

# The Stable Cannon Conjecture

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# Poincaré duality groups

## Definition (Finite Poincaré complex)

A (connected) finite  $n$ -dimensional CW-complex  $X$  is a **finite  $n$ -dimensional Poincaré complex** if there is  $[X] \in H_n(X; \mathbb{Z}^w)$  such that the induced  $\mathbb{Z}\pi$ -chain map

$$-\cap [X]: C^{n-*}(\tilde{X}) \rightarrow C_*(\tilde{X})$$

is a  $\mathbb{Z}\pi$ -chain homotopy equivalence.

## Theorem (Closed manifolds are Poincaré complexes)

*A closed  $n$ -dimensional manifold  $M$  is a finite  $n$ -dimensional Poincaré complex with  $w = w_1(X)$ .*

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A **Poincaré duality group**  $G$  of dimension  $n$  is a finitely presented group satisfying:

- $G$  is of type FP.
- $H^i(G; \mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

## Theorem (Wall)

*If  $G$  is a  $d$ -dimensional Poincaré duality group for  $d \geq 3$  and  $\tilde{K}_0(\mathbb{Z}G) = 0$ , then there is a model for  $BG$  which is a finite Poincaré complex of dimension  $d$ .*

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## Corollary

*If  $M$  is a closed aspherical manifold of dimension  $d$ , then  $\pi_1(X)$  is a  $d$ -dimensional Poincaré duality group.*

## Theorem (Bieri-Eckmann, Linnell)

*Every 2-dimensional Poincaré duality group is the fundamental group of a closed surface.*

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# Hyperbolic groups

## Definition (hyperbolic group)

A **hyperbolic group**  $G$  is a finitely generated group such that for one (and hence all) choice of symmetric finite set of generators the Cayley graph with the associated word metric is a hyperbolic geodesic metric space.

- A geodesic metric space is called **hyperbolic** if geodesic triangles are thin in comparison with geodesic triangles in  $\mathbb{R}^2$ .
- The property hyperbolic is a quasi-isometry invariant.
- Every hyperbolic group is finitely presented and has a finite model for its classifying space of proper actions.
- Every torsionfree hyperbolic group has a finite model for  $BG$ .

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- A random finitely presented group is hyperbolic.
- One can assign to hyperbolic group a topological space called **boundary  $\partial G$**  such that for any geodesic hyperbolic metric space  $X$  on which  $G$  acts properly and cocompactly by isometries there is a compactification  $\overline{X} = X \amalg \partial G$  such that  $\partial G$  is a  $Z$ -set in  $\overline{X}$ . This applies in particular to the Cayley graph.
- Notice that  $\partial G$  is independent of  $X$ .
- In general  $\partial G$  is totally disconnected, in other words, looks like a Cantor set.

## Theorem (Hadamard)

*If  $M$  is a closed smooth Riemannian manifold of dimension  $n$ , whose section curvature is negative, then  $\pi = \pi_1(M)$  is a torsionfree hyperbolic group with  $\partial\pi = S^{n-1}$ .*

- Actually  $\tilde{M}$  is a geodesic metric spaces on which  $\pi$  acts freely, properly and cocompactly by isometries.
- There is a diffeomorphism  $\tilde{M} \xrightarrow{\cong} \mathbb{R}^n$  and  $\partial\pi$  is the sphere  $S^{n-1}$  at infinity.

# The main conjectures

## Conjecture (Gromov)

*Let  $G$  be a torsionfree hyperbolic group whose boundary is a sphere  $S^{n-1}$ . Then there is a closed aspherical manifold  $M$  with  $\pi_1(M) \cong G$ .*

## Theorem (Bartels-Lück-Weinberger)

*Gromov's Conjecture is true for  $n \geq 6$ .*

## Conjecture (Wall)

*Every Poincaré duality group is the fundamental group of an aspherical closed manifold.*

## Conjecture (Cannon's Conjecture in the torsionfree case)

*A torsionfree hyperbolic group  $G$  has  $S^2$  as boundary if and only if it is the fundamental group of a closed hyperbolic 3-manifold.*

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## Theorem (Cannon-Cooper, Eskin-Fisher-Whyte, Kapovich-Leeb)

*A Poincaré duality group  $G$  of dimension 3 is the fundamental group of an aspherical closed 3-manifold if and only if it is quasi-isometric to the fundamental group of an aspherical closed 3-manifold.*

- A closed 3-manifold is a **Seifert manifold** if it admits a finite covering  $\bar{M} \rightarrow M$  such that there exists a  $S^1$ -principal bundle  $S^1 \rightarrow \bar{M} \rightarrow S$  for some closed orientable surface  $S$ .

## Theorem (Bowditch)

*If a Poincaré duality group of dimension 3 contains an infinite normal cyclic subgroup, then it is the fundamental group of a closed Seifert 3-manifold.*

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## Theorem (Bestvina)

*Let  $G$  be a hyperbolic 3-dimensional Poincaré duality group. Then its boundary is homeomorphic to  $S^2$ .*

## Theorem (Bestvina-Mess)

*Let  $G$  be an infinite torsionfree hyperbolic group which is prime, not infinite cyclic, and the fundamental group of a closed 3-manifold  $M$ . Then  $M$  is hyperbolic and  $G$  satisfies the Cannon Conjecture.*

- In order to prove the Cannon Conjecture, it suffices to show for a hyperbolic group  $G$ , whose boundary is  $S^2$ , that it is quasiisometric to the fundamental group of some aspherical closed 3-manifold.

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## Theorem

*Let  $G$  be the fundamental group of an aspherical oriented closed 3-manifold. Then  $G$  satisfies:*

- *$G$  is residually finite and Hopfian;*
- *All its  $L^2$ -Betti numbers  $b_n^{(2)}(G)$  vanish;*
- *Its deficiency is 0. In particular it possesses a presentation with the same number of generators and relations;*
- *Suppose that  $M$  is hyperbolic. Then  $G$  is virtually compact special and linear over  $\mathbb{Z}$ . It contains a subgroup of finite index  $G'$  which can be written as an extension  $1 \rightarrow \pi_1(S) \rightarrow G' \rightarrow \mathbb{Z} \rightarrow 1$  for some closed orientable surface  $S$ .*

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*A torsionfree hyperbolic  $G$  is a Poincaré duality group of dimension  $n$  if and only if its boundary  $\partial G$  and  $S^{n-1}$  have the same Čech cohomology.*

## Theorem

*If the boundary of a hyperbolic group contains an open subset homeomorphic to  $\mathbb{R}^n$ , then the boundary is homeomorphic to  $S^n$ .*

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# The main results

Theorem (Ferry-Lück-Weinberger, (preprint, 2018), Vanishing of the surgery obstruction)

Let  $G$  be a hyperbolic 3-dimensional Poincaré duality group. Then there is a normal map of degree one (in the sense of surgery theory)

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\bar{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \end{array}$$

satisfying

- 1 The space  $BG$  is a finite 3-dimensional CW-complex;
- 2 The map  $H_n(f, \mathbb{Z}): H_n(M; \mathbb{Z}) \xrightarrow{\cong} H_n(BG; \mathbb{Z})$  is bijective for all  $n \geq 0$ ;
- 3 The simple algebraic surgery obstruction  $\sigma(f, \bar{f}) \in L_3^s(\mathbb{Z}G)$  vanishes.

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## Theorem (Ferry-Lück-Weinberger, (preprint, 2018), Stable Cannon Conjecture)

Let  $G$  be a hyperbolic 3-dimensional Poincaré duality group. Let  $N$  be any smooth, PL or topological manifold respectively which is closed and whose dimension is  $\geq 2$ .

Then there is a closed smooth, PL or topological manifold  $M$  and a normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{f} & \xi \times TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \times N \end{array}$$

such that the map  $f$  is a simple homotopy equivalence.

## Theorem (Stable Cannon Conjecture, continued)

Moreover:

Let  $\widehat{M} \rightarrow M$  be the  $G$ -covering associated to the composite of the isomorphism  $\pi_1(f): \pi_1(M) \xrightarrow{\cong} G \times \pi_1(N)$  with the projection  $G \times \pi_1(N) \rightarrow G$ . Suppose additionally that  $N$  is aspherical and  $\dim(N) \geq 3$ .

Then  $\widehat{M}$  is homeomorphic to  $\mathbb{R}^3 \times N$ . Moreover, there is a compact topological manifold  $\overline{\widehat{M}}$  whose interior is homeomorphic to  $\widehat{M}$  and for which there exists a homeomorphism of pairs

$$(\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \rightarrow (D^3 \times N, S^2 \times N).$$



- The last two theorems follow from the Cannon Conjecture.
- By the product formula for surgery theory the second last theorem implies the last theorem.
- The manifold  $M$  appearing in the last theorem is unique up to homeomorphism by the **Borel Conjecture**, provided that  $\pi_1(N)$  satisfies the Farrell-Jones Conjecture.
- If we take  $N = T^k$  for some  $k \geq 2$ , then the Cannon Conjecture is equivalent to the statement that this  $M$  is homeomorphic to  $M' \times T^k$  for some closed 3-manifold  $M'$ .

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# The existence of a normal map

## Theorem (Existence of a normal map)

*Let  $X$  be a connected oriented finite 3-dimensional Poincaré complex. Then there are an integer  $a \geq 0$  and a vector bundle  $\xi$  over  $BG$  and a normal map of degree one*

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## Proof.

- Stable vector bundles over  $X$  are classified by the first and second Stiefel-Whitney class  $w_1(\xi)$  and  $w_2(\xi)$  in  $H^*(X; \mathbb{Z}/2)$ .
- Let  $\xi$  be a  $k$ -dimensional vector bundle over  $X$  such that  $w_1(\xi) = w_1(X)$  and  $w_2(\xi) = w_1(\xi) \cup w_1(\xi)$  holds.
- A spectral sequence argument applied to  $\Omega_3(X, w_1(X))$  shows that there is a closed 3-manifold  $M$  together with a map  $f: M \rightarrow X$  of degree one such that  $f^* w_1(X) = w_1(M)$ .
- Then  $w_1(f^*\xi) = w_1(M)$  and the Wu formula implies  $w_2(M) = w_1(f^*\xi) \cup w_1(f^*\xi)$ .
- Hence  $f^*\xi$  is stably isomorphic to the stable tangent bundle of  $M$  and we get the desired normal map.



# The total surgery obstruction

- Consider an aspherical finite  $n$ -dimensional Poincaré complex  $X$  such that  $G = \pi_1(X)$  is a **Farrell-Jones group**, i.e., satisfies both the  $K$ -theoretic and the  $L$ -theoretic Farrell-Jones Conjecture with coefficients in additive categories, and  $\mathcal{N}(X)$  is non-empty. (For simplicity we assume  $w_1(X) = 0$  in the sequel.)
- We have to find one normal map of degree one

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- Recall that the simple surgery obstruction defines a map

$$\sigma^S: \mathcal{N}(X) \rightarrow L_n^S(\mathbb{Z}G).$$

- Fix a normal map  $(f_0, \bar{f}_0)$ .
- Then there is a commutative diagram

$$\begin{array}{ccc}
 \mathcal{N}(X) & \xrightarrow{\sigma^S(-, -) - \sigma^S(f_0, \bar{f}_0)} & L_n^S(\mathbb{Z}G) \\
 s_0 \downarrow \cong & & \cong \uparrow \text{asmb}_n^S(X) \\
 H_n(X; \mathbf{L}_{\mathbb{Z}}^S\langle 1 \rangle) & \xrightarrow{H_n^G(\text{id}_X; \mathbf{i})} & H_n(X; \mathbf{L}_{\mathbb{Z}}^S)
 \end{array}$$

whose vertical arrows are bijections thanks to the Farrell-Jones Conjecture and the upper arrow sends the class of  $(f, \bar{f})$  to the difference  $\sigma^S(f, \bar{f}) - \sigma^S(f, \bar{f}_0)$  of simple surgery obstructions.

- An easy spectral sequence argument yields a short exact sequence

$$0 \rightarrow H_n(X; \mathbf{L}_{\mathbb{Z}}^s \langle 1 \rangle) \xrightarrow{H_n(\text{id}_X; \mathbf{i})} H_n(X; \mathbf{L}_{\mathbb{Z}}^s) \xrightarrow{\lambda_n^s(X)} L_0(\mathbb{Z}).$$

- Consider the composite

$$\mu_n^s(X): \mathcal{N}(X) \xrightarrow{\sigma^s} L_n^s(\mathbb{Z}G, w) \xrightarrow{\text{asmb}_n^s(X)^{-1}} H_n(X; \mathbf{L}_{\mathbb{Z}}^s) \xrightarrow{\lambda_n^s(X)} L_0(\mathbb{Z}).$$

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- We conclude that there is precisely one element, called the **total surgery obstruction**,

$$s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z}$$

such that for any element  $[(f, \bar{f})]$  in  $\mathcal{N}(X)$  its image under  $\mu_n^s(X)$  is  $s(X)$ .

### Theorem (Total surgery obstruction)

- *There exists a normal map of degree one  $(f, \bar{f})$  with target  $X$  and vanishing simple surgery obstruction  $\sigma^s(f, \bar{f}) \in L_n^s(\mathbb{Z}G)$  if and only if  $s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z}$  vanishes.*
- *The total surgery obstruction is a homotopy invariant of  $X$  and hence depends only on  $G$ .*

- We conclude that there is precisely one element, called the **total surgery obstruction**,

$$s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z}$$

such that for any element  $[(f, \bar{f})]$  in  $\mathcal{N}(X)$  its image under  $\mu_n^s(X)$  is  $s(X)$ .

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# ANR-homology manifolds

## Definition (Homology ANR-manifold)

A **homology ANR-manifold**  $X$  is an ANR satisfying:

- $X$  has a countable basis for its topology;
- The topological dimension of  $X$  is finite;
- $X$  is locally compact;
- for every  $x \in X$  we have for the singular homology

$$H_i(X, X - \{x\}; \mathbb{Z}) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$$

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- Every closed topological manifold is a closed ANR-homology manifold.
- Let  $M$  be homology sphere with non-trivial fundamental group. Then its suspension  $\Sigma M$  is a closed ANR-homology manifold but not a topological manifold.

# Quinn's resolution obstruction

## Theorem (Quinn (1987))

*There is an invariant  $\iota(M) \in 1 + 8\mathbb{Z}$  for homology ANR-manifolds with the following properties:*

- *if  $U \subset M$  is an open subset, then  $\iota(U) = \iota(M)$ ;*
- *$\iota(M \times N) = \iota(M) \cdot \iota(N)$ ;*
- *Let  $M$  be a homology ANR-manifold of dimension  $\geq 5$ . Then  $M$  is a topological manifold if and only if  $\iota(M) = 1$ .*
- *The Quinn obstruction and the total surgery obstruction are related for an aspherical closed ANR-homology manifold  $M$  of dimension  $\geq 5$  by*

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# Proof of the Theorem about the vanishing of the surgery obstruction

## Proof.

- We have to show for the aspherical finite 3-dimensional Poincaré complex  $X$  that its total surgery obstruction vanishes.
- The total surgery obstruction satisfies a product formula

$$s(X \times Y) = s(X) + s(Y).$$

- This implies

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## Proof (continued).

- There exists an aspherical closed ANR-homology manifold  $M$  and a homotopy equivalence  $f: M \rightarrow X \times T^3$ .
- There is a  $Z$ -compactification  $\widetilde{X}$  of  $X$  by the boundary  $\partial G = S^2$ .
- One constructs an appropriate  $Z$ -compactification  $\widetilde{M}$  of  $M$  so that we get a ANR-homology manifold  $\widetilde{M}$  whose boundary is a topological manifold and whose interior is  $\widetilde{M}$ .
- By adding a collar to  $\widetilde{M}$  one obtains a ANR-homology manifold  $Y$  which contains  $\widetilde{M}$  as an open subset and contains an open subset  $U$  which is homeomorphic to  $\mathbb{R}^6$ .



## Proof (continued).

- Hence we get

$$\begin{aligned}8s(X \times T^3) + 1 &= 8s(M) + 1 = i(M) = i(\tilde{M}) \\ &= i(Y) = i(U) = i(\mathbb{R}^6) = 1.\end{aligned}$$

- This implies  $s(X \times T^3) = 0$  and hence  $s(X) = 0$ .



# Appendix: Pulling back the boundary

## Notation

Let  $(\bar{Y}, Y)$  be a topological pair. Put  $\partial Y := \bar{Y} \setminus Y$ . Let  $X$  be a topological space and  $f: X \rightarrow Y$  be a continuous map. *Pulling back the boundary* is a construction of a topological pair  $(\bar{X}, X)$  and a continuous map  $\bar{f}: \bar{X} \rightarrow \bar{Y}$

- It has the desired universal property which we will not state here.
- Its basic properties are:



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- Its basic properties are:

## Lemma

- $Y \subseteq \bar{Y}$  is dense and the closure of the image of  $f$  in  $\bar{Y}$  contains  $\partial Y$ , then  $X \subseteq \bar{X}$  is dense;
- $\bar{Y}$  is compact,  $Y \subseteq \bar{Y}$  is open and  $f: X \rightarrow Y$  is proper. Then  $\bar{X}$  is compact;
- We have for the topological dimension of  $\bar{X}$

$$\dim(\bar{X}) \leq \dim(X) + \dim(\bar{Y}) + 1;$$

- The induced map  $\bar{f}$  induces a homeomorphism  $\partial f: \partial X \rightarrow \partial Y$ .

## Definition (Z-set)

A closed subset  $Z$  of a compact ANR  $X$  is called a **Z-set** if for every open subset  $U$  of  $X$  the inclusion  $U \setminus (U \cap Z) \rightarrow U$  is a homotopy equivalence.

- The boundary of a manifold is a Z-set in the manifold.

## Lemma

*Consider a pair  $(\bar{Y}, Y)$  of spaces such that  $\bar{Y}$  is a ANR and  $\partial Y$  is a Z-set in  $\bar{Y}$ . Consider a homotopy equivalence  $f: X \rightarrow Y$  which is continuously controlled. Let  $(\bar{f}, f): (\bar{X}, X) \rightarrow (\bar{Y}, Y)$  be obtained by pulling back the boundary along  $f$ .*

*Then  $\bar{X}$  is an ANR and  $\partial X \subseteq \bar{X}$  is a Z-set.*

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