Basis properties of the Haar system in various function spaces

Andreas Seeger (University of Wisconsin-Madison)

Workshop on approximation, sampling, and compression
in high dimensional problems
Isaac Newton Institute, June 2019

- Joint work with Gustavo Garrigós and Tino Ullrich

• A.S., T. Ullrich. Haar projection numbers and failure of unconditional convergence in Sobolev spaces. Mathematische Zeitschrift, 285 (2017), 91-119.

• ..., Lower bounds for Haar projections: Deterministic Examples. Constructive Approximation, 42 (2017), 227-242.

• G. Garrigós, A.S. and T. Ullrich. The Haar system as a Schauder basis in spaces of Hardy-Sobolev type. Journal of Fourier Analysis and Applications, 24(5) (2018), 1319-1339.

• ..., Basis properties of the Haar system in limiting Besov spaces. Preprint (arXiv).

• ..., The Haar system in Triebel-Lizorkin spaces: Endpoint cases. Manuscript.

The Haar system $\mathcal{H}$

Haar (1910): For $j \in \mathbb{N}_0$, $\mu \in \mathbb{Z}$ let $h_{j,\mu}$ be supported on $I_{j,\mu} = [2^{-j}\mu, 2^{-j}(\mu + 1))$ and

$$h_{j,\mu}(x) = \begin{cases} 1 & \text{on the left half of } I_{j,\mu} \\ -1 & \text{on the right half of } I_{j,\mu} \end{cases}$$

- The *Haar frequency* of $h_{j,\mu}$ is 2^j.
- The functions $2^{j/2} h_{j,\mu}$, together with the functions $h_{-1,\mu} := 1_{[\mu,\mu+1)}$ form an ONB of $L^2(\mathbb{R})$.
- Let $\mathcal{H}$ to be the collection of $h_{j,\mu}$, $j = -1, 0, 1, 2, \ldots$, $\mu \in \mathbb{Z}$.

Haar system on $[0, 1)$, or $\mathbb{T}$: Take only those Haar functions defined on $[0, 1)$.

Haar system in d dimensions

- Intervals are replaced by cubes. For every dyadic cube we have $2^d - 1$ Haar functions.

$$\text{Let} \quad u^{(0)} = \mathbb{1}_{[0,1)}, \qquad u^{(1)} = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}.$$

For every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{0,1\}^d$ let

$$h^{(\varepsilon)}(x_1, \dots, x_d) = u^{(\varepsilon_1)}(x_1) \cdots u^{(\varepsilon_d)}(x_d).$$

Finally, one sets

$$h^{(\varepsilon)}_{j,\ell}(x) = h^{(\varepsilon)}(2^j x - \ell), \quad j \in \mathbb{Z},\ \ell \in \mathbb{Z}^d,$$

The Haar system $\mathcal{H}_d$ is then given by

$$\mathcal{H}_d = \left\{ h^{(\vec{0})}_{0,\ell} \right\}_{\ell \in \mathbb{Z}^d} \cup \left\{ h^{(\varepsilon)}_{j,\ell} \mid j \in \mathbb{Z},\ \ell \in \mathbb{Z}^d,\ \varepsilon \in \{0,1\}^d \setminus \{\vec{0}\} \right\}.$$

Bases

Def. 1 Given a (quasi-)Banach space $\mathcal{X}$ of tempered distributions in $\mathbb{R}^d$ and an enumeration $\mathcal{U} = (u_1, u_2, \dots)$ of the Haar system $\mathcal{H}_d$, we say that $\mathcal{U}$ is a basic sequence on $\mathcal{X}$ if the orthogonal projections $P_n : \overline{\text{span}(\mathcal{U})} \to \text{span}(\{u_1, \dots, u_n\})$ are uniformly bounded.

Def. 2. If $\mathcal{U}$ is a basic sequence on $\mathcal{X}$ and if span($\mathcal{U}$) is dense in $\mathcal{X}$ then we say that $\mathcal{U}$ is a Schauder basis of $\mathcal{X}$.

Def. 3. $\mathcal{H}_d$ is an unconditional basis of $\mathcal{X}$ if

- The span of $\mathcal{H}_d$ is dense in $\mathcal{X}$
- The orthogonal projections to the spaces generated by (finite) subsets of $\mathcal{H}_d$ are uniformly bounded on $\mathcal{X}$.

Def. 4. $\mathcal{H}_d$ is a local basis ...

The Haar basis in $L^p(\mathbb{R})$

Schauder (28): $\mathcal{H}$ (with the natural lexicographic order) is a *basis* of $L^p([0,1))$ when $1 \leq p < \infty$.

$$f = \mathbb{E}_0 f + \sum_{j=0}^{\infty} \sum_{\mu=0}^{2^j-1} 2^j \langle f, h_{j,\mu} \rangle h_{j,\mu}$$

for $f \in L^p([0,1))$, with convergence in L^p.

• One works with conditional expectional operators $\mathbb{E}_N$ associated to dyadic intervals of length 2^{-N}.

• $\mathbb{E}_{N+1} - \mathbb{E}_N$ is the orthogonal projection to the space generated by the Haar functions with Haar frequency 2^N.

• Billard (1970's): $\mathcal{H}$ is a Schauder basis on the Hardy space $h^1(\mathbb{T})$.

The Haar basis in $L^p(\mathbb{R})$, $1 < p < \infty$

Marcinkiewicz (37): $\mathcal{H}$ is an *unconditional basis* of $L^p(\mathbb{R})$ when $1 < p < \infty$.

• For $f \in L^p$,

$$f = \sum_{j=-1}^{\infty} \sum_{\mu \in \mathbb{Z}} 2^j \langle f, h_{j,\mu} \rangle h_{j,\mu}$$

with *unconditional* convergence in L^p.

• By the UBP all projection operators are L^p bounded with uniform operator norm.

• Each bounded function is a multiplier for the Haar expansions:

$$\Big\| \sum_{j=-1}^{\infty} \sum_{\mu \in \mathbb{Z}} m(j,\mu)\, 2^j \langle f, h_{j,\mu} \rangle h_{j,\mu} \Big\|_p \leq C_p \|m\|_\infty \|f\|_p$$

• Pełczynski (61): L^1 cannot be imbedded in a Banach space with an unconditional basis.

Questions for function spaces measuring smoothness

Consider Triebel-Lizorkin spaces $F^s_{p,q}$, Besov spaces $B^s_{p,q}$.

Q1: For which spaces is $\mathcal{H}_d$ a basic sequence?

Q2: For which spaces is $\mathcal{H}_d$ a Schauder basis?

Q3: For which spaces is $\mathcal{H}_d$ an unconditional basis?

Q4: Haar system on unit cube or on $\mathbb{R}^d$: Does it matter for the outcomes?

- Obvious necessary condition: The Haar functions must belong to the space (mostly $s < 1/p$).
- Other necessary conditions by duality (e.g. mostly $s > -1 + 1/p$ when $1 < p < \infty$).
- Interpolation gives additional restrictions for cases with $p \leq 1$.
- We often disregard the cases $p = \infty$ or $q = \infty$ (Schwartz functions are not dense).

Some references to prior work

• Triebel (73), (78): $\mathcal{H}_d$ is an (unconditional) basis on $B^s_{p,q}$ if

$$\max\{\frac{d}{p} - d,\ \frac{1}{p} - 1\} < s < \min\{\frac{1}{p}, 1\}.$$

Result is sharp up to endpoints. Secondary smoothness parameter q plays no role.

• Many more results on splines, wavelets in Besov spaces (Ciesielski, Figiel, Ropela, Meyer, Sickel, Bourdaud, Oswald).

2010: Triebel's monograph :
$\mathcal{H}_d$ is an unconditional basis on $F^s_{p,q}$ if

$$\max\{\frac{1}{p} - 1,\ \frac{1}{q} - 1,\ \frac{d}{p} - d, \frac{d}{q} - d\} < s < \min\{\frac{1}{p}, \frac{1}{q}, 1\}.$$

Q: Is the additional restriction on q necessary?

A recurring picture

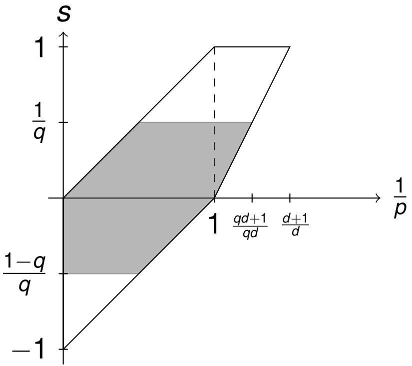


Figure: Parameter domains for the Haar system in $F^s_{p,q}$ spaces on $\mathbb{R}^d$, $1 < q < \infty$, here $q = 2$.

$\mathcal{H}_d$ is unconditional basis for $B^s_{p,q}$: interior of the entire domain.

A digression

Q: Is the additional restriction on q necessary? Yes.

• This is related to results by Christ-S. (PLMS 06) on necessary conditions for vector-valued convolution operators.

Phenomenon valid for many inequalities in function space theory. Most basic one for Peetre's maximal inequality: Let P_k Littlewood-Paley operator for frequencies $\approx 2^k$, or just $\lesssim 2^k$. Then

$$\Big\|\Big\{\sup_h \frac{P_k f_k(\cdot + h)}{(1+2^k|h|)^N}\Big\}\Big\|_{L^p(\ell^q)} \lesssim \big\|\{P_k f_k\}\big\|_{L^p(\ell^q)}$$

holds for $N > \max\{d/p, , d/q\}$.

• Condition on q is needed.

Counterexamples involve many scales.

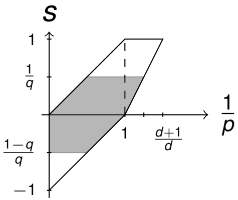
Schauder versus unconditional basis, I

Theorem (SU)

$\mathcal{H}_d$ is an unconditional basis on $F^s_{p,q}$ if and only if

$$\max\left\{\frac{1}{p}-1,\ \frac{1}{q}-1,\ \frac{d}{p}-d,\frac{d}{q}-d\right\} < s < \min\left\{\frac{1}{p},\ \frac{1}{q},\ 1\right\}.$$

Note: [SU] covers $d=1$, $1<p,q<\infty$; other cases in [GSU].



Proof also shows that outside the region of unconditionality we have $F^{s,\mathrm{dyad}}_{p,q} \neq F^s_{p,q}$.

Schauder versus unconditional basis, II

For suitable enumerations $\mathcal{U}$ of $\mathcal{H}_d$ we have

Theorem (GSU)

Let $0 < q < \infty$, *and* $p > \frac{d}{d+1}$. *Assume that*

$$-1 + \frac{1}{p} < s < \frac{1}{p} \quad \text{if } p > 1,$$
$$-d + \frac{d}{p} < s < 1 \quad \text{if } \frac{d}{d+1} < p \leq 1.$$

Then $\mathcal{U}$ *is a Schauder basis of* $F^s_{p,q}$.

- This result refers to *admissible* enumerations $\mathcal{U}$ of $\mathcal{H}_d$.

- Recall: For Besov spaces, Triebel had proved the result with unconditionality, so then admissibility is irrelevant.

Admissible enumerations

Admissibility means roughly: Mimicking the lexicographic order for Haar functions on the unit interval.

In endpoint cases one has to be careful with the definition of admissibility (characteristic function of cubes may not be pointwise multipliers). Here we use:

Def. An enumeration $\mathcal{U} = \{u_1, u_2, ...\}$ of the Haar system is (strongly) *admissible* if the following condition holds for some $b \in \mathbb{N}$. Whenever

u_n, $u_{n'}$ supported in 5-fold dilate of a dyadic unit cube,

$$|\mathrm{supp}\ (u_n)| \geq 2^b |\mathrm{supp}\ u_{n'}|$$

then $n \leq n'$.

Conditional expectations

Work with conditional expectional operators $\mathbb{E}_N$ associated to dyadic intervals of length 2^{-N}.

$\mathbb{E}_{N+1} - \mathbb{E}_N$ is the orthogonal projection to the space generated by the Haar functions with Haar frequency 2^N.
Then the Schauder basis property is deduced from

Theorem. Let $\max\{-d + d/p, \ -1 + 1/p\} < s < \min\{1/p, 1\}$, then

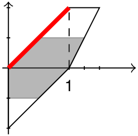
$$\|\mathbb{E}_N f\|_{F^s_{p,q}} \lesssim \|f\|_{F^s_{p,q}} .$$

Better estimate, with $\Pi_N = \Phi_0(2^{-N}D)$, $r > 0$:

$$\|\mathbb{E}_N f - \Pi_N f\|_{B^s_{p,r}} \lesssim \|f\|_{F^s_{p,\infty}} .$$

Approximation of $\mathbb{E}_N$ by Π_N is harder in the endpoint cases.

Endpoint case I: $1 < p < \infty$, $s = 1/p$.

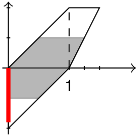


- Haar functions do not belong to $F^{1/p}_{p,q}$, $q \leq \infty$.
- Haar functions do not belong to $B^{1/p}_{p,q}$, $q < \infty$.
- But Haar functions belong to $B^{1/p}_{p,\infty}$.

Proposition

Let $1 < p < \infty$, $s = 1/p$. Then the operators $\mathbb{E}_N$ are uniformly bounded on $B^{1/p}_{p,\infty}$.

Endpoint case II: $p = \infty$, $-1 < s \leq 0$.

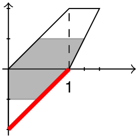


- Separability fails for $p = \infty$.

Proposition

Let $p = \infty$. (i) If $-1 < s < 0$ then the operators $\mathbb{E}_N$ are uniformly bounded on $B^s_{\infty,q}$, $0 < q < \infty$.
(ii) If $-1 < s \leq 0$ then the operators $\mathbb{E}_N$ are uniformly bounded on $B^s_{\infty,\infty}$.
(iii) If $-1 < s \leq 0$ then the operators $\mathbb{E}_N$ are uniformly bounded on $F^s_{\infty,q}$, $0 < q < \infty$.

Endpoint case III: $1 < p < \infty$, $s = 1/p - 1$.



Let $\mathcal{U}$ an admissible enumeration.

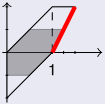
Theorem

Let $1 < p < \infty$, $s = 1/p - 1$.
(i) $\mathcal{U}$ not a basic sequence on $F^s_{p,q}$, $q > 0$, or $B^s_{p,q}$, for any $q > 1$.
(ii) Schauder basis property on $B^s_{p,q}(\mathbb{R}^d)$ fails for some admissible $\mathcal{U}$, any $q > 0$.
(iii) All admissible $\mathcal{U}$ are local Schauder bases on $B^s_{p,q}(\mathbb{R}^d)$ if and only if $0 < q \le 1$.

- In (iii) unconditionality fails.
- In the cases $1 < p, q < \infty$ negative results follow by duality.

Endpoint case IV: $p \le 1$, $s = d/p - d$

Theorem

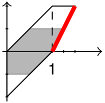


Let $\frac{d}{d+1} < p \le 1$, $s = \frac{d}{p} - d$. *Then*

(i) All admissible $\mathcal{U}$ *are Schauder bases on* $F^s_{p,q}$ *for* $0 < q < \infty$, *and basic sequences on* $F^s_{p,\infty}$.
(ii) All admissible $\mathcal{U}$ *are Schauder bases (basic sequences) on* $B^s_{p,q}$, $s = \frac{d}{p} - d$, *if and only if* $q = p$.
(iii) All admissible $\mathcal{U}$ *are local Schauder bases (basic sequences) on* $B^s_{p,q}$, $s = \frac{d}{p} - d$, *if and only if* $0 < q \le p$, *but uniform boundedness of the* $\mathbb{E}_N$ *fails for* $p < q \le 1$.

- In all cases above $\mathcal{H}_d$ is not an unconditional basis.
- The positive result in (iii), on $[0,1)^d$ was also obtained by Oswald.

Endpoint case IV, cont.



Def. Let Q be a large cube. X function space.

$$Op(T, X, Q) := \sup \left\{ \|Tf\|_X : \|f\|_X \leq 1, \mathrm{supp}\,(f) \subset Q \right\}.$$

Theorem

Let $\frac{d}{d+1} < p \leq 1$, $s = \frac{d}{p} - d$. Then for cubes of side length ≥ 1, and $p < q \leq 1$,

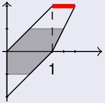
$$Op(\mathbb{E}_N, B^{\frac{d}{p}-d}_{p,q}, Q) \approx (2^{Nd}|Q|)^{1/p-1/q}.$$

Ex.: Let $g_l(x) = 2^{ld}\eta(2^l x)$, $\int \eta = 0$, N large, $\{a_m\} \in \ell^q$. Set $F_N(x) := \sum_{m \in \mathbb{Z}^d} a_m g_{N+m}(x - 2^{-N}m)$.

Endpoint case V: $s = 1$.

Let $\mathcal{U}$ be admissible enumeration of $\mathcal{H}_d$.

Theorem



Let $\frac{d}{d+1} \leq p < 1$.

(i) $\mathbb{E}_N$ are uniformly bounded on $F^1_{p,q}$ if and only if $0 < q \leq 2$.

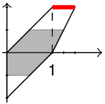
(ii) $\mathbb{E}_N$ are uniformly bounded on $B^1_{p,q}$ if and only if $0 < q \leq p$.

(iii) span $(\mathcal{H}_d)$ is not dense on these spaces.

(iv) All admissible $\mathcal{U}$ are basic sequences in case (i) and local basic sequences in case (ii), global if $p = q$.

• There are Schwartz functions for which $E_N f \not\to f$ in all spaces with $s = 1$. Also obtained by Oswald.

Endpoint case V: $\mathbb{E}_N$ for $s = 1$, cont.



Observe independence of Q, $|Q| > 1$, in:

Theorem

Let $\frac{d}{d+1} \leq p < 1$ *(or* $p = 1, q = \infty$ *in B-case). Then*

$$\text{(i)} \quad Op(\mathbb{E}_N, B^1_{p,q}) \approx N^{1/p-1/q}, \quad p \leq q \leq \infty.$$

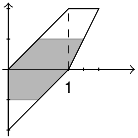
$$\text{(ii)} \quad Op(\mathbb{E}_N, F^1_{p,q}) \approx N^{1/2-1/q}, \quad 2 \leq q \leq \infty.$$

Example for (ii): $f_N(x) = \sum_{N/4<j<N/2} (\pm 1) 2^{-j} e^{2\pi i 2^j x} \psi(x)$ for random choices of sequences of ± 1.

Questions:

- What can we say about density of $\mathcal{H}_d$ in $B^s_{p,q}$ when $q > p$?
- What can we say about density of $\mathcal{H}_d$ in $F^s_{p,q}$ when $q > 2$?

Failure of unconditionality in $F^s_{p,q}(\mathbb{R})$: A multiplier question



$$T_m f := \sum_{j=0} m(j) \sum_{\mu} 2^j \langle f, h_{j,\mu} \rangle h_{j,\mu} = \sum_j m(j) \mathbb{D}_j f$$

where $\mathbb{D}_j = \mathbb{E}_{j+1} - \mathbb{E}_j$.

Note: $\mathcal{H}_1$ unconditional basis $\iff$ every bounded sequence m is a multiplier.

Q: What are the conditions on m that T_m is bounded on $F^s_{p,q}$ for (p^{-1}, s) in the non-shaded regions?

Multiplier question, II

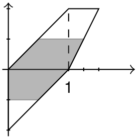
V^u: u-variation space:

$$\|m\|_{V^u} = \sup_N \sup_{j_1 < \cdots < j_N} \Big(\sum_{i=1}^{N-1} |m(j_{i+1}) - m(j_i)|^u \Big)^{1/u}$$

By a summation by parts argument it is easy to see: If the $\mathbb{E}_k$ are bounded on $\mathcal{X}$ then

$$\|T_m\|_{\mathcal{X}} \lesssim \|m\|_{V_1} \|f\|_{\mathcal{X}}.$$

Multiplier question, III



Theorem

Let $1 < p < q < \infty$ *and* $1/q \leq s < 1/p$. *Then*

$$\|T_m f\|_{F^s_{p,q}} \leq C\|m\|_{V_u}\|f\|_{F^s_{p,q}}, \quad 1/u > s - 1/q\,.$$

Essentially sharp up to endpoints: Lower bounds for Haar projection numbers in [SU] give the existence of sets $E \subset 2\mathbb{N}$ depending on s such that $\#E \geq 2^N$, and thus $\|\mathbb{1}_E\|_{V^u} \geq 2^{N/u}$, and such that

$$\|T_{\mathbb{1}_E}\|_{F^s_{p,q} \to F^s_{p,q}} \gtrsim \begin{cases} 2^{N(s-\frac{1}{q})} & \text{if } \frac{1}{q} < s < \frac{1}{p}, \\ N & \text{if } \frac{1}{q} = s < \frac{1}{p}. \end{cases}$$

Thank you for your attention.