A Homogenization Result in the Gradient Theory of Phase Transitions

Irene Fonseca

Kirk Distinguished Visiting Fellow, INI Program on Mathematical Design of New Materials

Center for Nonlinear Analysis (CNA)

Department of Mathematical Sciences
Carnegie Mellon University

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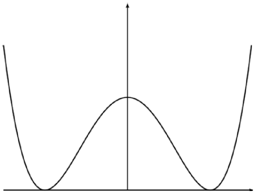
Van Der Waals (1893), Cahn and Hilliard (1958), Gurtin (1987)

. . . it started in 2003 . . .

Equilibrium behavior of a fluid with two stable phases may be described by the Gibbs free energy per unit volume

$$I(u) := \int_\Omega W(u)\, dx$$

$W : \mathbb{R} \to [0, +\infty)$ is a double well potential



$W(p) := (p^2 - 1)^2$, $\{W = 0\} = \{-1, 1\}$

- $\Omega \subset \mathbb{R}^N$ open, bounded, container
- $u : \Omega \to \mathbb{R}$ density of a fluid
- $\int_\Omega u\, dx = m \ldots m$ total mass of the fluid
- W double-well potential energy per unit volume
- $W^{-1}(\{0\}) = \{a, b\} \ldots a < b$ two phases of the fluid

Problem

Minimize total energy

$$I(u) = \int_\Omega W(u)\, dx$$

subject to $\int_\Omega u\, dx = m$

Solution

Assume $|\Omega| = 1$ *and* $a < m < b$. *Then minimizers are of the form*

$$u_E(x) = \begin{cases} a & \textit{if } x \in E, \\ b & \textit{if } x \in \Omega \setminus E, \end{cases}$$

where $E \subseteq \Omega$ *is any measurable set with* $|E| = \frac{b-m}{b-a}$

NONUNIQUENESS OF SOLUTIONS

Selection via singular perturbations:

$$I_\varepsilon(u) := \int_\Omega \left[W(u) + \varepsilon^2 |\nabla u|^2 \right] dx, \quad u \in C^1(\Omega), \varepsilon > 0$$

$\varepsilon^2 \int_\Omega |\nabla u|^2 \, dx \ldots$ surface energy penalization

Modica–Mortola, 1977

$\{W = 0\} = \{a, b\}$

Gurtin's 1985 conjecture:

Asymptotic behavior of minimizers to E_ε described via Γ-convergence. Scaling by ε^{-1} yields

$$\varepsilon^{-1} I_\varepsilon \xrightarrow{\Gamma} I_0,$$

$$F_0(u) := \begin{cases} c_W \ P(A_0; \Omega) & u \in BV(\Omega; \{a, b\}), \\ +\infty & u \in L^1(\Omega) \setminus BV(\Omega; \{a, b\}) \end{cases}$$

where

$$A_0 := \{u(x) = a\}, \ c_W := 2 \int_a^b \sqrt{W(s)} ds$$

$$I_\varepsilon(u) := \int_\Omega \left[W(u) + \varepsilon^2 |\nabla u|^2\right] dx, \quad u \in C^1(\Omega)$$

Gurtin Conjecture (1987): Minimizers u_ε

$$\min \left\{ I_\varepsilon(u) : u \in C^1(\Omega), \quad \int_\Omega u \, dx = m \right\}$$

converge to u_{E_0}, where

$$\mathrm{Per}_\Omega(E_0) \leq \mathrm{Per}_\Omega(E)$$

over all $E \subseteq \Omega$ measurable with $|E| = \frac{b-m}{b-a}$

$$F_\varepsilon(u) := \frac{1}{\varepsilon} I_\varepsilon(u) = \int_\Omega \left[\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right] dx$$

F_ε and I_ε have the same minimizers

So ... if we know the Γ-limit of $\{F_\varepsilon\}$ then we know where the minimizers of I_ε converge to ...

$$F_\varepsilon(u) = \int_\Omega \left[\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right] dx, \quad u \in C^1(\Omega)$$

Theorem (Modica (1987), Sternberg (1988), F. and Tartar (1989),...)

$F_\varepsilon \xrightarrow{\Gamma} F_0$ *with respect to strong convergence in* $L^1(\Omega)$, *where*

$$F_0(u) := \begin{cases} c_W \operatorname{Per}_\Omega\left(u^{-1}(\{a\})\right) & \text{if } u \in BV(\Omega;\{a,b\}), \int_\Omega u\,dx = m, \\ +\infty & \text{otherwise} \end{cases}$$

$$c_W := 2\int_a^b \sqrt{W(s)}\,ds$$

What about higher order nonlocal regularizations?

- G. Dal Maso, I.F. and G. Leoni, *Trans. Amer. Math. Soc.* (2018)

$$F_\varepsilon(u) := \begin{cases} \int_\Omega \frac{1}{\varepsilon} W(u)\, dx + \mathcal{J}_\varepsilon(u) & \text{if } u \in W^{1,2}_{\text{loc}}(\Omega) \cap L^2(\Omega) \\ +\infty & \text{otherwise} \end{cases},$$

where

$$\mathcal{J}_\varepsilon(u) := \varepsilon \int_\Omega \int_\Omega J_\varepsilon(x-y)|\nabla u(x) - \nabla u(y)|^2 dxdy \quad \text{for } u \in W^{1,2}_{\text{loc}}(\Omega)$$

$$J_\varepsilon(x) := \frac{1}{\varepsilon^N} J\left(\frac{x}{\varepsilon}\right)$$

$J : \mathbb{R}^N \to [0, +\infty)$... even measurable function

$$\int_{\mathbb{R}^n} J(x)(|x| \wedge |x|^2)\, dx < +\infty$$

where $a \wedge b := \min\{a, b\}$.

Nonlocal higher order singular perturbations

$J : \mathbb{R}^N \to [0, +\infty)\ldots$ even measurable function

$$\int_{\mathbb{R}^N} J(x)(|x| \wedge |x|^2)\, dx < +\infty$$

For example

$$J(x) := |x|^{-N-2s}, \quad \frac{1}{2} < s < 1$$

leads to Gagliardo's seminorm for the fractional Sobolev space $H^s(\mathbb{R})$
In this case

$$J_\varepsilon(x) = \varepsilon^{2s}|x|^{-N-2s}$$

• G. Alberti and G. Belletini, *Math. Ann.* (1998)

$$F_\varepsilon(u) := \begin{cases} \int_\Omega \frac{1}{\varepsilon} W(u)\, dx + \tilde{\mathcal{J}}_\varepsilon(u) & \text{if } u \in W^{1,2}_{\text{loc}}(\Omega) \cap L^2(\Omega) \\ +\infty & \text{otherwise,} \end{cases},$$

$$\tilde{\mathcal{J}}_\varepsilon(u) := \frac{1}{\varepsilon}\int_\Omega\int_\Omega J_\varepsilon(x-y)(u(x)-u(y))^2 dxdy \quad \text{for } u \in W^{1,2}_{\text{loc}}(\Omega)$$

$$J_\varepsilon(x) := \frac{1}{\varepsilon^N} J\left(\frac{x}{\varepsilon}\right)$$

(statistical mechanics) free energies of continuum limits of Ising spin systems on lattices

u . . . macroscopic magnetization

J . . . ferromagnetic Kac potential

but dependence on ∇u in place of u adds **remarkable** difficulties!

Relevant Spaces:

$\nu \in \mathbb{S}^{N-1} := \partial B_1(0)$

$\nu_1, \dots, \nu_N$ ... orthonormal basis in $\mathbb{R}^N$ with $\nu_N = \nu$

$$V^\nu := \{x \in \mathbb{R}^N : \ |x \cdot \nu_i| < 1/2 \text{ for } i = 1, \dots, N-1\}$$
$$Q^\nu := \{x \in \mathbb{R}^N : \ |x \cdot \nu_i| < 1/2 \text{ for } i = 1, \dots, N\}$$

$$W^{1,2}_{\nu_1,\dots,\nu_{N-1}} := \{v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N) : v(x+\nu_i) = v(x) \text{ for a.e. } \in \mathbb{R}^N, i = 1, \dots, N-1\}$$

$$X^\nu := \{v \in W^{1,2}_{\nu_1,\dots,\nu_{N-1}} : \ v(x) = \pm 1 \text{ for a.e. } x \in \mathbb{R}^N \text{ with } \pm x \cdot \nu \geq 1/2\}$$

When $N = 1$ take $\nu = \pm 1$, $V^\nu := \mathbb{R}$, $Q^\nu := (-1/2, 1/2)$

$X^\nu := \{v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}) : v(x) = \pm 1 \text{ for a.e. } x \in \mathbb{R} \text{ with } \pm x \geq 1/2\}$

Anisotropic Surface Energy

$$\psi(\nu) := \inf_{0<\varepsilon<1} \inf_{v \in X^\nu} \mathcal{F}_\varepsilon^\nu(v)$$

where

$$\mathcal{F}_\varepsilon^\nu(u) := \frac{1}{\varepsilon} \int_{Q^\nu} W(u(x))\,dx + \varepsilon \int_{V^\nu} \int_{\mathbb{R}^N} J_\varepsilon(x-y)|\nabla u(x) - \nabla u(y)|^2 dxdy$$

Define $\mathcal{F} : L^2(\Omega) \to [0, +\infty]$ by

$$\mathcal{F}(u) := \begin{cases} \int_{S_u} \psi(\nu_u)\; d\mathcal{H}^{N-1} & \text{if } u \in BV(\Omega; \{-1, 1\}) \ , \\ +\infty & \text{otherwise in } L^2(\Omega) \end{cases}$$

Compactness in L^2 of energy bounded sequences

$\{\mathcal{F}_\varepsilon\}$ Γ-converges to $\mathcal{F}$ in $L^2(\Omega)$

Localized energies:

$$\mathcal{W}_\varepsilon(u, A) := \frac{1}{\varepsilon} \int_A W(u(x))\ dx$$

$$\mathcal{J}_\varepsilon(u, A, B) := \varepsilon \int_A \int_B J_\varepsilon(x-y)|\nabla u(x) - \nabla u(y)|^2 dxdy$$

When $A = B$ we set

$$\mathcal{F}_\varepsilon(u, A) := \mathcal{W}_\varepsilon(u, A) + \mathcal{J}_\varepsilon(u, A, A) \quad \text{and} \quad \mathcal{J}_\varepsilon(u, A) := \mathcal{J}_\varepsilon(u, A, A)$$

Theorem (Interpolation Inequality)

$$\varepsilon \int_A |\nabla u(x)|^2 dx \leq C\mathcal{F}_\varepsilon(u, (A)^{2\varepsilon\gamma_J})$$

for every $\varepsilon > 0$*, for every open set* $A \subset \mathbb{R}^N$*, and for every* $u \in W^{1,2}_{\mathrm{loc}}((A)^{2\varepsilon\gamma_J})$

$$(A)^\eta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, A) < \eta\}$$

$$\varepsilon \int_A |\nabla u(x)|^2 dx \leq C\mathcal{F}_\varepsilon(u, (A)^{2\varepsilon\gamma_J})$$

$(A)^\eta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, A) < \eta\}$

γ_J: For all $\xi \in \mathbb{S}^{N-1}$ there exist $-\gamma_J < \alpha(\xi) < \beta(\xi) < \gamma_J$ s.t.

$$\int_{\alpha(\xi)}^{\beta(\xi)} \frac{1}{J(t\xi)|t|^{N-1}}\, dt \leq C_J$$

Next ... "modification lemma" ... proof 11 pages long ...

Interaction Phase Transition/Homogenization

Consider fluids which exhibit periodic heterogeneity at small scales, i.e.

$$F_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} W\left(\frac{x}{\delta(\varepsilon)}, u \right) + \varepsilon |\nabla u|^2 \right] dx$$

where

$$W(x,p) = 0 \iff p \in \{a,b\},$$
$$W(\cdot, p) \text{ is } Q\text{-periodic for every } p,$$

and

$$\delta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Example: $W(x,p) = \chi_E(x) W_1(p) + \chi_{Q \setminus E} W_2(p)$

Goal: Identify Γ-limit of F_ε

Ansini, Braides, Piat (2003): W homogeneous, regularization $f\left(\frac{x}{\delta(\varepsilon)}, \nabla u\right)$

Braides, Zeppieri (2009): $\int_0^1 \left[W^{(k)}\left(\frac{x}{\delta(\varepsilon)}, u\right) + \varepsilon^2 |u''|^2 \right] dx$

Scaling regime $\delta(\varepsilon) = \varepsilon$

Theorem (Cristoferi, F., Hagerty, Popovici. To appear: *Interfaces Free Bound.*(2019))

Let $\delta(\varepsilon) = \varepsilon$. *Then* $F_\varepsilon \xrightarrow{\Gamma} F$,

$$F(u) := \begin{cases} \displaystyle\int_{\partial^* A_0} \sigma(\nu) d\mathcal{H}^{N-1} & u \in BV(\Omega; \{a, b\}), \\ +\infty & \textit{otherwise} \end{cases}$$

where

$$A_0 := \{u(x) = a\}, \ \nu \textit{ is the outward normal to } A_0,$$

and (anisotropic surface energy)

$$\sigma(\nu) := \lim_{T\to\infty} \inf_{u \in \mathcal{A}_{\nu,T}} \left\{ \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[W(y, u(y)) + |\nabla u(y)|^2 \right] dy \right\}$$

Cell Problem

$$\sigma(\nu) = \lim_{T\to\infty} \inf_{u\in\mathcal{A}_{\nu,T}} \left\{ \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[W(y,u(y)) + |\nabla u(y)|^2 \right] dy \right\}$$

where

$$\mathcal{A}_{\nu,T} := \left\{ u \in H^1(TQ_\nu;\mathbb{R}^d) : u(x) = (\rho_T * u_0)(x\cdot\nu) \text{ on } \partial TQ_\nu \right\}$$

$$u_0(t) := \begin{cases} b & \text{if } t > 0, \\ a & \text{if } t < 0 \end{cases}$$

$$\rho_T(x) := T^N \rho(Tx),\ \ \rho \in C_c^\infty(\mathbb{R}) \text{ with } \int_{\mathbb{R}} \rho = 1$$

Outline of Proof

- **Compactness: Bounded energy** $\rightarrow BV$ **structure**
  - Reduction to classical Modica-Mortola technique
  - $W(x,p) = 0$ iff $p \in \{a, b\}$
  - $(x,p) \rightarrow W(x,p)$ Carathéodory, only measurability in x
  - $W(x,p) \geq \tilde{W}(p)$, $\tilde{W}(p) = 0$ iff $p \in \{a, b\}$, $\tilde{W}(p) \geq C|p|$ for $|p| >> 1$
- Γ**-liminf: "Lower-semicontinuity" result using blow-up techniques**
- $\frac{|p|^q}{C} - C \leq W(x,p) \leq C(1 + |p|^q)$, some $q \geq 2$
  - "Blow up" at points in jump set
  - De Giorgi's slicing method $\rightarrow$ prescribe boundary conditions from σ
  - Compare with optimal profiles given by σ
- Γ**-limsup: Recovery sequences**
  - Blow-Up Method
  - Recovery sequences for polyhedral sets with $\nu \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$
  - Density result and upper semicontinuity of σ

Compactness

Reduce to

$$E_\varepsilon(u) := \int_\Omega \left[\tilde{W}(u) + \varepsilon^2 |\nabla u|^2\right] dx$$

Use F. and Tartar (1989)

$u \in BV(\Omega; \{a, b\})$, $A_0 := \{u = a\}$, $\varepsilon_n \to 0^+$

Claim: there exists $\{u_n\} \subset H^1(\Omega; \mathbb{R}^d)$ s.t. $u_n \to u$ in L^1 and

$$\limsup F_{\varepsilon_n}(u_n) \le F(u) = \int_{\partial^* A_0} \sigma(\nu) d\mathcal{H}^{N-1}$$

Localization: for $U \subset \Omega$ open

$$\mathcal{F}_{\{\varepsilon_n\}}(u; U) := \inf\{\liminf F_{\varepsilon_n}(u_n, U) : u_n \to u \text{ in } L^1(U; \mathbb{R}^d)\}$$

Up to a subsequence

$$\lambda : \mathcal{A}(\Omega) \to [0, +\infty), \qquad \lambda(B) := \mathcal{F}_{\{\varepsilon_n\}}(u; B), \quad B \text{ Borel set}$$

is a positive finite measure, and

$$\lambda << \mu := \mathcal{H}^{N-1} \lfloor \partial^* A_0$$

Done if

$$\frac{d\lambda}{d\mu}(x_0) \leq \sigma(\nu(x_0))$$

To prove it:

$$\frac{d\lambda}{d\mu}(x_0) = \lim \frac{\lambda(Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}$$

$$\frac{\lambda(Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} \leq \liminf_{n\to\infty} \frac{1}{\varepsilon^{N-1}} F_{\varepsilon_n}(u_{n,\varepsilon}, Q_\nu(x_0,\varepsilon))$$

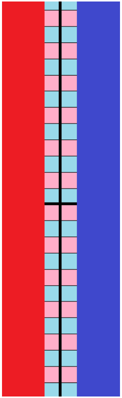
with $u_{n,\varepsilon} \to u$, $n \to \infty$, in $L^1(Q_\nu(x_0,\varepsilon))$

How do we construct these approximating sequences?

Easy Case: Transition Layer Aligned with Principal Axes

If $\nu \in \{e_1, \dots, e_N\}$, create recovery sequence by tiling optimal profiles from definition of σ.

Say $\nu = e_N$



Pick $T_k \subset \mathbb{N}$ and u_k s.t.

$$\sigma(e_N) = \lim_{k\to\infty} \frac{1}{T_k^{N-1}} \int_{T_k Q} \left[W(y, u_k(y)) + |\nabla u_k(y)|^2\right] dy,$$

$$v_k(x) := u_k(T_k x), \text{ extended by } Q'\text{-periodicity},$$

$$v_{k,\varepsilon,r}(x) := \begin{cases} u_0(x) & |x_N| \geq \frac{\varepsilon T_k}{2r} \\ v_k\left(\frac{rx}{\varepsilon T_k}\right) & |x_N| < \frac{\varepsilon T_k}{2r} \end{cases}$$

$$u_{k,\varepsilon,r}(x) := v_{k,\varepsilon,r}\left(\frac{x}{r}\right) \to u \text{ in } L^1(rQ)$$

Transition Layer Aligned with Principal Axes, cont.

Blow up:

$$\lim_{r\to 0}\frac{F(u;rQ)}{r^{N-1}} \le \lim_{r\to 0}\lim_{\varepsilon\to 0}\frac{1}{r^{N-1}}\int_{rQ}\left[\frac{1}{\varepsilon}W(x,u_{k,\varepsilon,r})+\varepsilon|\nabla u_{k,\varepsilon,r}|^2\right]dx$$

$$= \lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{Q'}\int_{-\varepsilon T_k/2r}^{\varepsilon T_k/2r}\left[\frac{r}{\varepsilon}W\left(\frac{r}{\varepsilon}y, v_k\left(\frac{ry}{\varepsilon T_k}\right)\right)\right.$$

$$\left. + \frac{r}{\varepsilon T_k^2}\left|\nabla v_k\left(\frac{ry}{\varepsilon T_k}\right)\right|^2\right]dy$$

$$= \lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{Q'}\int_{-1/2}^{1/2}\left[T_k W\left(\left(T_k\frac{rz'}{\varepsilon T_k}, T_k z_N, v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right)\right)\right.$$

$$\left. + \frac{1}{T_k}\left|\nabla v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right|^2\right]dz$$

$$\lim_{r\to 0}\frac{F(u;rQ)}{r^{N-1}} \le \lim_{r\to 0}\lim_{\varepsilon\to 0}\frac{1}{r^{N-1}}\int_{rQ}\left[\frac{1}{\varepsilon}W(x,u_{k,\varepsilon,r})+\varepsilon|\nabla u_{k,\varepsilon,r}|^2\right]dx$$

$$=\lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{Q'}\int_{-\varepsilon T_k/2r}^{\varepsilon T_k/2r}\left[\frac{r}{\varepsilon}W\left(\frac{r}{\varepsilon}y, v_k\left(\frac{ry}{\varepsilon T_k}\right)\right)\right.$$

$$\left.+\frac{r}{\varepsilon T_k^2}\left|\nabla v_k\left(\frac{ry}{\varepsilon T_k}\right)\right|^2\right]dy$$

$$=\lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{Q'}\int_{-1/2}^{1/2}\left[T_k W\left(\left(T_k\frac{rz'}{\varepsilon T_k}, T_k z_N, v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right)\right.\right.$$

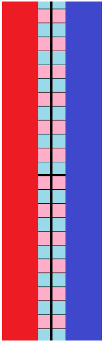
$$\left.+\frac{1}{T_k}\left|\nabla v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right|^2\right]dz$$

Transition Layer aligned with Principal Axes, cont.

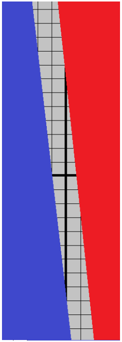
Since W and v_k are **BOTH** Q'-periodic and $T_k \in \mathbb{N}$, we can use the Riemann Lebesgue Lemma:

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{Q'}\int_{-1/2}^{1/2}\left[T_k W\left(\left(T_k\frac{rz'}{\varepsilon T_k}, T_k z_N\right), v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right)\right.$$
$$\left. + \frac{1}{T_k}\left|\nabla v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right|^2\right]dz$$
$$= \lim_{r\to 0}\int_{Q'}\int_{-1/2}^{1/2}\left[T_k W((T_k y', T_k z_N), v_k(y', z_N))\right.$$
$$\left. + \frac{1}{T_k}|\nabla v_k(y', z_N)|^2 dz_N\right]dy'$$
$$= \frac{1}{T_k^{N-1}}\int_{T_k Q}\left[W(x, u_k(x)) + |\nabla u_k(x)|^2\right]dx$$

Other Transition Directions?



(a)
Aligned



(b)
Misaligned

Figure: Since W is Q-periodic, can tile along principal axes. What if the transition layer is not aligned?

Q-periodic implies $\lambda_\nu Q_\nu$-periodic

Key observation: Periodic microstructure in principal directions $\rightarrow$ periodicity in other directions.

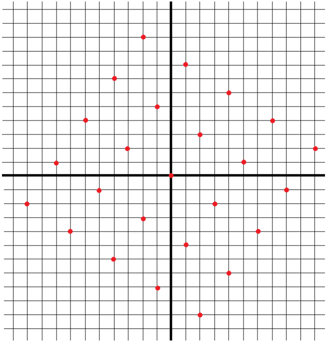


Figure: Integer lattice contains copies of itself, rotated and scaled

$\triangleright$ W is $\lambda_\nu Q_\nu$-periodic for some $\lambda_\nu \in \mathbb{N}$, and for $\nu \in \Lambda := \mathbb{Q}^N \cap \mathbb{S}^{N-1}$: Dense!

Orthonormal Bases in $\mathbb{Q}^N$

Important: Every face of Q_ν has rational normal.

Need an orthonormal basis using rational vectors:

Theorem (Witt, '37)

Any isometry between two subspaces F_1 and F_2 of a finite-dimensional vector space V defined over a field $\mathbb{K}$ of characteristic different from 2 and provided with a metric structure induced from a nondegenerate symmetric or skew-symmetric bilinear form $B[\cdot,\cdot]$ may be extended to a metric automorphism of the entire space V.

In particular:

$$V = \mathbb{Q}^N,\ F_1 := \operatorname{span}_{\mathbb{Q}}(e_N),\ F_2 := \operatorname{span}_{\mathbb{Q}}(\nu),\ B[x,y] := x \cdot y$$

Then, the mapping $e_N \mapsto \nu$ extends to an isometry!

Theorem (Cristoferi, F., Hagerty, Popovici (2018))

Let $\nu_N \in \Lambda = \mathbb{Q}^N \cap \mathbb{S}^{N-1}$. *There exist* $\nu_1, \ldots, \nu_{N-1} \in \Lambda$, $\lambda_\nu \in \mathbb{N}$, *s.t.*

$$\nu_1, \ldots, \nu_{N-1}, \nu_N$$

o.n. basis of $\mathbb{R}^N$ *and*

$$W(x + n\lambda_\nu \nu_i, p) = W(x, p)$$

a.e. $x \in Q$, *all* $n \in \mathbb{N}$, $p \in \mathbb{R}^d$.

Also use:

$\varepsilon > 0$, $\nu \in \Lambda$, $S : \mathbb{R}^N \to \mathbb{R}^N$ rotation, $Se_N = \nu$.

Then there is a rotation $R : \mathbb{R}^N \to \mathbb{R}^N$ s.t. $Re_N = \nu$, $Re_i \in \Lambda$ all $i = 1, \ldots, N-1$, $||R - S|| < \varepsilon$

Properties of σ are important

$$\sigma(\nu) = \lim_{T\to\infty} \inf_{u\in\mathcal{A}_{Q_\nu,T}, Q_\nu\in\mathcal{Q}_\nu} \left\{ \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[W(y,u(y)) + |\nabla u(y)|^2 \right] dy \right\}$$

where

$$\mathcal{A}_{Q_\nu,T} := \left\{ u \in H^1(TQ_\nu;\mathbb{R}^d) : u(x) = (\rho_T * u_0)(x\cdot\nu) \text{ on } \partial TQ_\nu \right\}$$

$$u_0(t) := \begin{cases} b & \text{if } t > 0 \\ a & \text{if } t < 0 \end{cases}$$

$$\rho_T(x) := T^N \rho(Tx),\ \rho \in C_c^\infty(\mathbb{R}) \text{ with } \int_{\mathbb{R}} \rho = 1$$

$\mathcal{Q}_\nu$ . . . unit cubes centered at the origin with two faces orthogonal to ν

Properties of σ:

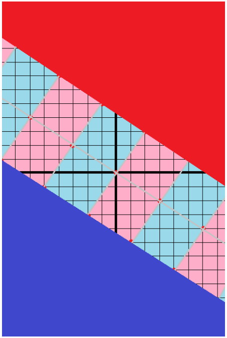
- σ is well defined and finite
- the definition of σ does not depend on the choice of the mollifier
- $\sigma : \mathbb{S}^{N-1} \to [0, +\infty)$ is upper semicontinuous
- if $\nu \in \Lambda$ then

$$\sigma(\nu) = \lim_{n\to\infty} \lim_{T\to\infty} \inf_{u\in\mathcal{A}_{Q_n,T}} \left\{ \frac{1}{T^{N-1}} \int_{TQ_n} \left[W(y, u(y)) + |\nabla u(y)|^2 \right] dy \right\}$$

where the normals to all faces of Q_n belong to Λ

Transition Layer aligned with $\nu \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$

Same periodic tiling technique: Use $T_k \in \lambda_\nu \mathbb{N}$.



$\triangleright$ Blow up method $\rightarrow$ Recovery sequences for polyhedral sets A_0 with normals to its facets in Λ

Blow up method $\to$ Recovery sequences for polyhedral sets A_0 with normals to its facets in Λ

$x_0 \in \Omega \cap \partial^* A_0$, $\nu := \nu_{A_0}(x_0)$.

Find rotation R_ν, $\lambda_\nu \in \mathbb{N}$, s.t. with $Q_\nu := R_\nu(x_0 + Q)$

$$W(x + n\lambda_\nu v, p) = W(x, p)$$

a.e. $x \in \Omega$, every $n \in \mathbb{N}$, every $p \in \mathbb{R}^d$, every v orthogonal to one face of Q_ν

As before, done if

$$\frac{d\lambda}{d\mu}(x_0) \leq \sigma(\nu(x_0))$$

To prove it:

$$\frac{d\lambda}{d\mu}(x_0) = \lim \frac{\lambda(Q_\nu(x_0, \varepsilon))}{\varepsilon^{N-1}}$$

. . . and work a little harder . . .

Recovery sequences for arbitrary $u \in BV(\Omega; \{a, b\})$

- For $u \in BV(\Omega; \{a, b\})$, we can find $u^{(n)} \in BV(\Omega; \{a, b\})$ such that $A_0^{(n)}$ are polyhedral,
$$u^{(n)} \to u \text{ in } L^1$$
$$|Du^{(n)}|(\Omega) \to |Du|(\Omega).$$
Since $\mathbb{Q}^N \cap \mathbb{S}^{N-1}$ dense, can require $\nu^{(n)} \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$.
- Since σ upper-semicontinuous, by Reshetnyak's,
$$\int_{\partial^* A_0} \sigma(\nu) d\mathcal{H}^{n-1} \leq \limsup_{n\to\infty} \int_{\partial^* A_0^{(n)}} \sigma\left(\nu^{(n)}\right) d\mathcal{H}^{n-1}$$
- Find recovery sequences $u_\varepsilon^{(n)}$ for the $u^{(n)}$ so that
$$\int_{\partial^* A_0^{(n)}} \sigma\left(\nu^{(n)}\right) d\mathcal{H}^{n-1} \leq \limsup_{\varepsilon\to 0^+} F_\varepsilon\left(u_\varepsilon^{(n)}\right)$$
- Diagonalize!

Other scaling regimes

Recently considered the case where the scale of homogenization is much **smaller** than the scale of the phase transition

$$F_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} W\left(\frac{x}{\delta(\varepsilon)}, u \right) + \varepsilon |\nabla u|^2 \right] dx.$$

If $\delta(\varepsilon)$ is **sufficiently small** compared to ε, the homogenization effects are effectively **instantaneous**, and we can pass to a homogenized system

$$F_\varepsilon^H(u) = \int_\Omega \left[\frac{1}{\varepsilon} W_H(u) + \varepsilon |\nabla u|^2 \right] dx$$

where

$$W_H(p) := \int_Q W(y, p) dy$$

Scaling regime $\delta(\varepsilon) << \varepsilon$

Theorem (Cristoferi, F., Hagerty (2019))

Let $\delta(\varepsilon)$ *be such that*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{\frac{3}{2}}}{\delta(\varepsilon)} = +\infty.$$

Then, $F_\varepsilon \xrightarrow{\Gamma} F_0^H$*, where (isotropic surface energy)*

$$F_0^H(u) := \begin{cases} K_H \; P(A_0;\Omega) & u \in BV(\Omega;\{a,b\}), \\ +\infty & u \in L^1(\Omega) \setminus BV(\Omega;\{a,b\}) \end{cases}$$

$W_H(p) := \int_Q W(y,p)\,dy$, $A_0 := \{u(x) = a\}$

$$K_H := 2 \inf \left\{ \int_0^1 \sqrt{W_H(g(s))}|g'(s)|ds : g \text{ piecewise } C^1, g(0) = a, g(1) = b \right\}$$

Outline of Proof

- Homogenization Lemma
  - Compare the bulk energy to a homogenized bulk energy
  - Requires quantitative control on δ vs ε
- Use the result of F. and Tartar to identify Γ-limit of homogenized energy
  - Comparison with homogenized energy yields information about minimizing sequences $\to$ relaxed growth assumptions for W

Theorem (F., Tartar (1989))

Functionals of the form

$$G_\varepsilon(u) = \int_\Omega \left[\frac{1}{\varepsilon} \widetilde{W}(u) + \varepsilon |\nabla u|^2 \right] dx, \ \ u \in H^1(\Omega; \mathbb{R}^d)$$

have a Γ-limit

$$G_0(u) := K_G \ P(A_0; \Omega), \ \ u \in BV(\Omega; \{a, b\})$$

Homogenization Lemma

The key tool in comparing F_ε and F_ε^H is a Riemann-Lebesgue type result for all W uniformly bounded.

Lemma

Let ε_n, δ_n and $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ be such that

$$\sup_{n\in\mathbb{N}} \int_\Omega \varepsilon_n |\nabla u_n|^2 dx < \infty \text{ and } \lim_{n\to\infty} \varepsilon_n^{-\frac{3}{2}} \delta_n = 0.$$

Then,

$$\lim_{n\to\infty} \frac{1}{\varepsilon_n} \int_\Omega \left[W\left(\frac{x}{\delta_n}, u_n(x)\right) - W_H(u_n(x)) \right] dx = 0$$

- Uniform boundedness: NOT required for the main theorem- will be discussed later
- Scaling: More on this...

Scaling

- The homogenization lemma requires a particular exponent $\varepsilon^{\frac{3}{2}}$
  - If the regularization is of the form $|\nabla u|^p$, the exponent would be $\varepsilon^{1+\frac{1}{p}}$.
- This same exponent is necessary Ansini, Braides, Piat (2003) who homogenized the regularization term
- Unclear if this is purely technical or if truly different behavior is possible in the intermediate regime

Homogenization Lemma - Outline of Proof

At scale δ_n, decompose Ω into δ_n-cubes and a remainder R_n

$$\Omega = \bigcup_{i=1}^{M_n} Q(p_i, \delta_n) \cup R_n,$$

where p_i are on the lattice $\delta_n \mathbb{Z}^N$
R_n ... collection of cubes $Q(z, \delta_n)$, $z \in \delta_n \mathbb{Z}^N$, intersecting $\partial\Omega$

$$|R_n| \leq C\delta_n$$

Uniform boundedness:

$$\frac{1}{\varepsilon_n} \int_{R_n} W\left(\frac{x}{\delta_n}, u_n(x)\right) dx \leq C \frac{\delta_n}{\varepsilon_n} \to 0$$

Homogenization Lemma - Outline of Proof, cont.

Sufficient to control

$$\frac{1}{\varepsilon_n}\sum_{i=1}^{M_n}\left|\int_{Q(p_i,\delta_n)} W\left(\frac{x}{\delta_n}, u_n(x)\right) - W_H(u_n(x))dx\right|$$

Apply the substitution $x = p_i + \delta_n y$ and periodicity:

$$\frac{\delta_n^N}{\varepsilon_n}\sum_{i=1}^{M_n}\left|\int_Q W(y, u_n(p_i+\delta_n y)) - W_H(u_n(p_i+\delta_n y))dy\right|$$

Recast as the double integral

$$\frac{\delta_n^N}{\varepsilon_n}\sum_{i=1}^{M_n}\left|\int_Q\int_Q W(y, u_n(p_i+\delta_n y)) - W(z, u_n(p_i+\delta_n y))dzdy\right|$$

Homogenization Lemma - Outline of Proof, cont.

After another change of variables, this is

$$\frac{\delta_n^N}{\varepsilon_n}\sum_{i=1}^{M_n}\left|\int_Q\int_Q W(y,u_n(p_i+\delta_n y))-W(y,u_n(p_i+\delta_n z))dzdy\right|$$

and by Lipschitz behavior of W, enough to control

$$\frac{\delta_n^N}{\varepsilon_n}\sum_{i=1}^{M_n}\int_Q\int_Q |u_n(p_i+\delta_n y)-u_n(p_i+\delta_n z)|\,dzdy$$

By Poincaré, we can estimate via

$$\delta_n\frac{\delta^N}{\varepsilon_n}\sum_{i=1}^{M_n}\int_Q |\nabla u_n(p_i+\delta_n y)|dy \leq \frac{\delta_n}{\varepsilon_n}\int_\Omega |\nabla u_n|dx$$

$$\leq \frac{\delta_n}{\varepsilon_n}\varepsilon_n^{-1/2}\left(\varepsilon_n\int_\Omega |\nabla u_n|^2dx\right)^{1/2}$$

Uniform Boundedness

To apply the homogenization lemma to potentials which may be unbounded, we use a cut-off trick- possible because by F.-Tartar, the homogenized problem is based on the 1-dimensional optimization

$$K_H = 2\inf\left\{\int_0^1 \sqrt{W_H(g(s))}|g'(s)|ds\right\}$$

where the g are pointwise C^1 so that $g(0) = a,\ g(1) = b$. Pick $R > 0$ so that for optimal curves g, $|g(t)| \leq R$. Let

$$M = \operatorname*{ess\,sup}_{x\in\Omega} \max_{|p|\leq R} W(x,p)$$

and define the truncated potential

$$\widetilde{W}(x,p) := \min\{W(x,p), M\}$$

Future problems

- Scaling regime $\varepsilon << \delta(\varepsilon)$ ... homogenization of the "surface Cahn-Hilliard limiting energy". Forthcoming
- multiple wells
- More general regularization terms, i.e. $|\nabla u|^2 \to f(x, u, \nabla u)$
- Nonlocal stochastic homogenization
- Solid-solid phase transitions: $W\left(\frac{x}{\delta(\varepsilon)}, \nabla u(x)\right)$

Solid-sold phase transitions without homogenization:

$$W(F) \approx |F|^p, \text{ Conti, Fonseca, Leoni, '02.}$$

$$W(F) \approx \text{dist}^p(F, SO(N)A \cup SO(N)B)$$

only studied for N=2 (Conti–Schweizer, '06) ... and in arbitrary dimensions under a suitable anisotropic penalization of second variations
Elisa Davoli and Manuel Friedrich, 2018

A good place to stop . . .