Inertial Ericksen-Leslie model for liquid crystals and the zero inertia density limits

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March 6th, 2019

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Ericksen-Leslie's system for liquid crystals

- Hydrodynamic theory of liquid crystals: (Leslie's paper "Some constitutive equations for liquid crystals". Arch. Rational Mech. Anal. 28 (1968), no. 4, 265-283.)

$$\begin{cases} \rho\dot{\mathrm{u}} = \rho\mathrm{F} + \mathrm{div}\hat{\sigma}\,, \mathrm{div}\mathrm{u} = 0\,, & \text{(Balance of momentum)} \\ \rho_1\ddot{\mathrm{d}} = \rho_1\mathrm{G} + \hat{\mathrm{g}} + \mathrm{div}\pi\,. & \text{(Balance of angular momentum)} \end{cases} \quad (0.1)$$

- Unknown functions:
  - u: flow velocity.
  - d: direction field of the liquid molecules, $|\mathrm{d}| = 1$.
- ρ: fluid density (constant, for simplicity).
- $\rho_1 \geq 0$: the inertial density.
- $\hat{\mathrm{g}}$: intrinsic force associated with d.
- π: director stress.
- F and G: external body force and external director body force.

Ericksen-Leslie's system for liquid crystals

- $\mathrm{A} = \frac{1}{2}(\nabla \mathrm{u} + \nabla^{\top}\mathrm{u})$: rate of strain tensor.
- $\mathrm{B} = \frac{1}{2}(\nabla \mathrm{u} - \nabla^{\top}\mathrm{u})$: skew-symmetric part of the strain rate.
- $\omega = \dot{\mathrm{d}} = \partial_t \mathrm{d} + (\mathrm{u} \cdot \nabla)\mathrm{d}$: material derivative of d.
- $\mathbb{N} = \omega + \mathrm{Bd}$: rigid rotation part of director changing rate by fluid vorticity. ("Oldroyd derivative")
- The constitutive relations for $\hat{\sigma}$, π and $\hat{\mathrm{g}}$ are given by:

$$\begin{aligned} \hat{\sigma}_{ij} &= -p\delta_{ij} + \sigma_{ij} - \rho \frac{\partial W}{\partial \mathrm{d}_{k,i}} \mathrm{d}_{k,j} \,, \quad \pi_{ij} = \beta_i \mathrm{d}_j + \rho \frac{\partial W}{\partial \mathrm{d}_{j,i}} \,, \\ \hat{\mathrm{g}}_{ij} &= \gamma \mathrm{d}_i - \beta_j \mathrm{d}_{i,j} - \rho \frac{\partial W}{\partial \mathrm{d}_i} + \mathrm{g}_i \,. \end{aligned} \tag{0.2}$$

- p: the pressure, vector $\beta = (\beta_1, \cdots, \beta_n)^{\top}$ and the scalar function γ: Lagrangian multipliers for the constraint $|\mathrm{d}| = 1$.

Ericksen-Leslie's system for liquid crystals

- W: the Oseen-Frank energy functional

$$2W = k_1(\mathrm{div}\mathrm{d})^2 + k_2|\mathrm{d}\cdot(\nabla\times\mathrm{d})|^2 + k_3|\mathrm{d}\times(\nabla\times\mathrm{d})|^2$$
$$+(k_2+k_4)\left[\mathrm{tr}(\nabla\mathrm{d})^2 - (\mathrm{div}\mathrm{d})^2\right].$$

- g: the kinematic transport $\mathrm{g}_i = \lambda_1\mathbb{N}_i + \lambda_2\mathrm{d}_j\mathrm{A}_{ij}$: the effect of the macroscopic flow field on the microscopic structure. 1st term: rigid rotation of molecule; 2nd term: stretching of the molecule by the flow.
- σ: stress tensor $\sigma_{ij} = \mu_1\mathrm{d}_k\mathrm{A}_{kp}\mathrm{d}_p\mathrm{d}_i\mathrm{d}_j + \mu_2\mathbb{N}_i\mathrm{d}_j + \mu_3\mathrm{d}_i\mathbb{N}_j + \mu_4\mathrm{A}_{ij} + \mu_5\mathrm{A}_{ik}\mathrm{d}_k\mathrm{d}_j + \mu_6\mathrm{d}_i\mathrm{A}_{jk}\mathrm{d}_k$. The coefficients μ_i are called Leslie coefficients, $\mathrm{div}\sigma = \frac{1}{2}\mu_4\Delta\mathrm{u} + \mathrm{div}\tilde{\sigma}$.
- $\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6.$
- $\mu_2 + \mu_3 = \mu_6 - \mu_5$: *Parodi's relation.*
- For simplicity, assume $\rho = 1$, $\mathrm{F} = 0$, $\mathrm{G} = 0$. We also take $k_1 = k_2 = k_3 = 1$, $k_4 = 0$.

The dynamic theory and inertia

- The derivation of the dynamic equations was proposed by Ericksen and then adjusted by Leslie and started from an energy balance:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \{\frac{1}{2}|u|^2 + \rho_1|\dot{d}|^2 + W\}\,\mathrm{d}v$$
$$= \int_V (-\Delta)\,\mathrm{d}v + \text{boundary terms and “harmless terms”}$$

  where Δ dissipation function and W the Oseen-Frank energy.
- Concerning the inertial constant ρ_1, F. Leslie in *Adv. in Physics, v. 4, 1978* stated:
  "*the term involving σ represent rotational kinetic energy of the material element and therefore σ is an inertial constant. While this contribution to the kinetic energy is undoubtely negligible in most circumstances we retain it in the general form which follows, since it could conceivably play a nontrivial rote when the anisotropic axis is subjected to large accelerations...*"

Ericksen-Leslie's system for liquid crystals

- Ericksen-Leslie's hyperbolic liquid crystal model reduces to the following form:

$$\begin{cases} \partial_t \mathrm{u} + \mathrm{u}\cdot\nabla\mathrm{u} - \frac{1}{2}\mu_4\Delta\mathrm{u} + \nabla p = -\mathrm{div}(\nabla\mathrm{d}\odot\nabla\mathrm{d}) + \mathrm{div}\tilde{\sigma}\,, \\ \mathrm{div}\mathrm{u} = 0\,, \\ \rho_1\ddot{\mathrm{d}} = \Delta\mathrm{d} + \gamma\mathrm{d} + \lambda_1(\dot{\mathrm{d}} - \mathrm{B}\mathrm{d}) + \lambda_2\mathrm{A}\mathrm{d}\,, \end{cases} \tag{0.3}$$

on $\mathbb{R}^n\times\mathbb{R}^+$ with the constraint $|\mathrm{d}| = 1$, where the Lagrangian multiplier γ is given by

$$\gamma \equiv \gamma(\mathrm{u},\mathrm{d},\dot{\mathrm{d}}) = -\rho_1|\dot{\mathrm{d}}|^2 + |\nabla\mathrm{d}|^2 - \lambda_2\mathrm{d}^\top\mathrm{A}\mathrm{d}\,. \tag{0.4}$$

- Initial data:

$$\mathrm{u}|_{t=0} = \mathrm{u}^{in}(x),\ \dot{\mathrm{d}}|_{t=0} = \tilde{\mathrm{d}}^{in}(x),\ \mathrm{d}|_{t=0} = \mathrm{d}^{in}(x)\,, \tag{0.5}$$

where d^{in} and $\tilde{\mathrm{d}}^{in}$ satisfy the constraint and compatibility condition:

$$|\mathrm{d}^{in}| = 1\,,\quad \tilde{\mathrm{d}}^{in}\cdot\mathrm{d}^{in} = 0\,. \tag{0.6}$$

Special case: Navier-Stokes coupled with wave map

A particularly important special case of the hyperbolic system of Ericksen-Leslie's model is that the term $\mathrm{div}\tilde{\sigma}$ vanishes. Namely, the coefficients $\mu_i's$, $(1 \le i \le 6, i \ne 4)$ of $\mathrm{div}\tilde{\sigma}$ are chosen as 0, which immediately implies $\lambda_1 = \lambda_2 = 0$. Consequently, the system (0.3) reduces to a model which is Navier-Stokes equations coupled with a (damping) wave map from $\mathbb{R}^n$ to $\mathbb{S}^2$:

$$\begin{cases} \partial_t \mathrm{u} + \mathrm{u} \cdot \nabla \mathrm{u} + \nabla p = \frac{1}{2}\mu_4 \Delta \mathrm{u} - \mathrm{div}(\nabla \mathrm{d} \odot \nabla \mathrm{d}) \,, \\ \mathrm{div}\mathrm{u} = 0 \,, \\ \rho_1 \ddot{\mathrm{d}} = \Delta \mathrm{d} + (-\rho_1 |\dot{\mathrm{d}}|^2 + |\nabla \mathrm{d}|^2)\mathrm{d} \,. \end{cases} \tag{0.7}$$

- if $\mathrm{d} =$ constant vector, then u satisfies the standard Navier-Stokes equations.
- if $\mathrm{u} = 0$, then d is called "*twisted wave*" (proposed by J.L. Ericksen, 1968)

$\rho_1 = 0, \lambda_1 = -1$, parabolic model

- If $\rho_1 = 0$ and $\lambda_1 = -1$ in the 3rd equation of (0.3), the system reduces to the parabolic Ericksen-Leslie's system.
- The static analogue of the parabolic Ericksen-Leslie's system is Oseen-Frank model (Hardt-Kinderlehrer-Lin, 80's).
- Ginzburg-Landau approximation : partial regularity and regularity (Lin-Liu, 90's and early 00's).
- Global weak solutions with at most a finite number of singular times (2D, Lin-Lin-Wang, 2010, 3D, Lin-Wang, 2015).

$$\begin{cases} \partial_t \mathrm{u} + \mathrm{u}\cdot\nabla\mathrm{u} + \nabla p = \Delta\mathrm{u} - \mathrm{div}(\nabla\mathrm{d}\odot\nabla\mathrm{d}), \\ \mathrm{div}\mathrm{u} = 0, \\ \partial_t\mathrm{d} + \mathrm{u}\cdot\nabla\mathrm{d} = \Delta\mathrm{d} + |\nabla\mathrm{d}|^2\mathrm{d}, \quad |\mathrm{d}| = 1, \end{cases} \tag{0.8}$$

- More general parabolic Ericksen-Leslie's system: many people $\cdots$

$\rho_1 > 0$, hyperbolic model (inertial Ericksen-Leslie model)

- Very few analytical works for general model, except for very *special* case: $\mathrm{u} = 0$ and 1-D, even this simplest case is very subtle.
  - Singularities of variational wave equation. (Saxton 1989, Glassey-Hunter-Zheng 1996, Chen-Huang-Liu 2015 $\cdots$)
  - Orientational waves: splay and twist waves. (Ali-Hunter 2007)
  - Dissipative and energy conservative solution to variational wave equations. (Zheng-Zhang, Bressan, $\cdots$)

$$\mathrm{u}_{tt} - c(\mathrm{u})(c(\mathrm{u})\mathrm{u}_x)_x = 0 .$$

- Difficulties for *general* model:
  - Double material derivative on d is a nonlinear operator on d

$$\ddot{\mathrm{d}} = \partial_t^2 \mathrm{d} + 2\mathrm{u} \cdot \nabla \partial_t \mathrm{d} + \mathrm{u} \nabla \mathrm{u} \nabla \mathrm{d} + \mathrm{u}(\nabla^2 \mathrm{d})\mathrm{u} .$$

  - The most troublesome part is the Lagrange multiplier γ that generates high derivatives. In particular, if add the unit lengh constraint, the situation gets even worse.

Qian-Sheng model (Physical Review E 1998):

$$\dot{\mathrm{u}} + \nabla p - \frac{\beta_4}{2}\Delta \mathrm{u} = \mathrm{div}\,(-L\nabla Q \odot \nabla Q + \beta_1 Q\mathrm{tr}\{QA\} + \beta_5 AQ + \beta_6 QA)$$
$$+\mathrm{div}\Big(\frac{\mu_2}{2}(\dot{Q} - [\Omega, Q] + \mu_1[Q, \dot{Q} - [\Omega, Q]])\Big),$$
$$\mathrm{div}\mathrm{u} = 0\,,$$
$$J\ddot{Q} + \mu_1\dot{Q} = L\Delta Q - aQ + b\left(Q^2 - \frac{1}{d}|Q|^2 I_d\right) - cQ|Q|^2 + \frac{\tilde{\mu}_2}{2}A$$
$$+\mu_1[\Omega, Q]\,,$$

where $[A, B] = AB - BA$, $A_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \Omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ and $(\nabla Q \odot \nabla Q)_{ij} = Q_{kl,i}Q_{kl,j}$.

Mathematical works for inertial Qian-Sheng model

- Global existence and uniqueness for small initial data (De Anna-Zarnescu 2016).
  - key assumption 1: Newtonian viscosity β_4 is large enough.
  - key assumption 2: the coefficient $a > 0$ which gives "damping".
  - technical trick: higher-order commutator estimate.
- A class of global "twist waves" (solutions of the coupled system for which *the flow vanishes for all time*) (De Anna-Zarnescu 2016).
- A global existence of the *dissipative solution* which is inspired from that of incompressible Euler equation defined by P-L. Lions (Feireisl-Rocca-Schimperna-Zarnescu 2016).
- First results involving *second order material derivative*, many open questions left in this direction.

Compressible model

Ericksen-Leslie hyperbolic liquid crystal model for a compressible flow has the form:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathrm{u}) = 0\,, \\ \partial_t(\rho \mathrm{u}) + \mathrm{div}(\rho \mathrm{u} \otimes \mathrm{u}) + \nabla p = \mathrm{div}(\Sigma_1 + \Sigma_2 + \Sigma_3)\,, \\ \rho \ddot{\mathrm{d}} = \Delta \mathrm{d} + \Gamma \mathrm{d} + \lambda_1(\dot{\mathrm{d}} + \mathrm{Bd}) + \lambda_2 \mathrm{Ad}\,, \end{cases} \tag{0.9}$$

on $\mathbb{R}^N \times \mathbb{R}^+$, $(N = 2, 3)$ with the geometric constraint $|\mathrm{d}| = 1$. For the simplicity, we assume that the pressure p obeys the γ-law, $p(\rho) = a\rho^\gamma$ with $\gamma \geq 1$, $a > 1$, and

$$\Sigma_1 := \tfrac{1}{2}\mu_4(\nabla \mathrm{u} + \nabla^\top \mathrm{u}) + \xi \mathrm{div}\mathrm{u} I\,,$$

$$\Sigma_2 := \tfrac{1}{2}|\nabla \mathrm{d}|^2 I - \nabla \mathrm{d} \odot \nabla \mathrm{d}\,,$$

$$\Sigma_3 := \tilde{\sigma}\,.$$

Our goals

- Well-posedness in the framework of classical solutions;
  - Incompressible model, $\lambda_1 < 0$, (J-Luo, *SIAM Math. Anal. 2019*)
  - Incompressible model, $\lambda_1 = 0$, (Huang-J-Luo-Zhao, preprint 2018)
  - Compressible model, $\lambda_1 < 0$, (J-Luo-Tang, *Math. Models & Methods in App. Sci. 2019*)
  - Compressible model, $\lambda_1 = 0$, (open!)
- Zero inertia limit ($\rho_1 \to 0$) to the corresponding *parabolic* Ericksen-Leslie model.
  - Incompressible model, $\lambda_1 < 0$, $\mathrm{u} = 0$ (J-Luo-Tang-Zarnescu, to appear on *Commun. Math. Sci. 2019*)
  - Incompressible model, $\lambda_1 < 0$, (J-Luo, preprint 2019)

General principles: 1. $\lambda_1 < 0$ case is relatively easier, since it gives some "damping" effect. $\lambda_1 = 0$ case is essentially related to the wave map type equations with target manifold $\mathbb{S}^2$. 2. All above results, we need some restrictions on the coefficients.

Part I: Well-posedness

Proposition

(Basic energy-dissipation law) If (u, d) is a smooth solution to the system (0.3) with initial conditions (0.5), then

$$\frac{1}{2}\frac{\partial}{\partial t}\Big(|\mathrm{u}|^2_{L^2} + \rho_1|\dot{\mathrm{d}}|^2_{L^2} + |\nabla \mathrm{d}|^2_{L^2}\Big) + \frac{1}{2}\mu_4|\nabla \mathrm{u}|^2_{L^2} + \mu_1|\mathrm{d}^\top \mathrm{A}\mathrm{d}|^2_{L^2}$$
$$-\lambda_1|\dot{\mathrm{d}} + \mathrm{B}\mathrm{d}|^2_{L^2} - 2\lambda_2\langle\dot{\mathrm{d}} + \mathrm{B}\mathrm{d}, \mathrm{A}\mathrm{d}\rangle + (\mu_5 + \mu_6)|\mathrm{A}\mathrm{d}|^2_{L^2} = 0\,.$$

Moreover, the above basic energy law is *dissipated* if the Leslie coefficients satisfy that either

$$\mu_1 \geq 0\,, \mu_4 > 0\,, \lambda_1 < 0\,, \mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \geq 0\,,$$

or

$$\mu_1 \geq 0\,, \mu_4 > 0\,, \lambda_1 = 0\,, (1-\delta)\mu_4(\mu_5+\mu_6) \geq 2|\lambda_2|^2$$

for some $\delta \in (0, 1)$.

Lemma: Assume (u, d) is a classical solution to the Ericksen-Leslie's hyperbolic system (0.3)-(0.5) satisfying $\mathrm{u} \in L^\infty_T H^s_x \cap L^2_T H^{s+1}_x$, $\nabla \mathrm{d} \in L^\infty_T H^s_x$, $\dot{\mathrm{d}} \in L^\infty_T H^s_x$ and $|\mathrm{d}|_{L^\infty_{T,x}} < \infty$ for some $T \in (0, \infty)$, where $s > \frac{n}{2} + 1$.
If the constraint $|\mathrm{d}| = 1$ is required, then the Lagrangian multiplier γ is

$$\gamma = -\rho_1 |\dot{\mathrm{d}}|^2 + |\nabla \mathrm{d}|^2 - \lambda_2 \mathrm{d}^\top \mathrm{A}\mathrm{d}\,. \tag{0.10}$$

Conversely, if we give the form of γ as (0.10) and d satisfies the initial data conditions $\tilde{\mathrm{d}}^{in} \cdot \mathrm{d}^{in} = 0$, $|\mathrm{d}^{in}| = 1$, then $|\mathrm{d}| = 1$.
Proof: $h = |\mathrm{d}|^2 - 1$ solves for a given smooth vector field u:

$$\begin{cases} \rho_1 \ddot{h} - \lambda_1 \dot{h} - \Delta h = 2\gamma h\,, \\ \dot{h}|_{t=0} = 0\,, \quad h|_{t=0} = 0\,. \end{cases} \tag{0.11}$$

Our goal is to verify $h(t, x) = 0$ for later time $0 < t < T$.

Equations for $\mathrm{d}(t,x)$ with a given velocity $\mathrm{u}(t,x)$

For a given velocity field $\mathrm{u}(t,x)$, consider the equation of $\mathrm{d} \in \mathbb{S}^2$:

$$\begin{cases} \rho_1 \ddot{\mathrm{d}} = \Delta \mathrm{d} + \gamma(\mathrm{u}, \mathrm{d}, \dot{\mathrm{d}})\mathrm{d} + \lambda_1(\dot{\mathrm{d}} - \mathrm{B}\mathrm{d}) + \lambda_2 \mathrm{A}\mathrm{d}, \\ \mathrm{d}(0,x) = \mathrm{d}_0(x), \ \dot{\mathrm{d}}(0,x) = \tilde{\mathrm{d}}_0(x). \end{cases} \tag{0.12}$$

Proposition

For $s > \frac{n}{2} + 1$ and $T_0 > 0$, let vector fields $(\mathrm{d}^{in}, \tilde{\mathrm{d}}^{in}, \mathrm{u}) \in \mathbb{S}^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n$ satisfy $\nabla \mathrm{d}^{in} \in H^s$, $\tilde{\mathrm{d}}^{in} \in H^s$ and $\mathrm{u} \in C(0, T_0; H^s) \cap L^1(0, T_0; H^{s+1})$. Then there exists a number $0 < T \leq T_0$, depending only on d^{in}, $\tilde{\mathrm{d}}^{in}$ and u, such that the system (0.12) has a unique classical solution d satisfying $\nabla \mathrm{d}, \ \dot{\mathrm{d}} \in C(0, T; H^s)$. Moreover, there is a positive constant C_2, depending only on d^{in}, $\tilde{\mathrm{d}}^{in}$ and u, such that the solution d satisfies the following bound

$$\rho_1 |\dot{\mathrm{d}}|^2_{L^\infty(0,T;H^s)} + |\nabla \mathrm{d}|^2_{L^\infty(0,T;H^s)} \leq C_2 .$$

Proof of the Proposition

Construct the approximate system

$$\begin{cases} \rho_1 \partial_t \dot{\mathrm{d}}^\epsilon = -\rho_1 \mathcal{J}_\epsilon(\mathrm{u}\nabla \cdot \dot{\mathrm{d}}^\epsilon) + \Delta \mathrm{d}^\epsilon + \mathcal{J}_\epsilon\Big(\gamma(\mathrm{u}, \mathrm{d}^\epsilon, \dot{\mathrm{d}}^\epsilon)\mathrm{d}^\epsilon\Big) \\ \qquad\qquad + \lambda_1 \dot{\mathrm{d}}^\epsilon - \lambda_1 \mathcal{J}_\epsilon(\mathrm{B}\mathrm{d}^\epsilon) + \lambda_2 \mathcal{J}_\epsilon(\mathrm{A}\mathrm{d}^\epsilon)\,, \\ \partial_t \mathrm{d}^\epsilon = \dot{\mathrm{d}}^\epsilon - \mathcal{J}_\epsilon(\mathrm{u}\cdot\nabla \mathrm{d}^\epsilon)\,, \\ (\mathrm{d}^\epsilon, \dot{\mathrm{d}}^\epsilon)|_{t=0} = (\mathcal{J}_\epsilon \mathrm{d}_0, \mathcal{J}_\epsilon \tilde{\mathrm{d}}_0)\,. \end{cases} \tag{0.13}$$

Define the energy functional $E_\epsilon(t)$

$$E_\epsilon(t) = \rho_1 |\dot{\mathrm{d}}^\epsilon|^2_{H^s} + |\nabla \mathrm{d}^\epsilon|^2_{H^s} + |\mathrm{d}^\epsilon - \mathcal{J}_\epsilon \mathrm{d}_0|^2_{L^2}\,,$$

$$\frac{\partial}{\partial t} E_\epsilon(t) \le C_1(1 + |\mathrm{u}|_{H^{s+1}})[1 + E_\epsilon(t)]^2 \tag{0.14}$$

holds for all $t \in [0, T_\epsilon)$. We obtain the uniform energy bound

$$\rho_1 |\dot{\mathrm{d}}^\epsilon|^2_{H^s} + |\nabla \mathrm{d}^\epsilon|^2_{H^s} + |\mathrm{d}^\epsilon - \mathcal{J}_\epsilon \mathrm{d}_0|^2_{L^2} \le 2E^{in}\,, \tag{0.15}$$

for all $\epsilon > 0$ and $t \in [0, T]$. Pass to the limit $\epsilon \to 0$, and use the previous lemma to prove the limit satisfies $|\mathrm{d}| = 1$.

Main Theorem (I) (J-Luo)

Let the integer $s > \frac{n}{2} + 1$, and let the the initial data satisfy $\mathrm{u}^{in}, \tilde{\mathrm{d}}^{in} \in H^s(\mathbb{R}^n), \nabla \mathrm{d}^{in} \in H^s(\mathbb{R}^n), |\mathrm{d}^{in}| = 1, \tilde{\mathrm{d}}^{in} \cdot \mathrm{d}^{in} = 0$. The initial energy is defined as $E^{in} \equiv |\mathrm{u}^{in}|^2_{H^s} + \rho_1 |\tilde{\mathrm{d}}^{in}|^2_{H^s} + |\nabla \mathrm{d}^{in}|^2_{H^s}$. If the Leslie coefficients satisfy the relations that either

$$\mu_1 \geq 0\,, \mu_4 > 0\,, \lambda_1 < 0\,, \mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \geq 0\,, \quad \text{or} \tag{0.16}$$

$$\mu_1 \geq 0\,, \mu_4 > 0\,, \lambda_1 = 0\,, (1-\delta)\mu_4(\mu_5 + \mu_6) \geq 2|\lambda_2|^2 \tag{0.17}$$

for some $\delta \in (0,1)$, then the following statements hold:
(I). If the initial energy $E^{in} < \infty$, then there exists $T > 0$, depending only on E^{in} and Leslie coefficients, such that the system (0.3)-(0.5) admits a unique solution $\mathrm{u} \in L^\infty_T H^s_x \cap L^2_T H^{s+1}_x$, $\nabla \mathrm{d} \in L^\infty_T H^s_x$ and $\dot{\mathrm{d}} \in L^\infty_T H^s_x$. Moreover, the solution (u, d) satisfies

$$\sup_{0 \leq t \leq T} \Big(|\mathrm{u}|^2_{H^s} + \rho_1 |\dot{\mathrm{d}}|^2_{H^s} + |\nabla \mathrm{d}|^2_{H^s} \Big)(t) + \tfrac{1}{2}\mu_4 \int_0^T |\nabla \mathrm{u}|^2_{H^s}(\tau) \mathrm{d}\tau \leq C_0\,,$$

where C_0 depends only on E^{in}, Leslie coefficients and T.

Main Theorem (II)

(II). If in addition, $\lambda_1 < 0$, henceforth the coefficients constraints (0.16) hold, then there is a constant $\epsilon_0 > 0$, depending only on Leslie coefficients, such that if the initial data satisfy $E^{in} \leq \epsilon_0$, then the system (0.3)-(0.5) has a unique global solution $\mathrm{u} \in L^\infty(0,\infty; H^s_x) \cap L^2(0,\infty; H^{s+1}_x)$, $\nabla \mathrm{d} \in L^\infty(0,\infty; H^s_x)$ and $\dot{\mathrm{d}} \in L^\infty(0,\infty; H^s_x)$. Moreover, the solution (u,d) satisfies

$$\sup_{t\geq 0}\left(|\mathrm{u}|^2_{H^s} + \rho_1|\dot{\mathrm{d}}|^2_{H^s} + |\nabla \mathrm{d}|^2_{H^s}\right) + \tfrac{1}{2}\mu_4\int_0^\infty |\nabla \mathrm{u}|^2_{H^s}\mathrm{d}t \leq C_1 E^{in}\,,$$

where the constant $C_1 > 0$ depends upon the Leslie coefficients and inertia constant ρ_1.

The iterating approximate system: for all integer $k \geq 0$,

$$\begin{cases} \partial_t \mathrm{u}^{k+1} + \mathrm{u}^k \cdot \nabla \mathrm{u}^k - \frac{1}{2}\Delta \mathrm{u}^{k+1} + \nabla p^{k+1} = \\ -\mathrm{div}(\nabla \mathrm{d}^k \odot \nabla \mathrm{d}^k) + \mathrm{div}\tilde{\sigma}(\mathrm{u}^{k+1}, \mathrm{d}^k, \dot{\mathrm{d}}^k), \\ \mathrm{div}\mathrm{u}^{k+1} = 0, \\ \rho_1 \partial_t \dot{\mathrm{d}}^{k+1} + \rho_1 \mathrm{u}^k \cdot \nabla \dot{\mathrm{d}}^{k+1} = \Delta \mathrm{d}^{k+1} + \gamma(\mathrm{u}^k, \mathrm{d}^{k+1}, \dot{\mathrm{d}}^{k+1})\mathrm{d}^{k+1} \\ \qquad + \lambda_1(\dot{\mathrm{d}}^{k+1} + \mathrm{B}^k \mathrm{d}^{k+1}) + \lambda_2 \mathrm{A}^k \mathrm{d}^{k+1}, \\ (\mathrm{u}^{k+1}, \mathrm{d}^{k+1}, \dot{\mathrm{d}}^{k+1})\big|_{t=0} = (\mathrm{u}^{in}(x), \mathrm{d}^{in}(x), \tilde{\mathrm{d}}^{in}(x)) \in \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n, \end{cases} \tag{0.18}$$

where $\dot{\mathrm{d}}^{k+1} = \partial_t \mathrm{d}^{k+1} + \mathrm{u}^k \cdot \nabla \mathrm{d}^{k+1}$ is the iterating approximate material derivatives, the iterating approximate Lagrangian multiplier $\gamma(\mathrm{u}^k, \mathrm{d}^{k+1}, \dot{\mathrm{d}}^{k+1})$ is

$$\gamma(\mathrm{u}^k, \mathrm{d}^{k+1}, \dot{\mathrm{d}}^{k+1}) = -\rho_1 |\dot{\mathrm{d}}^{k+1}|^2 + |\nabla \mathrm{d}^{k+1}|^2 - \lambda_2 (\mathrm{d}^{k+1})^\top \mathrm{A}^k \mathrm{d}^{k+1}.$$

Uniform energy estimate

To simply state a priori estimate, we introduce the following energy functionals:

$$E(t) = |\mathrm{u}|_{H^s}^2 + \rho_1 |\dot{\mathrm{d}}|_{H^s}^2 + |\nabla \mathrm{d}|_{H^s}^2 \,,$$
$$D(t) = \frac{1}{2}\mu_4 |\nabla \mathrm{u}|_{H^s}^2 \,.$$

Lemma

Let $s > \frac{n}{2} + 2$. Assume (u, d) is a smooth solution to the system (0.3)-(0.5). *Then there exists a constant $C > 0$, depending only on Leslie coefficients and inertia density constant ρ_1, such that*

$$\frac{1}{2}\frac{\partial}{\partial t}E(t) + D(t) \le C E^{\frac{3}{2}}(t) + C \sum_{p=1}^{4} E^{\frac{p+1}{2}}(t) D^{\frac{1}{2}}(t) \,.$$

A key lemma

We define the following energy functionals:

$$E_{k+1}(t) = |\mathrm{u}^{k+1}|^2_{H^s} + \rho_1 |\dot{\mathrm{d}}^{k+1}|^2_{H^s} + |\nabla \mathrm{d}^{k+1}|^2_{H^s}\,,$$
$$D_{k+1}(t) = \tfrac{1}{2}\mu_4 |\nabla \mathrm{u}^{k+1}|^2_{H^s}\,.$$

Lemma

Assume that $(\mathrm{u}^{k+1}, \mathrm{d}^{k+1})$ *is the solution to the iterating approximate system* (0.18) *and we define*

$$T_{k+1} \equiv \left\{ \tau \in [0, T^*_{k+1});\ \sup_{t\in[0,\tau]} E_{k+1}(t) + \int_0^\tau D_{k+1}(t)\partial t \le M \right\},$$

where $T^*_{k+1} > 0$ *is the existence time of the iterating approximate system* (0.18). *Then for any fixed* $M > E^{in}$ *there is a constant* $T > 0$*, depending only on Leslie coefficients,* M *and* E^{in}*, such that*

$$T_{k+1} \ge T > 0\,.$$

Proof of Theorem (I)

Note $E_\epsilon(0) \le E^{in}$. If $E^{in} < \min\left\{1, \frac{\rho_1\beta^2}{[96C(\sqrt{\rho_1}(\mu_1+\mu_6)+|\lambda_1|-\lambda_2)]^2}\right\}$, we have $\mathbf{Q}(E_\epsilon(0)) \le \frac{1}{8}\beta$. Define

$$T_\epsilon^* = \sup\left\{T > 0; E_\epsilon(t) \le 2 \text{ and } \mathbf{Q}(E_\epsilon(t)) \le \tfrac{1}{4}\beta \text{ hold } \text{ for all } t \in [0, T]\right\}.$$

Lemma **??** implies that $E_\epsilon(t)$ is continuous, thus $T_\epsilon^* > 0$. So for any fixed $\epsilon > 0$ and for all $t \in [0, T_\epsilon^*]$

$$\frac{1}{2}\frac{\partial}{\partial t}E_\epsilon(t) + \left[\tfrac{1}{4}\beta - \mathbf{Q}(E_\epsilon(t))\right]F_\epsilon(t) \le 12C_1 E_\epsilon(t)\,,$$

which implies $E_\epsilon(t) \le E^{in}e^{24C_1 t}$ holds for all $\epsilon > 0$ and $t \in [0, T_\epsilon^*]$. Thus let $0 < T \le \frac{1}{48C_1}\ln\frac{1}{E^{in}}$ such that for all $t \in [0, \min\{T, T_\epsilon^*\}]$

$$E_\epsilon(t) \le E^{in}e^{24C_1 T} \le \sqrt{E^{in}} < 1\,,$$

which consequently implies that for all $t \in [0, \min\{T, T_\epsilon^*\}]$

$$\mathbf{Q}(E_\epsilon(t)) \le 12C\Big(\mu_1 + \mu_6 + \frac{|\lambda_1|-\lambda_2}{\sqrt{\rho_1}}\Big)(E^{in})^{\frac{1}{4}}\,.$$

Proof of Theorem (I)– continued

Let $\epsilon_0 = \min\left\{1, \frac{\rho_1\beta^2}{[96C(\sqrt{\rho_1}(\mu_1+\mu_6)+|\lambda_1|-\lambda_2)]^2}, \frac{\rho_1^2\beta^4}{[96C(\sqrt{\rho_1}(\mu_1+\mu_6)+|\lambda_1|-\lambda_2)]^4}\right\}$, s.t. if $E^{in} < \epsilon_0$, then for all $t \in [0, \min\{T, T_\epsilon^*\}]$, $E_\epsilon(t) \leq \sqrt{E^{in}} < 1$, and $\mathbf{Q}(E_\epsilon(t)) \leq \frac{1}{8}\beta$. Thus, by the continuity of $E_\epsilon(t)$, $T_\epsilon^* \geq T$. As a consequence, for all $t \in [0, T]$,

$$E^\epsilon(t) + \frac{1}{4}\beta \int_0^t |\nabla \mathrm{u}^\epsilon|_{H^s}^2(\tau)\partial\tau \leq \widetilde{C}_1(E^{in}, T)\,, \tag{0.19}$$

where $\widetilde{C}_1(E^{in}, T) = E^{in} + 12C_1 T\sqrt{E^{in}} > 0$. Thus the energy estimate is closed. The rest convergence proof is tedious but standard.

Proof of Theorem (III)–Global solution

Need a new *a priori* estimate: For $1 \le k \le s$, take ∇^k in 3rd equation of (0.3), multiply by $\nabla^k \mathrm{d}$. (we did it for $\dot{\mathrm{d}}$ in local existence.)

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\left(\rho_1|\dot{\mathrm{d}}+\mathrm{d}|^2_{\dot{H}^s}+(|\lambda_1|-\rho_1)|\mathrm{d}|^2_{\dot{H}^s}-\rho_1|\dot{\mathrm{d}}|^2_{\dot{H}^s}\right)-\rho_1|\dot{\mathrm{d}}|^2_{\dot{H}^s}+|\nabla\mathrm{d}|^2_{\dot{H}^s} \\ \le&(|\lambda_1|-7\lambda_2)|\nabla\mathrm{u}|_{H^s}|\nabla\mathrm{d}|_{\dot{H}^s}+C\rho_1|\dot{\mathrm{d}}|_{H^s}|\nabla\mathrm{u}|_{H^s}|\nabla\mathrm{d}|_{H^s} \\ &+C(1+\rho_1)(|\dot{\mathrm{d}}|^2_{H^s}+|\nabla\mathrm{d}|^2_{\dot{H}^s})(|\nabla\mathrm{d}|_{H^s}+|\nabla\mathrm{d}|^2_{H^s}) \\ &+C(|\lambda_1|-\lambda_2)|\nabla\mathrm{u}|_{H^s}|\nabla\mathrm{d}|_{\dot{H}^s}(|\nabla\mathrm{d}|_{H^s}+|\nabla\mathrm{d}|^2_{H^s}+|\nabla\mathrm{d}|^3_{H^s})\,. \end{aligned} \tag{0.20}$$

Taking a positive constant $\eta=\frac{1}{2}\min\left\{1,\frac{1}{\rho_1},\frac{|\lambda_1|}{\rho_1}\right\}\in(0,\frac{1}{2}]$, we multiply by η in the inequality (0.20) and then add it to the energy estimate obtained in the local existence.

Proof of Theorem (III)–Global solution

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\Big(|\mathrm{u}|^2_{H^s} + \rho_1(1-\eta)|\dot{\mathrm{d}}|^2_{H^s} + (1-\eta\rho_1)|\nabla \mathrm{d}|^2_{H^s} \\
&\qquad + \eta\rho_1|\dot{\mathrm{d}} + \mathrm{d}|^2_{\dot{H}^2} + \eta\rho_1|\nabla^{s+1}\mathrm{d}|^2_{L^2} + \eta\rho_1|\lambda_1||\mathrm{d}|^2_{\dot{H}^s}\Big) \\
&+\frac{1}{2}\alpha|\nabla \mathrm{u}|^2_{H^s} + (|\lambda_1| - \eta\rho_1)|\dot{\mathrm{d}}|^2_{H^s} + \eta|\nabla \mathrm{d}|^2_{\dot{H}^s} \\
&\le C'(1+\mu_1+|\lambda_1|-\lambda_2+\mu_6-\rho_1\lambda_2+\rho_1+\frac{1}{\sqrt{\rho_1}})\Big(|\mathrm{u}|_{H^s} + |\dot{\mathrm{d}}|_{H^s} + \sum_{i=1}^{4}|\nabla \mathrm{d}|^i_{H^s}\Big) \\
&\qquad \times\Big(|\nabla \mathrm{u}|_{H^s} + |\dot{\mathrm{d}}|_{H^s} + |\nabla \mathrm{d}|_{\dot{H}^s}\Big)|\nabla \mathrm{u}|_{H^s},
\end{aligned}
$$

where $\alpha = \mu_4 - 4\mu_6 - \frac{(|\lambda_1|-7\lambda_2)^2}{\eta} - \frac{2(7|\lambda_1|-2\lambda_2)^2}{|\lambda_1|} > 0.$

Proof of Theorem (III)–Global solution

We denote by

$$\mathcal{E}(t) \equiv |\mathrm{u}|^2_{H^s} + \rho_1(1-\eta)|\dot{\mathrm{d}}|^2_{H^s} + (1-\eta\rho_1)|\nabla \mathrm{d}|^2_{H^s} + \eta\rho_1|\dot{\mathrm{d}} + \mathrm{d}|^2_{\dot{H}^s} + \eta\rho_1|\nabla^{s+1}\mathrm{d}|^2_{L^2} + \eta\rho_1|\lambda_1||\mathrm{d}|^2_{\dot{H}^s} \tag{0.21}$$

and

$$\mathcal{D}(t) \equiv |\nabla \mathrm{u}|^2_{H^s} + |\dot{\mathrm{d}}|^2_{H^s} + |\nabla \mathrm{d}|^2_{\dot{H}^s}\,,$$

We have a new energy estimate

$$\frac{\partial}{\partial t}\mathcal{E}(t) + \theta\mathcal{D}(t) \le C_3 \sum_{q=1}^{4} [\mathcal{E}(t)]^{\frac{q}{2}} \mathcal{D}(t)\,, \tag{0.22}$$

where $\theta = \min\left\{\alpha, \eta, \frac{1}{2}|\lambda_1|\right\} > 0$. The rest is continuity argument similar as before.

$\lambda_1 = 0$ case: no damping

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u - \mathrm{div}(\nabla d \odot \nabla d) + \mathrm{div}\sigma, \\ \mathrm{div} u = 0, \\ \ddot{d} - \Delta d = (-|\dot{d}|^2 + |\nabla d|^2) d. \end{cases} \tag{0.23}$$

on $\mathbb{R}^3 \times \mathbb{R}^+$ with the constraint $|d| = 1$, where $u \in \mathbb{R}^3$ is the velocity field, $d \in \mathbb{R}^3$ is the orientation field and

$$\sigma_{ji} = \mu_1 d_k d_p A_{kp} d_i d_j + \mu_2 (d_j d_k A_{ki} + d_i d_k A_{kj}),$$

where $A_{ij} := \frac{1}{2}(\partial_i u_j + \partial_j u_i)$.
Remark: Cai-Wang treated the case $\mu_1 = \mu_2 = 0$.

Notations

Define the perturbed angular momentum operators by

$$\tilde{\Omega}_i u = \Omega_i u + A_i u,\ \tilde{\Omega}_i d = \Omega_i d,$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the rotation vector-field $\Omega = x \wedge \nabla$ and A_i is defined by

$$\begin{aligned} A_1 &= e_2 \otimes e_3 - e_3 \otimes e_2,\ \ A_2 = e_3 \otimes e_1 - e_1 \otimes e_3, \\ A_3 &= e_1 \otimes e_2 - e_2 \otimes e_1. \end{aligned}$$

We define the scaling vector-field S by

$$S = t\partial_t + x_i \partial_{x_i}.$$

Let

$$\Gamma \in \{\partial_t, \partial_1, \partial_2, \partial_3, \tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3\}$$

and $Z^a = S^{a_1}\Gamma^{a'}$, where $a = (a_1, a') := (a_1, a_2, \cdots, a_8) \in \mathbb{Z}_+^8$, $\Gamma^{a'} = \Gamma^{a_2}\Gamma^{a_3}\cdots\Gamma^{a_8}$, we define

$$u^{(a)} := Z^a u,\ \ d^{(a)} := Z^a d.$$

Main theorem (Huang-J-Luo-Zhao, 2018)

Theorem

Assume that $N_0 := 60, N_1 := 6, h := 6$, μ_1, μ_2 *satisfy*

$$\mu_1 > -4(\mu + \frac{\mu_2}{2}), \ \ \mu + \frac{\mu_2}{2} > 0, \tag{0.24}$$

and (u_0, d_0, d_1) *are initial data near equilibrium* $(\vec{0}, \vec{i}, \vec{0})$ *satisfying the smallness assumptions*

$$\sup_{|a| \le N_1} \{\|u_0^{(a)}\|_{H^{N(a)}} + \|\nabla d_0^{(a)}\|_{H^{N(a)}} + \|d_1^{(a)}\|_{H^{N(a)}}\} \le \epsilon_0, \tag{0.25}$$

where $N(a) = N_0 - |a|h$ *for* $0 \le |a| \le N_1$. *Then there exists a unique global solution* (u, d) *of the system (0.23) with initial data*

$$u(0) = u_0, \ \ d(0) = d_0, \ \partial_t d(0) = d_1,$$

satisfies the energy bounds

Main theorem (Huang-J-Luo-Zhao, 2018)

Theorem

$$\sup_{|a|\leq N_1}\{\|u^{(a)}\|_{H^{N(a)}}+\|\nabla u^{(a)}\|^2_{L^2([0,t]:H^{N(a)})}\}+\|\nabla d\|_{H^{N(0)}}+\|\partial_t d\|_{H^{N(0)}}\lesssim \epsilon_0,$$

$$\sup_{1\leq|a|\leq N_1}\{\|\partial_t d^{(a)}\|_{H^{N(a)}}+\|\nabla d^{(a)}\|_{H^{N(a)}}\}\lesssim \epsilon_0(1+t)^{\bar{\delta}}.$$

for any $t\in[0,\infty)$, *where* $\bar{\delta}<10^{-7}$ *depends on* ϵ_0, N_0 *and* N_1.

Remark: 2-*D* case will be much harder!

Part II, Zero inertia density limit: from hyperbolic to parabolic E-L ($\lambda_1 < 0$)

Formally, the inertia constant $\rho_1 \to 0$, the hyperbolic E-L system converges to the parabolic E-L. Our goal is to *rigorously justify* this limiting process. Main difficulties: 1. changing type of equations; 2. coupled with NS.

- Special case 1: the velocity flow u is *given*. (J-Luo-Tang, 2017)
- Special case 2: completely forget u, redues to the scaling limit from wave map to heat flow into $\mathbb{S}^2$ (J-Luo-Tang-Zarnescu, 2017)
- General case: global-in-time, small initial data, from hyperbolic to parabolic E-L. (J-Luo, 2019)

Wave map

We consider a hyperbolic system for functions $\mathrm{d} : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{S}^2$:

$$\partial_t \mathrm{d} = -\Box \mathrm{d} + (|\nabla \mathrm{d}|^2 - |\partial_t \mathrm{d}|^2)\mathrm{d}\,, \tag{0.26}$$

subject to initial data: for $x \in \mathbb{R}^3$,

$$\mathrm{d}|_{t=0} = \mathrm{d}^0(x) \in \mathbb{S}^2\,,\ \partial_t \mathrm{d}|_{t=0} = \tilde{\mathrm{d}}^0(x) \in \mathbb{R}^3\,, \mathrm{d}^0(x)\cdot\tilde{\mathrm{d}}^0(x) = 0\,, \tag{0.27}$$

The system (0.26) is a wave map from $\mathbb{R}^3$ to the unit sphere $\mathbb{S}^2$, with a damping term $\partial_t \mathrm{d}$. One way of interpreting this system is as follows: setting the righthand side of (0.26) equal to 0, we obtain $\Box \mathrm{d} = (|\nabla \mathrm{d}|^2 - |\partial_t \mathrm{d}|^2)\mathrm{d}$. This is the well-known *wave map*, which can be characterized variationally as a critical point of the functional

$$\mathcal{A}(\mathrm{d}) = \frac{1}{2}\iint (|\nabla \mathrm{d}|^2 - |\partial_t \mathrm{d}|^2)\, \mathrm{d}x\mathrm{d}t\,, \tag{0.28}$$

among maps d satisfying the target constraint, $\mathrm{d} : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{S}^2$. Thus the full system (0.26) can be viewed as a "*gradient flow*" of the functional (0.28).

The *heat flow* from $\mathbb{R}^3$ to $\mathbb{S}^2$:

$$\partial_t \mathrm{d} = \Delta \mathrm{d} + |\nabla \mathrm{d}|^2 \mathrm{d}\,. \tag{0.29}$$

The corresponding harmonic map $\Delta \mathrm{d} + |\nabla \mathrm{d}|^2 \mathrm{d} = 0$, is a critical point of the energy functional $E(\mathrm{d}) = \frac{1}{2}\int |\nabla \mathrm{d}|^2 \,\mathrm{d}x$. Relation between (0.26) and (0.29): let $\mathrm{d}^\epsilon(t,x) := \mathrm{d}(\frac{t}{\epsilon}, \frac{x}{\sqrt{\epsilon}})$, d^ϵ satisfies the scaled wave map:

$$\partial_t \mathrm{d}^\epsilon = -(\epsilon\partial_{tt} - \Delta)\mathrm{d}^\epsilon + (|\nabla \mathrm{d}^\epsilon|^2 - \epsilon|\partial_t \mathrm{d}^\epsilon|^2)\mathrm{d}^\epsilon\,, \tag{0.30}$$

$$\mathrm{d}^\epsilon\big|_{t=0} = \mathrm{d}^{in}(x) \in \mathbb{S}^2\,,\ \partial_t \mathrm{d}^\epsilon\big|_{t=0} = \tilde{\mathrm{d}}^{in}(x) \in \mathbb{R}^3\,. \tag{0.31}$$

It is easy to see that letting $\epsilon = 0$ in (0.30) will formally give the heat flow (0.29). Question: Justify it.

Initial layer

The wave map is a system of hyperbolic equations with two initial conditions, while the heat flow is a parabolic system with only one initial condition. Usually the solution of the heat flow does not satisfy the second initial condition in (0.31). This disparity between the initial conditions of the wave map (0.30) and of the heat flow indicates that in one should expect an "initial layer" in time, appearing in the limiting process $\epsilon \to 0$. A formal derivation indicates that this should be of the form:

$$\mathrm{d}_0^I\Big(\frac{t}{\epsilon}, x\Big) = -\epsilon D(x) \exp(-\tfrac{t}{\epsilon}),$$

where

$$D(x) \equiv \tilde{\mathrm{d}}^{in}(x) - \Delta \mathrm{d}^{in}(x) - |\nabla \mathrm{d}^{in}(x)|^2 \mathrm{d}^{in}(x).$$

Limit from wave map to heat flow (J-Luo-Tang-Zarnescu)

Theorem

$\nabla \mathrm{d}^{in} \in H^6$, $\tilde{\mathrm{d}}^{in} \in H^5$, and let $T > 0$ be the time interval of existence of the solution of the heat flow with initial condition d^{in}. Then, there exists an $\epsilon_0 \equiv \epsilon_0(|\nabla \mathrm{d}^{in}|_{H^6}, |\tilde{\mathrm{d}}^{in}|_{H^5}, T) \in (0, \frac{1}{2})$ s.t. for all $\epsilon \in (0, \epsilon_0)$ we have that on the interval $[0, T]$ the wave map equation (0.30) *with the initial conditions* (0.31) *admits a unique solution with the form:*

$$\mathrm{d}^{\epsilon}(t,x) = \mathrm{d}_0(t,x) + \mathrm{d}_0^I\Big(\frac{t}{\epsilon}, x\Big) + \sqrt{\epsilon}\mathrm{d}_R^{\epsilon}(t,x)\,,$$

where d_0 is the solution of the heat flow with initial condition d^{in} and $\mathrm{d}_0^I(\frac{t}{\epsilon}, x)$ is the initial layer. Moreover, there exists $C_0 = C_0(\mathrm{d}^{in}, \tilde{\mathrm{d}}^{in}, T) > 0$, s.t. the remainder term d_R^{ϵ} satisfies the bound

$$|\partial_t \mathrm{d}_R^{\epsilon}|^2_{L^\infty(0,T;H^2)} + \frac{1}{\epsilon}|\mathrm{d}_R^{\epsilon}|^2_{L^\infty(0,T;H^3)} \le C_0 \tag{0.32}$$

for all $\epsilon \in (0, \epsilon_0)$.

$$\begin{cases} \partial_t \mathrm{u}^\epsilon + \mathrm{u}^\epsilon \cdot \nabla \mathrm{u}^\epsilon - \frac{1}{2}\mu_4 \Delta \mathrm{u}^\epsilon + \nabla p^\epsilon = -\mathrm{div}(\nabla \mathrm{d}^\epsilon \odot \nabla \mathrm{d}^\epsilon) + \mathrm{div}\sigma^\epsilon \,, \\ \mathrm{div}\mathrm{u}^\epsilon = 0 \,, \\ \epsilon \mathrm{D}^2_{\mathrm{u}^\epsilon} \mathrm{d}^\epsilon = \Delta \mathrm{d}^\epsilon + \gamma^\epsilon \mathrm{d}^\epsilon + \lambda_1 (\mathrm{D}_{\mathrm{u}^\epsilon} \mathrm{d}^\epsilon + \mathrm{B}^\epsilon \mathrm{d}^\epsilon) + \lambda_2 \mathrm{A}^\epsilon \mathrm{d}^\epsilon \,, \\ |\mathrm{d}^\epsilon| = 1 \,, \end{cases} \tag{0.33}$$

where $\mathrm{D}_{\mathrm{u}^\epsilon}\mathrm{d}^\epsilon = \partial_t \mathrm{d}^\epsilon + \mathrm{u}^\epsilon \cdot \nabla \mathrm{d}^\epsilon$, and Lagrangian multiplier and stress

$$\gamma^\epsilon = -\epsilon |\mathrm{D}_{\mathrm{u}^\epsilon}\mathrm{d}^\epsilon|^2 + |\nabla \mathrm{d}^\epsilon|^2 - \lambda_2 \mathrm{A}^\epsilon : \mathrm{d}^\epsilon \otimes \mathrm{d}^\epsilon \,,$$

$$\begin{aligned} \sigma^\epsilon = & \mu_1 (\mathrm{A}^\epsilon : \mathrm{d}^\epsilon \otimes \mathrm{d}^\epsilon) \mathrm{d}^\epsilon \otimes \mathrm{d}^\epsilon + \mu_2 (\mathrm{D}_{\mathrm{u}^\epsilon}\mathrm{d}^\epsilon + \mathrm{B}^\epsilon \mathrm{d}^\epsilon) \otimes \mathrm{d}^\epsilon \\ & + \mu_3 \mathrm{d}^\epsilon \otimes (\mathrm{D}_{\mathrm{u}^\epsilon}\mathrm{d}^\epsilon + \mathrm{B}^\epsilon \mathrm{d}^\epsilon) + \mu_5 (\mathrm{A}^\epsilon \mathrm{d}^\epsilon) \otimes \mathrm{d}^\epsilon + \mu_6 \mathrm{d}^\epsilon \otimes (\mathrm{A}^\epsilon \mathrm{d}^\epsilon) \,. \end{aligned}$$

the initial data

$$\mathrm{u}^\epsilon|_{t=0} = \mathrm{u}^{in} \,, \ \mathrm{d}^\epsilon|_{t=0} = \mathrm{d}^{in} \,, (\mathrm{D}_{\mathrm{u}^\epsilon}\mathrm{d}^\epsilon)|_{t=0} = \tilde{\mathrm{d}}^{in}$$

with the compatibilities

$$\mathrm{div}\mathrm{u}^{in} = 0 \,, \ \mathrm{d}^{in} \cdot \tilde{\mathrm{d}}^{in} = 0 \,.$$

General parabolic E-L

Formally $\mathrm{u}^\epsilon \to \mathrm{u}$ and $\mathrm{d}^\epsilon \to \mathrm{d}$ as $\epsilon \to 0$, the hyperbolic liquid crystal system converges to the parabolic liquid crystal model

$$\begin{cases} \partial_t \mathrm{u} + \mathrm{u} \cdot \nabla \mathrm{u} - \frac{1}{2}\mu_4 \Delta \mathrm{u} + \nabla p = -\mathrm{div}(\nabla \mathrm{d} \odot \nabla \mathrm{d}) + \mathrm{div}\sigma \,, \\ \mathrm{div}\mathrm{u} = 0 \,, \\ -\lambda_1(\mathrm{D_u d_0} + \mathrm{Bd}) = \Delta \mathrm{d} + \gamma \mathrm{d} + \lambda_2 \mathrm{Ad} \,, \\ |\mathrm{d}| = 1 \,, \end{cases}$$

where the Lagrangian multiplier γ is

$$\gamma = |\nabla \mathrm{d}|^2 - \lambda_2 \mathrm{A} : \mathrm{d} \otimes \mathrm{d} \,,$$

with initial data $\mathrm{u}|_{t=0} = \mathrm{u}^{in}$, $\mathrm{d}|_{t=0} = \mathrm{d}^{in}$. Again, because of the disparity between the initial conditions, an initial layer is needed. The ansatz

$$\begin{aligned} \mathrm{u}^\epsilon(t,x) &= \mathrm{u}_0(t,x) + \sqrt{\epsilon}\mathrm{u}_R^\epsilon(t,x) \,, \\ \mathrm{d}^\epsilon(t,x) &= \mathrm{d}_0(t,x) + \epsilon^\beta \mathrm{d}_I(\tfrac{t}{\epsilon^\beta}, x) + \sqrt{\epsilon}\mathrm{d}_R^\epsilon(t,x) \end{aligned} \qquad (0.34)$$

for a fixed $\beta > 0$ to be determined.

Initial layer

Define $\tau = \frac{t}{\epsilon^\beta}$, and the disparity

$$\mathrm{D}^{in} = \tilde{\mathrm{d}}^{in} - \mathrm{D}_{\mathrm{u}_0}\mathrm{d}_0|_{t=0} = \tilde{\mathrm{d}}^{in} + \mathrm{B}^{in}\mathrm{d}^{in} + \frac{1}{\lambda_1}(\Delta \mathrm{d}^{in} + \gamma_0^{in}\mathrm{d}^{in} + \lambda_2 \mathrm{A}^{in}\mathrm{d}^{in}) .$$

$$\begin{cases} \partial^2_{\tau\tau}\mathrm{d}_I + \frac{-\lambda_1}{\epsilon^{1-\beta}}\partial_\tau \mathrm{d}_I = \epsilon^{2\beta-1}\Delta \mathrm{d}_I \,, \\ \mathrm{d}_I(\infty, x) = \lim_{\tau\to\infty} \mathrm{d}_I(\tau, x) = 0 \,, \\ \partial_\tau \mathrm{d}_I(0, x) = \mathrm{D}^{in}(x) \,. \end{cases} \tag{0.35}$$

If $\beta > \frac{1}{2}$, $\epsilon^{2\beta-1}\Delta \mathrm{d}_I$ is a higher order term. Solve (0.35),

$$\begin{aligned} &\epsilon^\beta \mathrm{d}_I(\tfrac{t}{\epsilon^\beta}, x) = \epsilon \mathrm{D}_I^\epsilon(t, x) \\ &= 2\epsilon\Big(\lambda_1 - \sqrt{\lambda_1^2 + 4\epsilon\Delta}\Big)^{-1} \exp\Big(\frac{\lambda_1 - \sqrt{\lambda_1^2 + 4\epsilon\Delta}}{2\epsilon} t\Big)\mathrm{D}^{in}(x) \,. \end{aligned} \tag{0.36}$$

So in the ansatz, indeed, $\beta = 1$.

Main Theorem (J-Luo, 2019)

Theorem

$\mathrm{u}^{in}, \tilde{\mathrm{d}}^{in}, \nabla\mathrm{d}^{in} \in H^{2S_N}$, $\mu_4 > 0$, $\lambda_1 < 0$, $\mu_1 \geq 0$, $\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \geq 0$, *and there exist* $\epsilon_0, \xi_0 \in (0,1]$, *s.t.*

$$E^{in} \triangleq \|\mathrm{u}^{in}\|^2_{H^{2S_N}} + \|\tilde{\mathrm{d}}^{in}\|^2_{H^{2S_N}} + \|\nabla\mathrm{d}^{in}\|^2_{H^{2S_N}} \leq \xi_0 \tag{0.37}$$

for all $\epsilon \in (0, \epsilon_0]$, *then system* (0.33) *admits a unique solution*

$$\mathrm{u}^\epsilon, \nabla\mathrm{d}^\epsilon, \mathrm{D}_{\mathrm{u}^\epsilon}\mathrm{d}^\epsilon \in L^\infty(\mathbb{R}^+; H^N), \ \nabla\mathrm{u}^\epsilon \in L^2(\mathbb{R}^+; H^N). \tag{0.38}$$

Moreover, the solution $(\mathrm{u}^\epsilon, \mathrm{d}^\epsilon)$ *is of the form*

$$\begin{cases} \mathrm{u}^\epsilon(t,x) = \mathrm{u}_0(t,x) + \sqrt{\epsilon}\mathrm{u}^\epsilon_R(t,x), \\ \mathrm{d}^\epsilon(t,x) = \mathrm{d}_0(t,x) + \epsilon\mathrm{D}^\epsilon_I(t,x) + \sqrt{\epsilon}\mathrm{d}^\epsilon_R(t,x), \end{cases} \tag{0.39}$$

where $(\mathrm{u}_0, \mathrm{d}_0)$ *is the solution to parabolic E-L.*

the remainder $(\mathrm{u}_R^\epsilon, \mathrm{d}_R^\epsilon)$ satisfies the following system

$$\begin{cases} \partial_t \mathrm{u}_R^\epsilon - \frac{1}{2}\mu_4 \Delta \mathrm{u}_R^\epsilon + \nabla p_R^\epsilon = \mu_1 \mathrm{div}\Big[(\mathrm{A}_R^\epsilon : \mathrm{d}_0 \otimes \mathrm{d}_0)\mathrm{d}_0 \otimes \mathrm{d}_0\Big] \\ \qquad + \mathcal{K}_\mathrm{u} + \mathrm{div}(\mathcal{C}_\mathrm{u} + \mathcal{T}_\mathrm{u} + \sqrt{\epsilon}\mathcal{R}_\mathrm{u}) + \epsilon \mathrm{div} \mathcal{Q}_\mathrm{u}(\mathrm{D}_I) , \\ \mathrm{div} \mathrm{u}_R^\epsilon = 0 , \\ \mathrm{D}^2_{\mathrm{u}_0 + \sqrt{\epsilon}\mathrm{u}_R^\epsilon} \mathrm{d}_R^\epsilon + \frac{-\lambda_1}{\epsilon} \mathrm{D}_{\mathrm{u}_0 + \sqrt{\epsilon}\mathrm{u}_R^\epsilon} \mathrm{d}_R^\epsilon - \frac{1}{\epsilon}\Delta \mathrm{d}_R^\epsilon + \partial_t(\mathrm{u}_R^\epsilon \cdot \nabla \mathrm{d}_0 + \sqrt{\epsilon}\mathrm{u}_R^\epsilon \cdot \nabla \mathrm{D}_I^\epsilon) \\ \qquad = \frac{1}{\epsilon}\mathcal{C}_\mathrm{d} + \frac{1}{\epsilon}\mathcal{S}_\mathrm{d}^1 + \frac{1}{\sqrt{\epsilon}}\mathcal{S}_\mathrm{d}^2 + \mathcal{R}_\mathrm{d} + \mathcal{Q}_\mathrm{d}(\mathrm{D}_I) \end{cases}$$

with the constraint

$$2\mathrm{d}_0 \cdot (\mathrm{d}_R^\epsilon + \sqrt{\epsilon}\mathrm{D}_I^\epsilon) + \sqrt{\epsilon}|\mathrm{d}_R^\epsilon + \sqrt{\epsilon}\mathrm{D}_I^\epsilon|^2 = 0 ,$$

with initial data

$$\begin{cases} \mathrm{u}_R^\epsilon(0, x) = 0 , \\ \mathrm{d}_R^\epsilon(0, x) = -\sqrt{\epsilon}\mathrm{D}_I^\epsilon(0, x) = -\sqrt{\epsilon}\widetilde{\mathrm{D}}_\epsilon^{in}(x) , \\ (\mathrm{D}_{\mathrm{u}_0 + \sqrt{\epsilon}\mathrm{u}_R^\epsilon} \mathrm{d}_R^\epsilon)(0, x) = -\sqrt{\epsilon}(\mathrm{u}_0 \cdot \nabla \mathrm{D}_I^\epsilon)(0, x) = -\sqrt{\epsilon}(\mathrm{u}^{in} \cdot \nabla \widetilde{\mathrm{D}}_\epsilon^{in})(x) . \end{cases}$$

The key of this work is to prove the existence for the system of $(\mathrm{u}_R^\epsilon, \mathrm{d}_R^\epsilon)$, furthermore, they satisfy the uniform bound

$$\begin{aligned}&\Big(\tfrac{1}{\epsilon}\|\mathrm{u}_R^\epsilon\|_{H^N}^2+\|\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon\|_{H^N}^2+\tfrac{1}{\epsilon}\|\mathrm{d}_R^\epsilon\|_{H^{N+1}}^2\Big)(t)\\&+\tfrac{1}{\epsilon}\int_0^t\|\nabla\mathrm{u}_R^\epsilon\|_{H^N}^2(\tau)\mathrm{d}\tau\le C\xi_0\end{aligned} \tag{0.40}$$

for all $t \geq 0$, $\epsilon \in (0, \epsilon_0]$ and for some constant $C > 0$, independent of ϵ and t.

Energy and energy-dissipation functionals-1

We now introduce the following energy functional $\mathcal{E}_{N,\epsilon}(t)$

$$\mathcal{E}_{N,\epsilon}(t) = \frac{1}{\epsilon}\|\mathrm{u}_R^\epsilon\|_{H^N}^2 + (1-\delta)\|\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon\|_{H^N}^2 + \frac{1}{\epsilon}\|\nabla \mathrm{d}_R^\epsilon\|_{H^N}^2$$
$$+(\frac{-\delta\lambda_1}{\epsilon} - \frac{5}{4}\delta)\|\mathrm{d}_R^\epsilon\|_{H^N}^2 + \|\mathrm{u}_R^\epsilon\cdot\nabla\mathrm{d}_0 + \frac{\delta}{2}\mathrm{d}_R^\epsilon\|_{H^N}^2$$
$$+\delta\|\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon + \mathrm{d}_R^\epsilon\|_{H^N}^2 + 2\sum_{|m|\le N}\langle\partial^m(\mathrm{u}_R^\epsilon\cdot\nabla\mathrm{d}_0), \partial^m\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon\rangle$$

and the energy dissipative rate $\mathcal{D}_{N,\epsilon}(t)$

$$\mathcal{D}_{N,\epsilon}(t) = \frac{3\mu_4}{8\epsilon}\|\nabla\mathrm{u}_R^\epsilon\|_{H^N}^2 + \frac{\delta}{2\epsilon}\|\nabla\mathrm{d}_R^\epsilon\|_{H^N}^2 - \delta\|\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon\|_{H^N}^2$$
$$+\frac{\mu_1}{\epsilon}\sum_{|m|\le N}\|(\partial^m\mathrm{A}_R^\epsilon):\mathrm{d}_0\otimes\mathrm{d}_0\|_{L^2}^2 + \frac{1}{\epsilon}(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1})\sum_{|m|\le N}\|(\partial^m\mathrm{A}_R^\epsilon)\mathrm{d}_0\|_{L^2}^2$$
$$+\frac{-\lambda_1}{\epsilon}\sum_{|m|\le N}\|\partial^m\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon + (\partial^m\mathrm{B}_R^\epsilon)\mathrm{d}_0 + \frac{\lambda_2}{\lambda_1}(\partial^m\mathrm{A}_R^\epsilon)\mathrm{d}_0\|_{L^2}^2,$$

where $\delta\in(0,\frac{1}{2}]$ is a fixed constant, depending only on λ_1, λ_2 and N.

Energy and energy-dissipation functionals-2

Lemma

There is a small $\epsilon_0 > 0$*, such that the energy* $\mathcal{E}_{N,\epsilon}(t)$ *and the energy dissipative rate* $\mathcal{D}_{N,\epsilon}(t)$ *are both nonnegative for any* $\epsilon \in (0, \epsilon_0)$*. Moreover, for all* $\epsilon \in (0, \epsilon_0)$*, we have*

$$\mathcal{E}_{N,\epsilon}(t) \sim \tfrac{1}{\epsilon}\|\mathrm{u}_R^\epsilon\|_{H^N}^2 + \|\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon\|_{H^N}^2 + \tfrac{1}{\epsilon}\|\nabla \mathrm{d}_R^\epsilon\|_{H^N}^2 + \tfrac{1}{\epsilon}\|\mathrm{d}_R^\epsilon\|_{H^N}^2 \,,$$

and

$$\begin{aligned}\mathcal{D}_{N,\epsilon}(t) \sim & \tfrac{1}{\epsilon}\|\nabla \mathrm{u}_R^\epsilon\|_{H^N}^2 + \tfrac{1}{\epsilon}\|\nabla \mathrm{d}_R^\epsilon\|_{H^N}^2 + \|\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon\|_{H^N}^2 \\ & + \tfrac{1}{\epsilon}\sum_{|m|\leq N}\|\partial^m \mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon + (\partial^m \mathrm{B}_R^\epsilon)\mathrm{d}_0 + (\partial^m \mathrm{A}_R^\epsilon)\mathrm{d}_0\|_{L^2}^2 \,.\end{aligned}$$

Here the small positive constant $\beta_{S_N,0}$ *is needed in requirement of the initial data of the existence of* $(\mathrm{u}_0, \mathrm{d}_0)$*.*

Energy and energy-dissipation functionals-3

Lemma

Let $(\mathrm{u}_R^\epsilon, \mathrm{d}_R^\epsilon)$ be a sufficiently smooth solution to the remainder system on $[0, T]$. Then there are constants $C > 0$ and $\theta_0 \gg 1$, depending only on the Leslie coefficients and $\beta_{S_N,0}$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathcal{E}_{N,\epsilon}(t) + \theta_0 \mathcal{E}_{S_N,0}(t)\Big] + \mathcal{D}_{N,\epsilon}(t) + \frac{\theta_0}{2}\mathcal{D}_{S_N,0}(t)$$
$$\leq C\Big[\mathcal{E}^2_{N,\epsilon}(t) + \mathcal{E}^{\frac{1}{2}}_{N,\epsilon}(t) + \mathcal{E}^{\frac{1}{2}}_{S_N,\epsilon}(t)\Big]\Big[\mathcal{D}_{N,\epsilon}(t) + \frac{\theta_0}{2}\mathcal{D}_{S_N,0}(t)\Big]$$

holds for all $t \in [0, T]$ and $\epsilon \in (0, \epsilon_0]$, where the small positive constant ϵ_0 is mentioned in the last Lemma.

Future work and open problems

- Domains with boundaries.
- Global in time weak solutions.
- 2-D case for $\lambda_1 = 0$.
- Inertia density limit for $\lambda_1 = 0$.
- Twist solutions (i.e. $\mathrm{u} = 0$).
- Q-tensor analogue.
- *Active* liquid crystals analogue (both Ericksen-Leslie and Q-tensor models). Chen-Majumdar-Wang-Zhang: “parabolic” Q-tensor case.