}$
2. Admissible cuts $E = \{e_1, e_2\}$ such that $E_A \cap E = \emptyset$.

Extraction then cuts

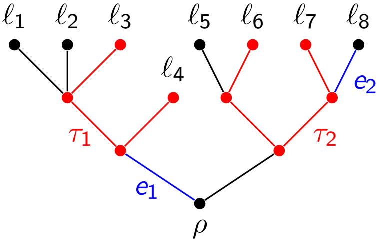


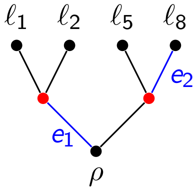
Cuts then extraction



Extended structure $\mathcal{T}_{\mathrm{ex}}$

The same cut is performed in a different order:





The length of the Taylor expansion may differ.

Extended structure $\mathcal{T}_{\text{ex}}$

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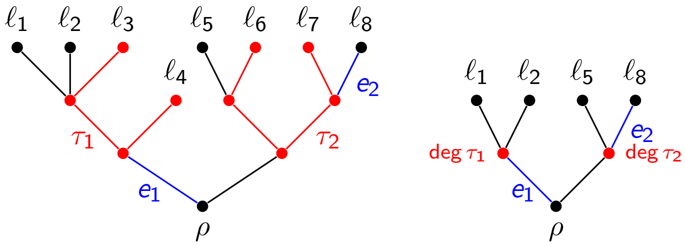


The length of the Taylor expansion may differ. We need to add an extra information: $\deg \tau_1$ and $\deg \tau_2$.

Two groups

We set

$$\mathcal{G}_+ := \{g \in \mathcal{T}_+^* : g(\tau_1 \star \tau_2) = g(\tau_1)g(\tau_2),\ \forall\, \tau_1, \tau_2 \in \mathcal{T}_+\}$$
$$\mathcal{G}_- := \{\ell \in \mathcal{T}_-^* : \ell(\tau_1 \cdot \tau_2) = \ell(\tau_1)\ell(\tau_2),\ \forall\, \tau_1, \tau_2 \in \mathcal{T}_-\}$$

Theorem (B., Hairer, Zambotti, 2016)

For every $g \in \mathcal{G}_-$ the renormalised model is described by:

$$\Pi_z^{M_g} = \Pi_z M_g,$$

where $\Pi_z = (\boldsymbol{\Pi} \otimes f_z)\Delta^+$ for some $f_z \in \mathcal{G}_+$ and $M_g = (g \otimes \mathrm{id})\Delta^-$.

Renormalised equation

Theorem (B., Chandra, Chevyrev, Hairer 2017)

There exist some constants $(c^{\tau}_{\varrho,\varepsilon})_{\tau \in T_-}$ such that the renormalised equation for u_ε is given by

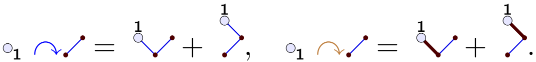
$$\partial_t u^\alpha_\varepsilon = \partial^2_x u^\alpha_\varepsilon + \Gamma^\alpha_{\beta\gamma}(u_\varepsilon)\,\partial_x u^\beta_\varepsilon \partial_x u^\gamma_\varepsilon + \sigma^\alpha_i(u_\varepsilon)\xi^{(\varepsilon)}_i + \sum_{\tau \in T_-} c^{\tau}_{\varrho,\varepsilon}\left(\Upsilon^\alpha_{\Gamma,\sigma}\tau\right)(u_\varepsilon, \partial_x u_\varepsilon)\,.$$

A possible choice of these constants called BPHZ renormalisation is

$$c^{\tau}_{\varrho,\varepsilon} = \mathbb{E}(\mathbf{\Pi}^{(\varepsilon)}\tilde{\mathcal{A}}_-\tau)(0).$$

Application of Pre-Lie structures

We consider the space of decorated trees as a 2-pre-Lie algebra with generators $\mathcal{G} = \{\bullet, \circ_i\}$ and the following two grafting operators:



- Derivation of the renormalised equation. First used in the framework of rough paths in [BCFP17] then extended to singular SPDEs in [BCCH17].
- Symmetry properties.

Computation of $\Upsilon^{\alpha}_{\Gamma,\sigma}\tau$

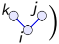
We define $\tau \mapsto (\Upsilon^{\alpha}_{\Gamma,\sigma}\tau)(x, v)$ as the unique 2-pre-Lie morphism satisfying:

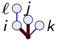
$$\Upsilon^{\alpha}_{\Gamma,\sigma}(\circ_i) = \sigma^{\alpha}_i, \quad \Upsilon^{\alpha}_{\Gamma,\sigma}(\bullet) = \Gamma^{\alpha}_{\beta\gamma} v_\beta v_\gamma,$$

$$\Upsilon^{\alpha}_{\Gamma,\sigma}(\tau_1 \curvearrowright \tau_2) = \Upsilon^{\beta}_{\Gamma,\sigma}(\tau_1) \frac{d}{dx_\beta} \Upsilon^{\alpha}_{\Gamma,\sigma}(\tau_2),$$

$$\Upsilon^{\alpha}_{\Gamma,\sigma}(\tau_1 \curvearrowright \tau_2) = \Upsilon^{\beta}_{\Gamma,\sigma}(\tau_1) \frac{d}{dv_\beta} \Upsilon^{\alpha}_{\Gamma,\sigma}(\tau_2).$$

Some examples of coefficients:

$\Upsilon^{\alpha}_{\Gamma,\sigma}\Big($  $\Big) = \sigma^{\beta}_j \partial_\beta \sigma^{\alpha}_i,$ $\qquad$ $\Upsilon^{\alpha}_{\Gamma,\sigma}\Big($  $\Big) = \sigma^{\gamma}_k \sigma^{\beta}_j \partial_\beta \partial_\gamma \sigma^{\alpha}_i,$

$\Upsilon^{\alpha}_{\Gamma,\sigma}\Big($  $\Big) = 2\sigma^{\eta}_k \partial_\eta \Gamma^{\alpha}_{\beta\gamma} \sigma^{\beta}_j \sigma^{\mu}_\ell \partial_\mu \sigma^{\gamma}_i.$

Symmetry Properties

- Space $\mathcal{T}_{\text{geo}}$: $\varphi \cdot (\Upsilon_{\Gamma,\sigma}\tau) = \Upsilon_{\varphi\cdot\Gamma,\varphi\cdot\sigma}\tau$.
- Space $\mathcal{T}_{\text{Itô}}$: $\Upsilon_{\Gamma,\sigma}\tau = \Upsilon_{\Gamma,\bar{\sigma}}\tau$.
- Space $\mathcal{T}_{\text{both}}$: $\varphi \cdot (\Upsilon_{\Gamma,\sigma}\tau - \Upsilon_{\Gamma,\bar{\sigma}}\tau) = \Upsilon_{\varphi\cdot\Gamma,\varphi\cdot\sigma}\tau - \Upsilon_{\varphi\cdot\Gamma,\varphi\cdot\bar{\sigma}}\tau$.

The spaces $\mathcal{T}_{\text{geo}}$ and $\mathcal{T}_{\text{Itô}}$ can be characterised as kernels of "deformed" 2-pre-Lie infinitesimal morphisms.

Proposition (B., Gabriel, Hairer, Zambotti 2018+)

One has $\mathcal{T}_{\text{both}} = \mathcal{T}_{\text{geo}} + \mathcal{T}_{\text{Itô}}$.