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recent progress in the theory of stochastic/pathwise solutions

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work with Pierre-Louis Lions and (in some projects) Gassiat and Gess

- plan of the talk
  - review of stochastic/pathwise solutions
  - domain of dependence and speed of propagation of stochastic Hamilton-Jacobi (HJ) equations
  - homogenization
  - intermittent regularity
  - long time behavior
  - loss of regularity

- pathwise/stochastic viscosity solutions

$$du = H(Du, u, x) \cdot d\omega + F(D^2u, Du, u, x)dt$$

$$du = \mathrm{div}\Phi(u, x) \cdot d\omega$$

ω continuous (Brownian or, more generally, rough path)

$u \in \mathbb{R}$ F degenerate elliptic

F and Φ and H may depend on t

if ω depends on x it must be regular KPZ is outside the scope of the theory

applications

- pathwise control vs control in the mean
- phase transitions – asymptotics of stochastically perturbed reaction-diffusion equations
- stochastic selection principle – convergence to a distinct solution
- mean field games

- review of pathwise viscosity solutions $du = H(Du) \cdot d\omega \quad u(\cdot, 0) = u_0$

  - "deterministic viscosity solutions" $\omega \in \mathrm{BV} \Rightarrow \exists!$ solution $u \in C_{x,t}$ and comparison

    $\|(u - v)_\pm(\cdot, t)\| \leq \|(u_0 - v_0)_\pm\| \qquad \|Du(\cdot, t)\| \leq \|Du_0\|$

    in general shocks (discontinuities of Du) appear in finite time

is it possible to extend to $\omega \in C$?

main difficulties: no pointwise formulation (singularities), no martingale theory (not enough compactness), choice of stochastic calculus due to lack of regularity (Ito's calculus does not work well with maximum principle), ...

- $du = u_x dB$ ill posed

$$du(x - B(t), t) \underset{\text{Itô's formula and equation}}{=} -\frac{1}{2} u_{xx}(x - B(t), t) dt$$

- $d_I \max(u, v) \geq \max(d_I u, d_I v)$

- $H(p) = |p| \quad u_0(x) = |x|$ density in $\omega \Rightarrow$

$$u(x, t) = \max\left[(|x| + \omega(t))_+, \omega(t) - \min_{0 \leq s \leq t} \omega(s)\right]$$

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"THEOREM" If H is the difference of two convex functions, then $\exists!$ solution with the same properties as in the classical case

- solutions are continuous in H and ω
- solutions to problems with regularized H and ω converge to the same limit

  if $u_{\varepsilon,t} = H_\varepsilon(Du_\varepsilon)\dot{\omega}_\varepsilon$ with $H_\varepsilon, \omega_\varepsilon$ smooth approximations to H and ω, then $\|u_\varepsilon - u\| \underset{\varepsilon \to 0}{\to} 0$
- the contraction property

  $$\|(u-v)_\pm(\cdot,t)\| \leq \|(u_0 - v_0)_\pm\| \qquad \|Du(\cdot,t)\| \leq \|Du_0\|$$
- $\|u(\cdot,t)\|$ and $\mathrm{osc}\, u(\cdot,t)$ decrease in t
- $H(0) = 0 \Rightarrow \max u(\cdot,t)$ and $\min u(\cdot,t)$ decrease in t

  formally $d[\max u(\cdot,t)] \leq 0$ and $d[\max u(\cdot,t)] \geq 0$

- domain of dependence and finite speed of propagation Gassiat, Gess, Lions and S.

is there a domain of dependence property for $du = H(Du, x) \cdot d\omega$?

$$u_1(\cdot, 0) = u_2(\cdot, 0) \text{ in } B(0, R_0) \quad \Rightarrow \quad u_1(\cdot, t) = u_2(\cdot, t) \text{ in } B(0, R(t))?$$

- a partial positive result

$$H(p) = H_1(p) - H_2(p) \quad H_1, H_2 \text{ convex and } H_1(0) = H_2(0) = 0$$

$$u(\cdot, 0) \equiv A \text{ in } B(0, R) \quad \Rightarrow \quad u(\cdot, t) \equiv A \text{ in } B(0, R(t))$$

$$R(t) := R - L(\max_{s\in[0,T]} \omega(s) - \min_{s\in[0,T]} \omega(s))$$

- a negative result Gassiat

$$u_t = (|u_x| - |u_y|)\dot{\omega} \qquad u(x, y, 0) = |x - y| + \Theta(x, y) \qquad \begin{cases} \Theta \geq 0 \\ \Theta(x, y) \geq 1 \text{ if } x, y \geq R \end{cases}$$

$$u(0, 0, T) > 0 \quad \text{if} \quad \|\omega\|_{\mathrm{TV}_{[0,T]}} > R$$

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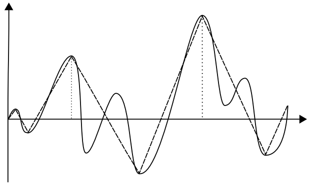
- finite speed of propagation

$du = H(Du, x) \cdot d\omega \quad H \text{ convex in } p \quad \omega \in C_0([0, T])$

$\rho_H(\xi, T) := \sup \left\{ R \geq 0 : \ u^1(\cdot, 0) = u^2(\cdot, 0) \text{ in } B_R(0) \text{ and } u^1(0, T) \neq u^2(0, T) \right\}$

skeleton $R_{0,T}(\omega)$



- positive results
  - $u^\omega(\cdot, T) = u^{R_{0,T}(\omega)}(\cdot, T) \quad \Rightarrow \quad \rho_H(\xi, T) \leq L \, \| R_{0,T}(\omega) \|_{TV([0,T])}$
  - B Brownian motion $\Rightarrow$

$$\begin{cases} \mathbb{P}\left(\| R_{0,T}(B) \|_{TV([0,T])} \geq x \right) \leq C \exp\left(-C x^\gamma\right) \\ \lim_{x \to \infty} \dfrac{\ln \mathbb{P}\left(\| R_{0,\theta(1)}(B) \|_{TV([0,\theta(1)])} \geq x \right)}{x \ln(x)} = -1 \end{cases}$$

$\theta(a) := \inf\{t \geq 0 : \ \max_{[0,t]} B - \min_{[0,t]} B = a\}$

$\Rightarrow \quad \| R_{0,T}(B) \|_{TV([0,T])} < \infty$ a.s.

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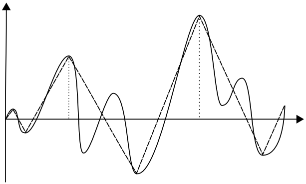

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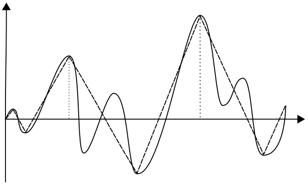

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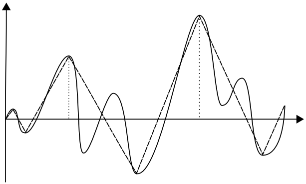

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main step of the proof

$S_{\pm H}(t)u_0$ solution of $u_t = \pm H(Du, x)$ in $\mathbb{R}^d \times (0, T]$ $u(\cdot, 0) = u_0$ in $\mathbb{R}^d$

$S_{-H}(t)u_0 = S_H(-t)$

$\xi_{s,t} := \xi_t - \xi_s$ increment of ξ over the interval $[s, t]$

$\xi(t) = \sum_{i=0}^{N-1} 1_{[t_i, t_{i+1})}(a_i(t - t_i) + b_i)$ piecewise linear path

$S_H^\xi(0, T) := S_H^\xi(t_{N-1}, t_N) \circ \cdots \circ S_H^\xi(t_0, t_1) = S_H(\xi_{t_{N-1}, t_N}) \circ \cdots \circ S_H(\xi_{t_0, t_1})$

- results
  - $S_H(c) \circ S_{-H}(b) \circ S_H(a) = S_H(a + c - b)$ for $a, c > b$
  - $\xi_t = 1_{t \in [0, t_1]}(a_0 t) + 1_{t \in [t_1, T]}(a_1(t - t_1) + a_0 t_1)$ $a_0 \geq 0$ and $a_1 \leq 0$

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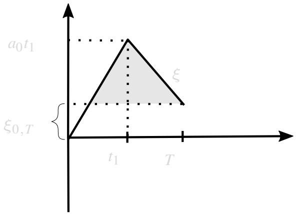

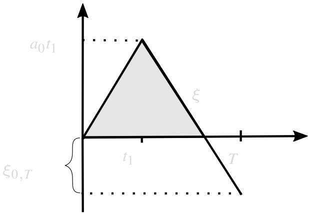

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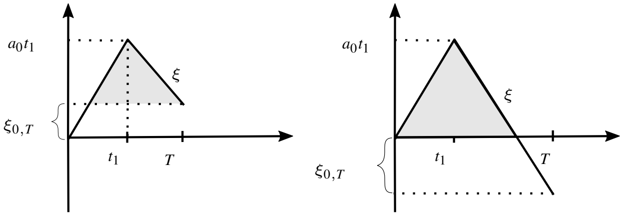

- homogenization   Ben Seeger

  - $u_t^\varepsilon = H(Du^\varepsilon, \frac{x}{\varepsilon})\dot{B}^\varepsilon$ H convex coercive periodic $B^\varepsilon \to B$ in distribution

    there exists $\overline{H}$ convex st $\begin{cases} u^\varepsilon \to \overline{u} \\ d\overline{u} = \overline{H}(D\overline{u}) \circ dB \end{cases}$

  - $u_t + H(Du, x) = f(x)\dot{\omega}$ ω piecewise constant with slope ± 1

    $u^\varepsilon = \varepsilon u(\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$ $\omega^\varepsilon(t) = \varepsilon^{1/2}\omega(\frac{t}{\varepsilon}) \underset{\varepsilon \to 0}{\longrightarrow} B$ Brownian motion

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    there exists $\overline{H}$ st $u^\varepsilon \to \overline{u}$ and $\overline{u}_t + \overline{H}(D\overline{u}) = 0$ BUT

    $\overline{H}$ does not come from the periodic homogenization

    $\varepsilon^{1/2}\omega(\frac{t}{\varepsilon})$ creates a stationary ergodic environment

  - $u_t^\varepsilon + |Du^\varepsilon| = \varepsilon^{1/2} f(\frac{x}{\varepsilon})\dot{\omega}^\varepsilon$

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- homogenization   Ben Seeger

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- intermittent regularity

$u_t = H(Du) \cdot d\omega \quad \omega \in C_0([0,\infty)) \quad \theta I \leq D^2 H \leq \Theta I$

$M(t) = \max_{0 \leq s \leq t} \omega(s) \quad m(t) = \min_{0 \leq s \leq t} \omega(s)$

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$H(p) = |p|^2$ bound true for $d \geq 1$ Gassiat and Gess

estimate not true when $d > 1$ and H anisotropic

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there exists an uncountable subset of $(0,\infty)$ with no isolated points and of Hausdorff measure $1/2$ off of which, $u(\cdot,t) \in C^{0,1}(\mathbb{R}^d)$ and, when $d = 1$, $u(\cdot,t) \in C^{1,1}(\mathbb{R})$

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- new regularity results for the "deterministic" problem and some proofs

$u_t = \pm H(Du) \quad \theta I \leq D^2 H \leq \Theta I$

regularizing effect $\quad D^2 u(\cdot, t) \geq -\frac{1}{\theta t} I$

regularity is propagated $\quad \begin{cases} D^2 u(\cdot, 0) \geq -CI \ \Rightarrow\ D^2 u(\cdot, t) \geq -CI \\ D^2 u(\cdot, 0) \leq CI \ \Rightarrow\ D^2 u(\cdot, t) \leq \dfrac{C}{(1-\Theta C t)_+} I \end{cases}$

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the proof of the regularizing effect

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$$u_t = H(Du) \quad u(\cdot,0)=u_0 \quad u(x,t) = \begin{cases} \sup \left[u_0(y) - tL(\frac{x-y}{t})\right] = u_0(\bar{y}) - tL(\frac{x-\bar{y}}{t}) \\ \sup \left[u_0(x-z) - tL(\frac{z}{t})\right] = u_0(x-\bar{z}) - tL(\frac{\bar{z}}{t}) \\ \sup \left[u_0(z) - tL(\frac{y}{t}) : y+z=x\right] = u_0(\bar{z}) - tL(\frac{\bar{y}}{t}) \end{cases}$$

- $u(x+h\chi,t) - 2u(x,t) + u(x-h\chi) \geq$
  $u_0(\bar{y}) - tL(\frac{x+h\chi-\bar{y}}{t}) - 2u_0(\bar{y}) + tL(\frac{x-\bar{y}}{t}) + u_0(\bar{y}) - tL(\frac{x-h\chi-\bar{y}}{t}) \geq -\frac{1}{\theta t}h^2|\chi|^2$
- $u(x+h\chi,t) - 2u(x,t) + u(x-h\chi) \geq$
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the proof of the regularizing effect

$$F(Du(\cdot,0))D^2u(\cdot,0)F(Du(\cdot,0)) \geq -CI \;\Rightarrow\; F(Du(\cdot,t))D^2u(\cdot,t)F(Du(\cdot,t)) \geq -\frac{C}{1+Ct}I$$

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- long time behavior

$$du = H(Du)\cdot d\omega \quad u(\cdot,0)=u_0 \qquad \begin{cases} H \text{ continuous}, \; u_0 \text{ continuous and “periodic” in } x \\ \omega \text{ continuous and } \omega(0)=0 \\ H(0)=0 \quad \Rightarrow \text{ constants are solutions} \end{cases}$$

$u(\cdot,t) \underset{t\to\infty}{\to} u^\infty$ with u^∞ constant in x and depending only on u_0 and ω ?

- $H(p) = v\cdot p \quad v\in\mathbb{R}^d$

  $u(x,t) = u_0(x+v\omega(t)) \not\to$ constant

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“nonlinearity” and monotonicity of $\omega \Rightarrow$ limit

what happens if ω oscillates ?

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- $t \to \infty$ proof

$d = 1$ $\omega =$ Brownian motion u_0 periodic

$$du = H(u_x) \underset{(S)}{\circ} dB = H(u_x) \underset{(I)}{\cdot} dB + \frac{1}{2} H'(u_x)^2 u_{xx} dt$$

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$M_t = \int_0^1 u(x,t) dx$ is a bounded martingale $\Rightarrow$

$$M_t \underset{t \to \infty}{\to} M_\infty \text{ and } \int_0^\infty \left(\int_0^1 H(u_x) dx\right)^2 dt < \infty \text{ a.s.}$$

if $H(z) > 0$ for $z \neq 0$, then $u(x,t,\omega) \underset{t \to \infty}{\to} M_\infty$ a.s.

argument false due to shocks!

what if H is more regular, for example, convex?

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- the long time behavior in the convex case $\quad H$ convex $\quad H(0)=0 \quad H(p)>0$ if $p\neq 0$

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$$u(\cdot,t) \underset{t\to\infty}{\to} u_\infty$$

  - true for Brownian motion

    off an uncountable set of times with no isolated points and of Hausdorff-dimension $1/2$

  - u_∞ is random

    $du=|u_x|\circ dB$ in $\mathbb{R}\times(0,\infty) \quad u(x,0)=1-|1-x|$ in $[0,2]$ extend with period 2

    there exist $A_\pm$ with positive probability st $u_\infty\le 1/4$ in A_+ and $u_\infty \ge 3/4$ in A_-

  - idea of the proof

    intermittent regularity (Lip bound) and monotonicity of Lip bound $\Rightarrow \lim\limits_{t\to\infty}\|Du(\cdot,t)\|=0$

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- some open problems about asymptotic behavior ($t \to \infty$)
  - H not convex or concave? x-dependent problems?
  - $du = \Sigma_{i=1}^{K} H_i(Du_i) \cdot \dot{\omega}_i \quad \omega_i$ "independent"

    $\dot{\omega}_2 = -\dot{\omega}_1 = \dot{\omega} \qquad du = (H_1 - H_2)(Du)\dot{\omega}$

    $H_1 = H_2 = H \qquad \dot{\omega} = \dot{\omega}_1 + \dot{\omega}_2 \qquad du = H(Du)(\omega_1 + \omega_2)$
  - systematic approach to ergodicity when ω Brownian

- lack of regularity
  - $du = H(u_x) \circ dB + \lambda u_{xx}\, dt$
    - how regular is the solution when $\lambda > 0$?
    - is the map $t \to \mathbb{E} \int u(x,t)\, dx$ a martingale?
  - (surprisingly ?) NO if λ is small
    - there exists $\lambda_0 > 0$ such that, for $\lambda > \lambda_0$, $u \in L^2_\omega((L^\infty_t(H^2_x)) \Rightarrow$ martingale property
    - result sharp

      $t \to \mathbb{E} \int u(x,t)\, dx$ martingale $\Rightarrow$ $\mathbb{E} \int u(x,t)\, dx$ is constant in time

      if regularity holds for all $\lambda > 0$, we must have (after letting $\lambda \to 0$) that

      $\mathbb{E} \int u(x,t)\, dx = \int u_0(x)\, dx$ if $du + H(u_x) \circ dB = 0$

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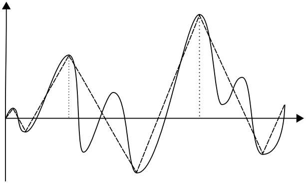
- sketch of the proof intermittent regualrity

skeleton $R_{0,T}(\omega)$

$u^\omega(\cdot, T) = u^{R_{0,T}(\omega)}(\cdot, T) \Rightarrow$

$Du^\omega(\cdot, T) = Du^{R_{0,T}(\omega)}(\cdot, T)$



decrease of gradient in time $\Rightarrow \|Du^{R_{0,T}(\omega)}(\cdot, T)\| \leq \|Du^{R_{0,T}(\omega)}(\cdot, m(T))\|$

$u^{R_{0,T}(\omega)}(\cdot, m(t)) = S_{-H}(M(T) - m(T))u^{R_{0,T}(\omega)}(\cdot, M(T))$

$D^2 u^{R_{0,T}(\omega)}(\cdot, m(t)) \leq \dfrac{1}{\theta(M(T) - m(T))}$

$\|u^{R_{0,T}(\omega)}\| \leq \|u\|$