Scaling limit and universal finite size corrections in 2D interacting Ising models

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Based on joint works with R. Greenblatt and V. Mastropietro

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Outline

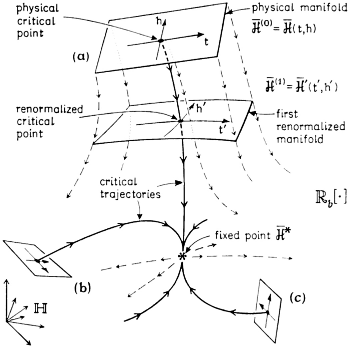
Universality, scaling limits and Renormalization Group

The scaling limit of the Gibbs measure of a critical stat-mech model is expected to be **universal**.

Conceptually, the route towards universality is clear:

1. Integrate out the small-scale d.o.f., rescale, show that the critical model reaches a fixed point (Wilsonian RG).

2. Use CFT to classify the possible fixed points (complete classification in 2D; recent progress in 3D).



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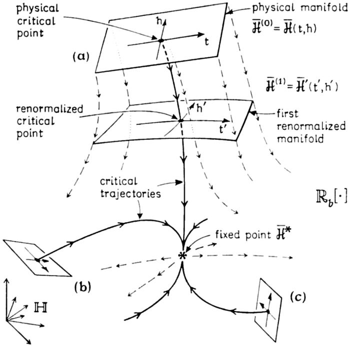

Challenge: Prove universality starting from an explicit class of microscopic Hamiltonians.

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where:

- $x \in a\mathbb{Z}^2 \cap \Omega$ ($a =$ lattice spacing; $\Omega \subset \mathbb{R}^2$);
- $\sigma_x = \pm 1$; $\langle x, y \rangle$ indicates n.n. sites in Λ;
- $v(x)$ finite-range; λ small (e.g., $\lambda, v \geq 0$).
- Boundary conditions: open, or $+/-$ (say).

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Gibbs measure:

$$\mu_{\beta;a,\Omega}(A) = \frac{1}{Z_{\beta;a,\Omega}} \sum_{\underline{\sigma}} e^{-\beta H(\sigma)} A(\sigma).$$

Phase transition

Take a fixed, $\Omega \nearrow \mathbb{R}^2$. Known:

- Small enough β: unique infinite volume Gibbs state with exp. decaying correlations.
- Large enough β: two distinct pure states with $m_\beta \neq 0$ and exp. decaying truncated corr.
- Unique value of β, called β_c, separating magnetized from non-magnetized phase.

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- Unique value of β, called β_c, separating magnetized from non-magnetized phase.

Expected: at $\beta = \beta_c$, unique infinite volume Gibbs state, denoted

$$\mu_{\beta_c;a,\mathbb{R}^2} = \lim_{\Omega \nearrow \mathbb{R}^2} \mu_{\beta_c;a,\Omega},$$

with polynomially decaying correlations.

Critical theory

Expected: fix $\beta = \beta_c$, fix Ω, rescale spin variables:

$$\sigma(x) = a^{-1/8}\sigma_x, \quad \varepsilon_j(x) = a^{-1}(\sigma_x\sigma_{x+a\hat{e}_j} - \mu_{\beta_c;a,\mathbb{R}^2}(\sigma_x\sigma_{x+a\hat{e}_j}))$$

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Then take $a \to 0$: the limit

$$\lim_{a \to 0} \mu_{\beta_c;a,\Omega}(\sigma(x_1)\cdots\sigma(x_n)\varepsilon_{j_1}(y_1)\cdots\varepsilon_{j_m}(y_m))$$

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should exist and satisfy remarkable properties. E.g.,

- Be independent of λ, up to a finite multiplicative renormalization, and equal to the correlations of the minimal CFT model with $c = 1/2$.
- Be conformal covariant under Riemann mapping of $\Omega \to \Omega'$. Correlations in any domain obtained from those in the half-plane.

Universal finite-size corrections

Fix $\beta = \beta_c$, fix Ω, compute the free energy:

$$\log Z_{a,\Omega} = a^{-2}|\Omega| f(\lambda) + a^{-1}|\partial\Omega|\tau(\lambda) + c_\Omega(a, \lambda).$$

Here $f(\lambda)$ and $\tau(\lambda)$ are independent of Ω (bulk and surface free energies). Expected:

$$\lim_{a\to 0} c_\Omega(a, \lambda) = c_\Omega \qquad \text{independent of } \lambda.$$

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E.g., if $\Omega =$ torus with aspect ratio ξ,

$$c_\Omega = \log(\theta_2 + \theta_3 + \theta_4) - \frac{1}{3}\log(4\theta_2\theta_3\theta_4),$$

where $\theta_i = \theta_i(e^{-\pi\xi})$ (Jacobi theta func^n). If $\xi \to \infty$

$$c_\Omega \sim c\frac{\pi}{6}\xi, \qquad \text{with} \qquad c = \frac{1}{2} = \text{central charge.}$$

Recent results

In recent years, constructive RG methods allowed us to prove some aspects of the picture above:

- we constructed bulk scaling limit of energy corr.
- we computed c_Ω for Ω torus, as $\xi \to \infty$,

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Work in progress on:

- scaling limit of energy corr. on the cylinder,
- computation of c_Ω for generic aspect ratio.

Technical issues to be faced: finite-size corrections to RG procedure; RG flow of boundary conditions.

Outline

Nearest neighbor Ising

If $\lambda = 0$,

$$H = H^0 = -J \sum_{\langle x,y \rangle} \sigma_x \sigma_y ,$$

the model is exactly solvable: solution due to Onsager, Kaufman, Yang, Kasteleyn, Kac-Ward, Montroll-Potts-Ward, Schultz-Lieb-Mattis, McCoy--Wu, Tracy, Barouch, Perk, Kadanoff-Ceva, ...

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E.g., fix a and send $\Omega \nearrow \mathbb{R}^2$: closed formulas for the free energy, specific heat, magnetization, energy and spin correlations (along special directions) are known.

Nearest neighbor Ising: scaling limit

At $\beta_c = \frac{1}{2J}\log(\sqrt{2}+1)$, polynomial decay of corr^{ns}.

From explicit formulas, scaling limit in the plane:

$$\lim_{a\to 0} \mu_{\beta_c;a,\mathbb{R}^2}(\sigma(x)\sigma(y)) = \frac{A}{|x-y|^{1/4}}, \quad A = 0.70338016\cdots$$

$$\lim_{a\to 0} \mu_{\beta_c;a,\mathbb{R}^2}(\varepsilon_{j_1}(x)\varepsilon_{j_2}(y)) = \frac{1}{\pi^2}\frac{1}{|x-y|^2}.$$

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More in general, the $2n$-point energy correlations are

$$\lim_{a\to 0} \mu_{\beta_c;a,\mathbb{R}^2}(\varepsilon_{j_1}(x_1)\cdots\varepsilon_{j_{2n}}(x_{2n})) = \pi^{-2n}|\mathrm{Pf}K(z_1,\ldots,z_{2n})|^2,$$

where $z_j = (x_j)_1 + i(x_j)_2$ and $K_{ij}(z_1,\ldots,z_{2n}) = \frac{\mathbb{1}_{i\neq j}}{z_i - z_j}$.

Nearest neighbor Ising: spin-spin correlations

The general formula for the scaling limit of $2n$-point spin can be guessed by CG/CFT methods, but remained elusive for many years. Recently proved by Dubedat, Chelkak-Hongler-Izyurov:

$$\lim_{a\to 0} \mu^T_{\beta_c;a,\mathbb{R}^2}(\sigma(x_1)\cdots\sigma(x_{2n})) =$$

$$= \Big(\frac{A}{\sqrt{2}}\Big)^{2n}\Big[\sum_{\substack{\varepsilon_1,\dots,\varepsilon_{2n}=\pm\\ \varepsilon_1+\cdots+\varepsilon_{2n}=0}} \prod_{1\le i<j\le 2n} |z_i - z_j|^{\varepsilon_i\varepsilon_j/2}\Big]^{1/2}.$$

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Other domains: in the half-plane, similar formulas (via 'image method'). More general domains, via Riemann map (Smirnov, Chelkak-Hongler-Izyurov)

Nearest neighbor Ising: universal finite-size corrections

Finite-size corrections to critical free energy. Take Ω a torus with sides ℓ_1, ℓ_2: asymptotically as $a \to 0$,

$$\log Z_{\beta_c;a,\Omega} = \frac{\ell_1 \ell_2}{a^2} f_\infty + \log(\theta_2 + \theta_3 + \theta_4) - \frac{\log(4\theta_2\theta_3\theta_4)}{3} + o(1)$$

where

$$f_\infty = \log 2 + \frac{1}{2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{dk}{(2\pi)^2} \log(2 - \cos k_1 - \cos k_2)$$

and $\theta_i = \theta_i(e^{-\pi \frac{\ell_1}{\ell_2}})$. Computed by Ferdinand-Fisher. Re-derived via CFT by Di Francesco-Saleur-Zuber.

Outline

Theorem 1 (G-Greenblatt-Mastropietro 2012)

Let $v(x)$ be finite-range, rotation-symmetric and λ small enough. There exists $\beta_c(\lambda)$, analytic in λ, s.t., letting $\mu_{\beta_c(\lambda);a,\mathbb{R}^2} = \lim_{L\to\infty} \mu_{\beta_c(\lambda);a,\mathbb{T}^2_L}$,

$$\lim_{a\to 0} \mu_{\beta_c(\lambda);a,\mathbb{R}^2}(\varepsilon_{j_1}(x_1)\cdots\varepsilon_{j_{2n}}(x_{2n})) = \left(\frac{Z}{\pi}\right)^{2n} |\mathrm{Pf}K(z_1,\ldots,z_{2n})|^2$$

where $Z = Z(\lambda) = 1 + O(\lambda)$ is analytic in λ.

Remarks

- Strategy proposed by Spencer. Previous results on 2-pt funct^n by Pinson-Spencer, Mastropietro.

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- The proof of Thm 1 provides explicit bounds on the speed of convergence of the lattice n-point energy correlations to their limit.
- We also control the massive scaling limit $\beta(a) = \beta_c(\lambda) + am^*/4J$. In this case, the dressed propagator has mass $Z^* m^*$ and $Z^* = = 1 + O(\lambda)$ is an analytic mass renormalization.

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- If $\beta \neq \beta_c(\lambda)$, proof much easier. At finite a, exp. decay with correlation length $\propto a$.
- λ and v can have either signs.
- Rotational invariance unimportant.

Theorem 2 (G-Mastropietro 2013)

Let $v(x) \geq 0$; λ positive and small enough; $\Omega_\xi =$ rectangular torus, aspect ratio ξ. Then, as $\xi \to \infty$,

$$\lim_{a\to 0} \Big(\log Z_{\beta_c(\lambda);a,\Omega_\xi} - a^{-2}|\Omega_\xi| f_\infty(\lambda) \Big) = c\frac{\pi}{6}\xi + o(\xi),$$

where $\boxed{c = 1/2}$ *and*

$$f_\infty(\lambda) = \frac{1}{|\Omega_\xi|} \lim_{a\to 0} a^2 \log Z_{\beta_c(\lambda);a,\Omega_\xi},$$

which is independent of ξ and is a non-trivial analytic function of λ.

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- In Thm 2, the condition $\lambda, v \geq 0$ is technical; eliminating it requires better understanding of the finite-size corrections to RG flow.

  Work in progress: any sign of λ, v, any ξ,

$$\lim_{a\to 0}\Big(\log Z_{\beta_c(\lambda);a,\Omega_\xi} - a^{-2}|\Omega_\xi| f_\infty(\lambda)\Big) =$$

$$= \log(\theta_2+\theta_3+\theta_4) - \frac{\log(4\theta_2\theta_3\theta_4)}{3}, \quad \theta_i = \theta_i(e^{-\pi\xi}).$$

Interacting Ising in the half-plane

More work in progress (joint with G. Antinucci and R. Greenblatt). Scaling limit in the half-plane $\mathbb{H}$, starting from $\Omega =$ cylinder:

$$\lim_{a\to 0}\lim_{\Omega\nearrow\mathbb{H}} \mu_{\beta_c(\lambda);a,\Omega}(\varepsilon_{j_1}(x_1);\cdots;\varepsilon_{j_n}(x_n)) =$$

$$= \Big(\frac{Z}{\pi}\Big)^{2n} \mathrm{Pf} K(z_1,\ldots,z_{2n},\bar z_{2n},\ldots,\bar z_1),$$

where $Z = Z(\lambda)$ is the same as in Thm 1.

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Key new ingredient: control RG flow of boundary quadratic terms, which are dimensionally marginal. A cancellation produces a dimensional improvement, which makes them effectively irrelevant.

Outline

Nearest neighbor model $\leftrightarrow$ free fermions

Let $Z_0 = Z_{\beta;a,\mathbb{T}_L^2}\big|_{\lambda=0}$. If $\mathcal{F}_L$ is Fisher lattice on $\mathbb{T}_L^2$,

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After some algebraic manipulations,

$$Z_0 = N_0 \int \prod_{\substack{x\in\mathbb{T}_L \\ \omega=\pm}} d\psi_{x,\omega} d\chi_{x,\omega} e^{-\frac{1}{2}(\psi,C_-\psi)-\frac{1}{2}(\chi,C_+\chi)-(\psi,Q\chi)},$$

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where $N_0 = (2a^{-2}\cosh^2(\beta J))^{a^{-2}L^2}$, Q is a derivative,

$$\hat{C}_\pm(k) = \frac{1}{4\pi a}\begin{pmatrix} -i\sin(ak_1)+\sin(ak_2) & i\sigma_\pm(k) \\ -i\sigma_\pm(k) & -i\sin(ak_1)+\sin(ak_2) \end{pmatrix}$$

and, if $t := \tanh(\beta J)$,

$$\sigma_\pm(k) = \cos(ak_1) + \cos(ak_2) \pm 2(\sqrt{2}\pm 1)/t.$$

Nearest neighbor model $\leftrightarrow$ free fermions

In other words, n.n. model = free fermions:

$$Z_0 = N_0' \int P_-(d\psi)\, P_+(d\chi) e^{-(\psi, Q\chi)},$$

where $P_\pm$ is the normalized Grassmann Gaussian integral with propagator $C_\pm^{-1}$. Note:

- C_- singular ('critical') at $t = \sqrt{2} - 1$;
- C_+ not singular ('massive'), $\forall t \geq 0$.

Interacting model $\leftrightarrow$ interacting fermions

In the presence of a weak interaction, we use cluster expansion to get:

$$Z_\lambda = e^{a^{-2}L^2E(\lambda)} \int P_-(d\psi)\, P_+(d\chi) e^{-(\psi, Q\chi) + V(\psi,\chi)}.$$

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Now, since χ is massive, we can integrate it out:

$$Z_\lambda = e^{a^{-2}L^2 E^{(0)}(\lambda)} \int P(d\psi) e^{V^{(0)}(\psi)}.$$

$V^{(0)}$ is again quadratic + quartic + etc. Its kernels are analytic in λ and exp. decaying in space (thanks to fermionic cluster exp. and BBFK det. formula).

Multiscale integration

At dominant order in a, the propagator of $P(d\psi)$ is

$$g_{\omega,\omega'}(x-y) = \int P(d\psi)\psi_{x,\omega}\psi_{y,\omega'} \simeq \frac{\delta_{\omega,\omega'}\zeta^{-1}}{(x-y)_1 + i\omega(x-y)_2}$$

We decompose it in scales:

$$g(x) = \sum_{h\leq N} g^{(h)}(x), \qquad g^{(h)}(x) \simeq 2^h g^{(0)}(2^h x),$$

where $N = [\log_2 a^{-1}]$, $g^{(0)}$ is exp. decaying on scale 1.

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We rewrite $\psi = \sum_{h\le 0} \psi^{(h)}$ and we iteratively integrate out the fields on scale $0, -1, -2, \ldots$ At each step, we compute the effective potential by fermionic cluster expansion and BBFK formula.

Effective potential

After the integration of the fields on scales $> h$:

$$Z_\lambda = e^{a^{-2}L^2 E^{(h)}(\lambda)} \int P_{\leq h}(d\psi) e^{V^{(h)}(\psi)},$$

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$$V^{(h)}(\psi) = \sum_{n\geq 2} \sum_{\substack{x_1,\dots,x_n \\ \omega_1,\dots,\omega_n}} W^{(h)}_{n;\underline{\omega}}(x_1,\dots,x_n)\psi_{x_1,\omega_1}\cdots\psi_{x_n,\omega_n},$$

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$$\| W^{(h)}_{n;\underline{\omega}} \|_{\kappa,h} = \int dx_2 \cdots dx_n e^{\kappa 2^h d(\underline{x})} |W^{(h)}_{n;\underline{\omega}}(\underline{x})| \leq C^n 2^{h(2-\frac{n}{2})} |\lambda|^{\frac{n}{2}-1}$$

Relevant, marginal, irrelevant terms

Norm on the kernels of the effective potential:

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The scaling factor $2^{h(1-n/2)}$ defines the scaling dimension of the n-legged kernel: $D_n = 2 - n/2$.

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If $D_n < 0 \Leftrightarrow n \geq 6$ (irrelevant terms), the estimate on the norm is ‘for free’.

Relevant, marginal, irrelevant terms

Norm on the kernels of the effective potential:

$$\|W_{n;\underline{\omega}}^{(h)}\|_{\kappa,h} \leq C^n 2^{h(2-\frac{n}{2})}|\lambda|^{\frac{n}{2}-1}.$$

The scaling factor $2^{h(1-n/2)}$ defines the scaling dimension of the n-legged kernel: $D_n = 2 - n/2$.

If $D_n < 0 \Leftrightarrow n \geq 6$ (irrelevant terms), the estimate on the norm is 'for free'.

If $D_n > 0$ ($n = 2$, relevant) or $D_n = 0$ ($n = 4$, marginal), the estimate is not for free. We need to look more closely at these terms.

Localization of the quadratic terms

We decompose them into local+ irrelevant part: e.g.,

$$\sum_{\substack{x,y \\ \omega,\omega'}} W^{(h)}_{2;(\omega,\omega')}(x,y)\psi_{x,\omega}\psi_{y,\omega'} = 2^h \nu_h \sum_x i\omega\psi_{x,\omega}\psi_{x,-\omega}$$

$$+\zeta_h \sum_x \psi_{x,\omega}(-i\omega\partial_1 + \partial_2)\psi_{x,\omega} + \mathcal{R}V_2^{(h)}(\psi).$$

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The non-local contribution $\mathcal{R}V_2^{(h)}(\psi)$ automatically satisfies the desired dimensional bound. The local part is parametrized by ν_h, ζ_h (counterterms for β_c and ζ); we choose the free parameters t, ζ in such a way that they flow to zero as $h \to -\infty$.

Localization of the quartic terms

Quartic terms: $V^{(h)}(\psi) = \mathcal{L}V_4^{(h)}(\psi) + \mathcal{R}V_4^{(h)}(\psi)$.

The potentially dangerous term is the local part,

$$\mathcal{L}V_4^{(h)}(\psi) = \sum_{\omega_1,\ldots,\omega_4} \lambda_{h;(\omega_1,\ldots,\omega_4)} \sum_x \psi_{x,\omega_1}\psi_{x,\omega_2}\psi_{x,\omega_3}\psi_{x,\omega_4},$$

with

$$\lambda_{h;\underline{\omega}} = \sum_{x_2,x_3,x_4} W_{4;\underline{\omega}}^{(h)}(x, x_2, x_3, x_4).$$

However, $\lambda_{h;\underline{\omega}} = 0$, thanks to the Grassmann anti--commuting rules, because ω_i assumes only 2 values.

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What if we have a boundary? We decompose $W^{(h)}_{2;(\omega,\omega')}(x,y)$ into a bulk + boundary contribution.

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In general, this has the effect of change the boundary conditions at $\partial\Omega$. In our case, an additional cancellation shows that $\delta_{h,\omega}$ flows to zero: o.b.c. are asymptotically invariant under RG flow.

Open problems

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- Spin-spin correlator? It corresponds to partition function in the presence of a 'string defect'. Possible strategies: FFE (affected by UV div.); treat string defect in analogy w. boundary term.
- Conformal covariance? More general domains? Our method relies too heavily on the exact solution in the cylinder. Can we perform the RG without diagonalizing the propagator?
- Effect of boundary conditions in more general models (e.g., interacting dimers, 6V, 8V)? There, we need to study the RG flow of $\delta_{h,\omega}$ (anomalous flow? boundary critical exponents?).

Thank you!