Bogoliubov Excitations of dilute Bose-Einstein Condensates

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Scaling limits & SPDEs:
recent developments and future directions

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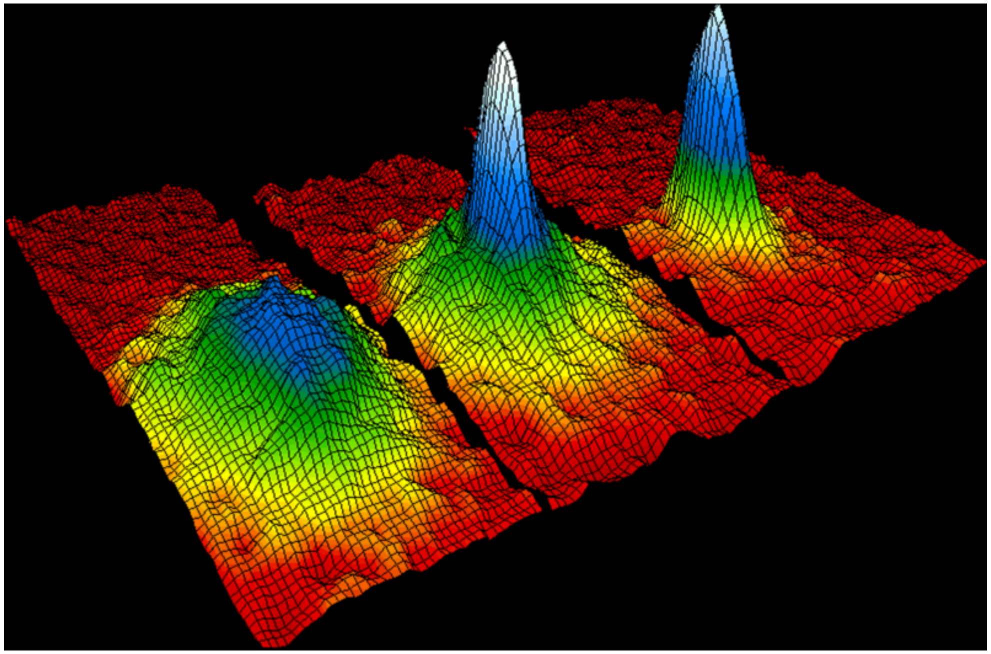
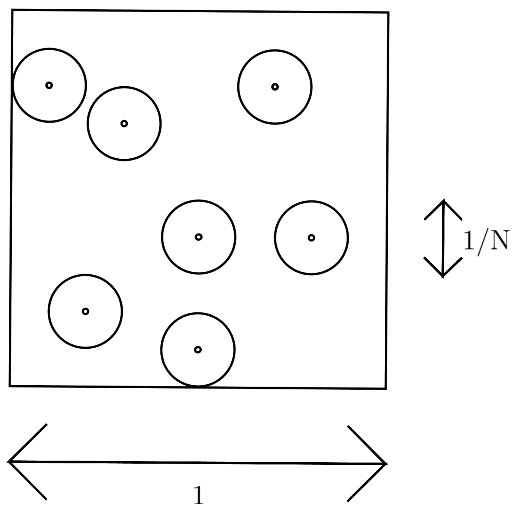
Introduction

Bose-Einstein condensates: in the last two decades, BEC have become accessible to experiments.

Goal of this talk: understand low-energy properties of trapped BEC, starting from microscopic description.



Gross-Pitaevskii regime: N bosons in $\Lambda = [0;1]^3$, interacting through potential with effective range of order N^{-1}, as $N \to \infty$.



Range of interaction much shorter than typical distance among particles: collisions rare, **dilute gas**.

Hamilton operator: has form

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} N^2 V(N(x_i - x_j)), \qquad \text{on } L^2_s(\Lambda^N)$$

$V \geq 0$ with **compact support**.

Scattering length: defined by zero-energy **scattering equation**

$$\left[-\Delta + \frac{1}{2}V(x)\right] f(x) = 0, \qquad \text{with} \qquad f(x) \to 1 \quad \text{as } |x| \to \infty$$

$$\Rightarrow \qquad f(x) = 1 - \frac{\mathfrak{a}_0}{|x|}, \qquad \text{for large } |x|$$

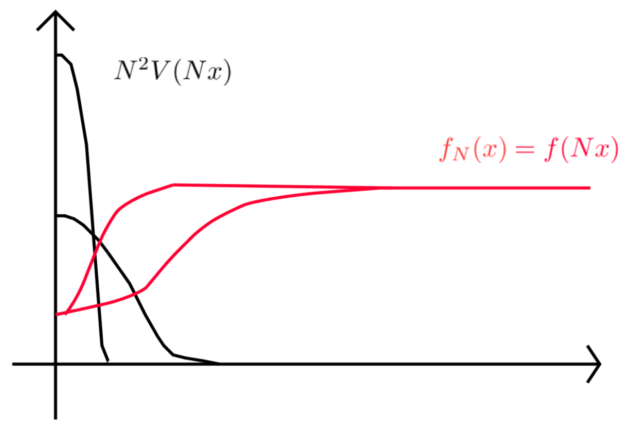
Equivalently,

$$8\pi\mathfrak{a}_0 = \int V(x) f(x) dx$$

By **scaling**,

$$\left[-\Delta + \frac{1}{2}N^2 V(Nx)\right] f(Nx) = 0$$

Rescaled potential has scattering length $\mathfrak{a}_0/N$.



Ground state energy: **[Lieb-Yngvason '98]** proved ground state energy given to leading order by

$$E_N = 4\pi\mathfrak{a}_0 N + o(N)$$

BEC: [Lieb-Seiringer '02, '06] showed that $\psi_N \in L^2_s(\Lambda^N)$ with

$$\langle \psi_N, H_N \psi_N \rangle \leq 4\pi\mathfrak{a}_0 N + o(N)$$

exhibits BEC, i.e.

$$\gamma_N(x;y) = \int dx_2 \dots dx_N \, \psi_N(x, x_2, \dots, x_N) \overline{\psi}_N(y, x_2, \dots, x_N)$$

is such that

$$\lim_{N\to\infty} \langle \varphi_0, \gamma_N \varphi_0 \rangle = 1$$

with $\varphi_0(x) = 1$ for all $x \in \Lambda$.

Warning: this does not mean that $\psi_N \simeq \varphi_0^{\otimes N}$. In fact

$$\langle \varphi_0^{\otimes N}, \, H_N \, \varphi_0^{\otimes N} \rangle = \frac{(N-1)}{2} \widehat{V}(0) \gg 4\pi\mathfrak{a}_0 N$$

Correlations play crucial role!!

Main results

Theorem 1: There exists $C > 0$ such that

$$|E_N - 4\pi\mathfrak{a}_0 N| \leq C$$

uniformly in N.

Furthermore, if $\psi_N \in L^2_s(\Lambda^N)$ such that

$$\langle \psi_N, H_N \psi_N \rangle \leq 4\pi\mathfrak{a}_0 N + \zeta$$

we have

$$1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \leq \frac{C(\zeta + 1)}{N}$$

Interpretation: in low-energy states, condensation holds with optimal rate, with **bounded** number of excitations.

Question: Is it possible to **resolve** order one contributions to the ground state energy?

Theorem 2: Let $\Lambda_+^* = 2\pi\mathbb{Z}^3 \backslash \{0\}$. Then

$$E_N = 4\pi\mathfrak{a}_0(N-1) + e_\Lambda \mathfrak{a}_0^2 - \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right] + \mathcal{O}(N^{-1/4})$$

where

$$e_\Lambda = 2 - \lim_{M\to\infty} \sum_{\substack{p\in\mathbb{Z}^3\backslash\{0\}: \\ |p_1|,|p_2|,|p_3|\le M}} \frac{\cos(|p|)}{p^2}$$

Remark: $e_\Lambda \mathfrak{a}_0^2$ describes **finite volume correction** to scattering length, ie.

$$4\pi\mathfrak{a}_0(N-1) + e_\Lambda\mathfrak{a}_0^2 = 4\pi\mathfrak{a}_N(N-1) + \mathcal{O}(N^{-1})$$

where $\mathfrak{a}_N$ is scattering length defined on box Λ.

Theorem 3: The **spectrum** of $H_N - E_N$ below ζ consists of eigenvalues

$$\sum_{p \in \Lambda_+^*} n_p \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} + \mathcal{O}(N^{-1/4}(1+\zeta^3))$$

where $n_p \in \mathbb{N}$ for all $p \in \Lambda_+^*$.

Interpretation: every excitation with momentum $p \in \Lambda_+^*$ "costs" energy $\varepsilon(p) = \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}$.

Remark: excitation spectrum is **crucial** to understand the low-energy properties of Bose gas.

The **linear dependence** of $\varepsilon(p)$ on $|p|$ for small p can be used to explain the emergence of **superfluidity**.

Previous works: mathematically simpler models described by

$$H_N^\beta = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N}\sum_{i<j}^N N^{3\beta} V(N^\beta(x_i - x_j))$$

for $\beta \in [0;1)$.

In **mean field regime**, $\beta = 0$, excitation spectrum determined in [Seiringer, '11], [Grech-Seiringner, '13], [Lewin-Nam-Serfaty-Solovej, '14], [Derezinski-Napiorkowski, '14], [Pizzo, '16].

Dispersion of excitations given by $\varepsilon_{\text{mf}}(p) = \sqrt{|p|^4 + 2\widehat{V}(p)p^2}$.

For **intermediate regimes**, $\beta \in (0;1)$ (and V small enough) excitations spectrum determined in [BBCS, '17].

Dispersion of excitations given by $\varepsilon_\beta(p) = \sqrt{|p|^4 + 2\widehat{V}(0)p^2}$.

For **Gross-Pitaevskii regime**, $\beta = 1$, and V small, excitations spectrum determined in [BBCS, '18].

Thermodynamic limit: consider N bosons in $\Lambda_L = [0; L]^3$, with $N, L \to \infty$ but fixed density $\rho = N/L^3$.

Lee-Huang-Yang formula: as $\rho \to 0$, expect

$$\lim_{\substack{N,L\to\infty \\ N/L^3=\rho}} \frac{E_N}{N} = 4\pi\mathfrak{a}_0\rho \left[1 + \frac{128}{15\sqrt{\pi}}(\rho\mathfrak{a}_0^3)^{1/2} + \text{smaller order}\right]$$

Leading order known from [**Lieb-Yngvason, '98**].

Upper bound to second order in [**Yau-Yin, '09**].

Gross-Pitaevskii limit corresponds to $\rho = N^{-2}$.

Bose-Einstein condensation: more ambitious goal is proof of BEC in thermodynamic limit.

Ongoing program of **Balaban-Feldman-Knörrer-Trubowitz** via renormalization group analysis.

Bogoliubov approximation

Fock space: define $\mathcal{F} = \bigoplus_{n\geq 0} L^2_s(\Lambda^n)$.

Creation and annihilation operators: for $p \in 2\pi\mathbb{Z}^3$, introduce a_p^*, a_p creating and annihilating particle with momentum p.

Canonical commutation relations: for any $p, q \in 2\pi\mathbb{Z}^3$,

$$\left[a_p, a_q^*\right] = \delta_{p,q}, \qquad \left[a_p, a_q\right] = \left[a_p^*, a_q^*\right] = 0$$

Number of particles: $a_p^* a_p$ measures number of particles with momentum p,

$$\mathcal{N} = \sum_{p\in\Lambda^*} a_p^* a_p = \text{total } \textbf{number of particles} \text{ operator}$$

Hamilton operator: we write

$$H_N = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{1}{N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r}$$

Number substitution: **BEC** implies that

$$a_0, a_0^* \simeq \sqrt{N} \gg 1 = [a_0, a_0^*]$$

Bogoliubov replaced a_0^*, a_0 by factors of $\sqrt{N}$. He found

$$\begin{aligned} H_N \simeq \; & \frac{(N-1)}{2} \widehat{V}(0) + \sum_{p \neq 0} p^2 a_p^* a_p + \widehat{V}(0) \sum_{p \neq 0} a_p^* a_p \\ & + \frac{1}{2} \sum_{p \neq 0} \widehat{V}(p/N) \left[2 a_p^* a_p + a_p^* a_{-p}^* + a_p a_{-p} \right] \\ & + \frac{1}{\sqrt{N}} \sum_{p,q \neq 0} \widehat{V}(p/N) \left[a_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} a_{p+q} \right] \\ & + \frac{1}{N} \sum_{p,q,r \neq 0} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r} \end{aligned}$$

Diagonalization: neglecting cubic and quartic terms, and using appropriate **Bogoliubov transformation**

$$T = \exp\Big\{ \sum_{p\in\Lambda_+^*} \tau_p \left(a_p^* a_{-p}^* - a_p a_{-p} \right) \Big\}$$

one finds

$$\begin{aligned} T^* H_N T \simeq \; & \frac{(N-1)}{2} \widehat{V}(0) - \frac{1}{2} \sum_{p\neq 0} \frac{\widehat{V}^2(p/N)}{2p^2} \\ & - \frac{1}{2} \sum_{p\neq 0} \left[p^2 + \widehat{V}(0) - \sqrt{|p|^4 + 2\widehat{V}(0)p^2} - \frac{\widehat{V}(0)^2}{2p^2} \right] \\ & + \sum_{p\neq 0} \sqrt{|p|^4 + 2\widehat{V}(0)p^2}\, a_p^* a_p \end{aligned}$$

Born series: for small potentials, scattering length given by

$$\begin{aligned} 8\pi \mathfrak{a}_0 = \; & \widehat{V}(0) \\ & + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n N^n} \sum_{p_1,\dots,p_n \neq 0} \frac{\widehat{V}(p_1/N)}{p_1^2} \prod_{j=1}^{n-1} \frac{\widehat{V}((p_j - p_{j+1})/N)}{p_{j+1}^2} \widehat{V}(p_n/N) \end{aligned}$$

Scattering length: replacing

$$\widehat{V}(0) \to 8\pi\mathfrak{a}_0, \qquad \widehat{V}(0) - \frac{1}{N}\sum_p \frac{\widehat{V}^2(p/N)}{2p^2} \to 8\pi\mathfrak{a}_0$$

Bogoliubov obtained

$$\begin{aligned} T^* H_N T \simeq\ & 4\pi\mathfrak{a}_0(N-1) \\ & - \frac{1}{2}\sum_{p\neq 0}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right] \\ & + \sum_{p\neq 0}\sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}\, a_p^* a_p \end{aligned}$$

Hence

$$E_N = 4\pi\mathfrak{a}_0(N-1) - \frac{1}{2}\sum_{p\neq 0}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right]$$

and excitation energies have form

$$\sum_{p\neq 0} n_p \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}, \qquad n_p \in \mathbb{N}$$

Final replacement makes up for **missing** cubic and quartic terms!

Some ideas from proof

Orthogonal excitations: for $\psi_N \in L^2_s(\Lambda^N)$ and $\varphi_0 \equiv 1$ on Λ, write

$$\psi_N = \alpha_0 \varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes (N-1)} + \alpha_2 \otimes_s \varphi_0^{\otimes (N-2)} + \cdots + \alpha_N$$

where $\alpha_j \in L^2_{\perp \varphi_0}(\Lambda)^{\otimes_s j}$.

As in [**Lewin-Nam-Serfaty-Solovej, '12**], define unitary map

$$U : L^2_s(\Lambda^N) \to \mathcal{F}_+^{\leq N} := \bigoplus_{j=0}^{N} L^2_{\perp \varphi_0}(\Lambda)^{\otimes_s j}$$
$$\psi_N \to U\psi_N = \{\alpha_0, \alpha_1, \ldots, \alpha_N\}$$

Excitation Hamiltonian: we use unitary map U to define

$$\mathcal{L}_N = U H_N U^* : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

For $p, q \in \Lambda_+^* = 2\pi\mathbb{Z}^3 \backslash \{0\}$, we have

$$\begin{aligned} U\, a_p^* a_q \, U^* &= a_p^* a_q, \\ U\, a_0^* a_0 \, U^* &= N - \mathcal{N}_+ \\ U\, a_p^* a_0 \, U^* &= a_p^* \sqrt{N - \mathcal{N}_+} =: \sqrt{N}\, b_p^*, \\ U a_0^* a_p \, U^* &= \sqrt{N - \mathcal{N}_+}\, a_p =: \sqrt{N}\, b_p \end{aligned}$$

Hence, similarly to **Bogoliubov substitution**,

$$\begin{aligned} \mathcal{L}_N = \; & \frac{(N-1)}{2} \widehat{V}(0) + \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p + \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) a_p^* a_p \\ & + \frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left[b_p^* b_{-p}^* + b_p b_{-p} \right] \\ & + \frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N) \left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q} \right] \\ & + \frac{1}{2N} \sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq -p,-q} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r} \end{aligned}$$

Renormalized excitation Hamiltonian: let $w = 1 - f$ and

$$\eta_p = -\frac{1}{N^2}\,\widehat{w}(p/N) \qquad \Rightarrow \qquad \eta_p \simeq \frac{C}{p^2}\,e^{-|p|/N}$$

We introduce **cutoff** $\mu > 0$ and define

$$T = \exp\Big\{ \sum_{|p|>\mu} \eta_p \left(b_p^* b_{-p}^* - b_p b_{-p} \right) \Big\}$$

For $|p| > \mu$,

$$T^* b_p T \simeq \cosh(\eta_p) b_p + \sinh(\eta_p) b_{-p}^*$$

Observation: let $\mathcal{K} = \sum p^2 a_p^* a_p$. Then

$$\langle \Omega, T^* \mathcal{N}_+ T \Omega \rangle \simeq \sum_{|p|>\mu} \sinh^2(\eta_p) \leq C \sum_{|p|>\mu} \eta_p^2 \leq C$$

$$\langle \Omega, T^* \mathcal{K} T \Omega \rangle \simeq \sum_{|p|>\mu} p^2 \sinh^2(\eta_p) \simeq \sum_{|p|>\mu} p^2 \eta_p^2 \simeq C N$$

T generates **finitely many** excitations but **macroscopic** energy.

Renormalized excitation Hamiltonian: define

$$\mathcal{G}_N = T^* \mathcal{L}_N T = T^* U H_N U^* T : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

Action of T: with $\mathcal{K} = \sum p^2 a_p^* a_p$ (**kinetic energy**) and $\mathcal{V}_N = (2N)^{-1} \sum \widehat{V}(r/N) a_{p+r}^* a_q^* a_{q+r} a_p$ (**potential energy**), we have

$$T^* \mathcal{K} T \simeq \mathcal{K} + \sum p^2 \eta_p^2 + \sum p^2 \eta_p \left[a_p^* a_{-p}^* + a_{-p} a_p \right]$$

$$\begin{aligned} T^* \mathcal{V}_N T \simeq \mathcal{V}_N &+ \frac{1}{2N} \sum \widehat{V}(r/N) \eta_{q+r} \eta_q \\ &+ \frac{1}{2N} \sum \widehat{V}(r/N) \eta_{r+p} \left[a_p^* a_{-p}^* + a_p a_{-p} \right] \end{aligned}$$

Combine with

$$\begin{aligned} T^* \sum \widehat{V}(p/N) \left[a_p^* a_{-p}^* + a_p a_{-p} \right] T \\ \simeq \sum \widehat{V}(p/N) \eta_p + \sum \widehat{V}(p/N) \left[a_p^* a_{-p}^* + a_p a_{-p} \right] \end{aligned}$$

to get rid of off-diagonal quadratic term.

Bounds on $\mathcal{G}_N$: with $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$, we arrive at

$$\mathcal{G}_N = 4\pi\mathfrak{a}_0 N + \mathcal{H}_N + \sum_{|p|\leq\mu} \widehat{V}(0) a_p^* a_p + 4\pi\mathfrak{a}_0 \sum_{|p|\leq\mu} \left[b_p^* b_{-p}^* + b_p b_{-p} \right]$$
$$+ \frac{1}{\sqrt{N}} \sum \widehat{V}(p/N) \left[b_{p+q}^* a_{-p}^* a_q + \text{h.c.} \right] + \mathcal{E}_N$$

with

$$\pm\mathcal{E}_N \leq \frac{C}{\mu^\alpha} \mathcal{H}_N + C\mu^\beta, \quad \text{for some } \alpha, \beta > 0$$

Main problem: T renormalizes bad quadratic contributions, but it leaves cubic term **essentially unchanged**.

Small potentials: with $\mathcal{N}_+ \leq C\mathcal{K} \leq C\mathcal{H}_N$, we can estimate

$$\frac{1}{\sqrt{N}} \sum \widehat{V}(p/N) \left[b_{p+q}^* a_{-p}^* a_q + \text{h.c.} \right] \geq -\frac{1}{2} \mathcal{H}_N - C$$

Thus

$$\mathcal{N}_+ \leq C\mathcal{H}_N \leq C(\mathcal{G}_N - 4\pi\mathfrak{a}_0 N) - C \quad \Rightarrow \quad \textbf{BEC}$$

For large interaction, this approach **fails**.

Cubic conjugation: for a cutoff $\nu < \mu$, define

$$A = \frac{1}{\sqrt{N}} \sum_{|r|>\mu, |v|<\nu} \eta_r \Big[b^*_{r+v} a^*_{-r} a_v - \text{h.c.} \Big]$$

Set $S = e^A$ and introduce **new excitation Hamiltonian**

$$\mathcal{R}_N = S^* \mathcal{G}_N S = S^* T^* \mathcal{L}_N T S : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

Bounds for $\mathcal{R}_N$: we obtain

$$\mathcal{R}_N = 4\pi\mathfrak{a}_0 N + \mathcal{H}_N + 8\pi\mathfrak{a}_0 \sum_{|p|<\mu} a^*_p a_p + 4\pi\mathfrak{a}_0 \sum_{|p|<\mu} \Big[b^*_p b^*_{-p} + b_p b_{-p} \Big]$$
$$+ \frac{8\pi\mathfrak{a}_0}{\sqrt{N}} \sum_{|p|<\mu, q \in \Lambda^*_+} \Big[b^*_{p+q} a^*_{-p} a_q + \text{h.c.} \Big] + \widetilde{\mathcal{E}}_N$$

where

$$\pm \widetilde{\mathcal{E}}_N \leq C\mu^{-\alpha} \mathcal{H}_N + C\mu^{\beta}$$

Observe: $\mathcal{R}_N$ is almost excitation Hamiltonian for **mean field** potential $8\pi\mathfrak{a}_0 \chi(|p| < \mu)$

With BEC for mean-field interaction [**Seiringer, '11**], obtain

$$\mathcal{R}_N \geq 4\pi\mathfrak{a}_0 N + \frac{1}{2}\mathcal{H}_N - \frac{4\pi\mathfrak{a}_0}{N} \sum_{p,q\in\Lambda_+^*, |r|<\mu} a^*_{p+r} a^*_q a_{q+r} a_p - C$$

$$\geq 4\pi\mathfrak{a}_0 N + \frac{1}{2}\mathcal{H}_N - \mathcal{N}_+^2/N - C$$

Localization technique: [Lewin-Nam-Serfaty-Solovej,'14]
$f, g \in C^\infty(\mathbb{R})$ with $f^2 + g^2$, $f(s) = 1$ for $s \leq 1$, $f(s) = 0$ for $s > 2$.

For $M > 0$, set $f_M = f(\mathcal{N}_+/M)$, $g_M = g(\mathcal{N}_+/M)$. Then

$$\mathcal{R}_N \geq f_M \mathcal{R}_N f_M + g_M \mathcal{R}_N g_M - CM^{-2}(\mathcal{H}_N + 1)$$

Pick $M = cN$, for sufficiently small $c > 0$.

On g_M, use **[Lieb-Seiringer '06]**,**[Nam-Rougerie-Seiringer '16]**.

We conclude

$$S^*T^*UH_NU^*TS = \mathcal{R}_N \geq 4\pi\mathfrak{a}_0 N + c\mathcal{N}_+ - C$$

Hence: let $\psi_N \in L^2_s(\Lambda^N)$ with

$$\langle \psi_N, H_N \psi_N \rangle \leq E_N + \zeta$$

Then, **excitation vector** $\xi_N = S^*T^*U\psi_N$ is such that

$$\langle \xi_N, \mathcal{N}_+ \xi_N \rangle \leq C(\zeta + 1)$$

Stronger bounds: if $\psi_N = \chi(H_N \leq E_N + \zeta)\psi_N$, we also find

$$\langle \xi_N, (\mathcal{H}_N + 1)(\mathcal{N}_+ + 1)^k \xi_N \rangle \leq C(1+\zeta)^k$$

for any $k \in \mathbb{N}$.

Theorem: renormalized excitation Hamiltonian is such that

$$\mathcal{G}_N = C_N + \mathcal{Q}_N + \mathcal{C}_N + \mathcal{V}_N + \delta_N$$

where C_N is a **constant**, Q_N is **quadratic**,

$$\mathcal{C}_N = \frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda_+^*} \widehat{V}(p/N) \left[b_{p+q}^* b_{-p}^* \Big(\gamma_q b_q + \sigma_q b_{-q}^* \Big) + \text{h.c.} \right]$$

$$\mathcal{V}_N = \frac{1}{2N} \sum_{p,q \in \Lambda_+^*} \widehat{V}(r/N) a_{p+r}^* a_q^* a_{q+r} a_p$$

and, where,

$$\pm \delta_N \leq \frac{C}{\sqrt{N}} \left[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3 \right]$$

Problem: $\mathcal{G}_N$ still contains non-negligible **cubic** and **quartic** terms! This is substantial difference compared with case $\beta < 1$!

New cubic phase: we define

$$\widetilde{A} = \frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} \eta_r \Big[\sigma_v b^*_{r+v} b^*_{-r} b^*_{-v} + \gamma_v b^*_{r+v} b^*_{-r} b_v - \text{h.c.} \Big]$$

with $P_L = \{p \in \Lambda^*_+ : |p| \leq \sqrt{N}\}$, $P_H = \{p \in \Lambda^*_+ : |p| \geq \sqrt{N}\}$.

Set $\widetilde{S} = e^A$ and introduce **new excitation Hamiltonian**

$$\mathcal{J}_N = \widetilde{S}^* T^* U_N H_N U_N^* T \widetilde{S} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

Remark: a similar **cubic conjugation** was used in [Yau-Yin, 09].

Proposition: we can decompose

$$\mathcal{J}_N = \widetilde{C}_N + \widetilde{Q}_N + \mathcal{V}_N + \widetilde{\delta}_N$$

where $\widetilde{C}_N$ is a **constant**, $\widetilde{Q}_N$ is **quadratic** and where

$$\pm\widetilde{\delta}_N \leq \frac{C}{N^{1/4}}\left[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3\right]$$

Mechanism: we have

$$\mathcal{J}_N = e^{-\widetilde{A}}\mathcal{G}_N e^{\widetilde{A}} \simeq \mathcal{G}_N + [\mathcal{G}_N, \widetilde{A}] + \frac{1}{2}[[\mathcal{G}_N, \widetilde{A}], \widetilde{A}] + \dots$$

where

$$\mathcal{G}_N \simeq C_N + Q_N + \mathcal{C}_N + \mathcal{V}_N$$

Combine $[Q_N, \widetilde{A}], [\mathcal{V}_N, \widetilde{A}]$ with $\mathcal{C}_N$ (use **scattering equation**).

At same time, $[\mathcal{C}_N, \widetilde{A}]$ modifies **constant** and **quadratic** terms.

Diagonalization: with last **Bogoliubov transformation** R, set

$$\mathcal{M}_N = R^* \mathcal{J}_N R = R^* S^* T^* U_N H_N U_N^* T S R : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

Then

$$\begin{aligned} \mathcal{M}_N = \; & 4\pi \mathfrak{a}_N (N-1) \\ & + \frac{1}{2} \sum_{p \in \Lambda_+^*} \left[-p^2 - 8\pi \mathfrak{a}_0 + \sqrt{|p|^4 + 16\pi \mathfrak{a}_0 p^2} + \frac{(8\pi \mathfrak{a}_0)^2}{2p^2} \right] \\ & + \sum_{p \in \Lambda_+^*} \sqrt{|p|^4 + 16\pi \mathfrak{a}_0 p^2}\, a_p^* a_p + \mathcal{V}_N + \delta_N' \end{aligned}$$

where

$$\pm \delta_N' \leq C N^{-1/4} \left[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3 \right]$$

Main theorem follows from **min-max principle**, because on low-energy states of diagonal quadratic Hamiltonian, we find

$$\mathcal{V}_N \leq C N^{-1} (\zeta + 1)^{7/2}$$