# Integral norm discretization and related problems

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## Notations

- $(\Omega, \mu)$: Probability space. For simplicity, assume $\Omega$ is a compact topological space and $\mu$ is Borel.
- $L_q(\Omega)$ or $L_q(d\mu)$: the Lebesgue $L_q$-space on $(\Omega, \mu)$ with norm
$$\|f\|_q := \begin{cases} \left(\int_\Omega |f|^q d\mu\right)^{1/q}, & q < \infty, \\ \text{ess sup}_{x\in\Omega}|f(x)|, & q = \infty. \end{cases}$$
- $X_N$: $N$-dimensional subspace of functions in $L_q$.

## General assumptions

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- We will always assume every function in $X_N$ is defined everywhere on $\Omega$! Indeed, we may first fix a basis $\{\varphi_1, \cdots, \varphi_N\}$ of $X_N$, and assume that each $\varphi_i$ is defined EVERYWHERE on $\Omega$.

## Cubature formulas (C.F.s)

- **Question:** Look for good ways to Approximate an integral $\int_\Omega f(\mathbf{x})d\mu(\mathbf{x})$ via

$$\Lambda_m(f,\boldsymbol{\xi}) := \sum_{j=1}^{m} \lambda_j f(\xi^j), \quad \boldsymbol{\xi} = (\xi^1, \ldots, \xi^m) \in \Omega^m. \tag{1}$$

- $\Lambda_m(\cdot,\boldsymbol{\xi})$ is called a *cubature formula* (C.F.) with nodes $\boldsymbol{\xi} = (\xi^1, \ldots, \xi^m)$ and weights $\Lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$.
- A C.F. is called positive if $\lambda_1, \cdots, \lambda_m \geq 0$.
- Quasi-Monte Carlo C.F.s: $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \frac{1}{m}$.

## Error of CFs

- Typically, one assumes $f \in \mathbf{W} \subset L_1(\Omega) \cap C(\Omega)$ in the approximation

$$\int_\Omega f(\mathbf{x}) d\mu(\mathbf{x}) \approx \Lambda_m(f, \boldsymbol{\xi}) := \sum_{j=1}^{m} \lambda_j f(\xi^j),$$

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$$\Lambda_m(\mathbf{W}, \boldsymbol{\xi}) := \sup_{f \in \mathbf{W}} \left| \int_\Omega f d\mu - \Lambda_m(f, \boldsymbol{\xi}) \right|.$$

- How to estimate $\inf_{\Lambda_m} \Lambda_m(\mathbf{W}, \boldsymbol{\xi})$?

## Tchakaloff's theorem

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  - (I). Reduce the dimension: find a "good" $X_N \subset L_1(\Omega)$ to approximation functions from **W** with $N \approx m$.

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$$\int_\Omega f(x)\, d\mu(x) = \Lambda_m(f) :\approx \sum_{j=1}^{m} \lambda_j f(\xi^j), \quad \forall f \in X_N.$$

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**Tchakaloff's theorem.** If $\Omega$ is sequentially compact, then there exist exactly $N$ pts $\xi^j \in \Omega$ and numbers $\lambda_j \geq 0$, $1 \leq j \leq N$ such that

$$\int_\Omega f(x)\,d\mu(x) = \sum_{j=1}^{N} \lambda_j f(\xi^j), \qquad \forall f \in X_N. \tag{2}$$

## Integral norm discretization on $X_N \subset L_q$, $1 \le q < \infty$

- We say $X_N \in \mathcal{M}^w(m, q)$ if $\exists \, \{\xi^\nu\}_{\nu=1}^m \subset \Omega$ and $\{\lambda_j\}_{j=1}^m \subset \mathbb{R}$ s.t. $\forall f \in X_N$,

$$C_1(q)\|f\|_{L_q(\Omega,\mu)} \le \left(\sum_{\nu=1}^m \lambda_\nu |f(\xi^\nu)|^q\right)^{\frac{1}{q}} \le C_2(q)\|f\|_{L_q(\Omega,\mu)}, \tag{3}$$

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- Write $X_N \in \mathcal{M}(m,q)$ for $X_N \in \mathcal{M}^w(m,q)$ if $\lambda_1 = \cdots = \lambda_m = \frac{1}{m}$ .

## Marcinkiewicz problem with $\varepsilon \in [0, 1)$.

- We write $X_N \in \mathcal{M}^w(m, q, \varepsilon)$ if

$$(1-\varepsilon)\|f\|_q^q \le \sum_{\nu=1}^{m} \lambda_\nu |f(\xi^\nu)|^q \le (1+\varepsilon)\|f\|_q^q, \quad \forall f \in X_N$$

- The notations $\mathcal{M}_+^w(m, q, \varepsilon)$ and $\mathcal{M}(m, q, \varepsilon)$ can be defined similarly.
- Exact discretization $\mathcal{M}^w(m, q, 0)$:

$$\|f\|_q^q = \sum_{\nu=1}^{m} \lambda_\nu |f(\xi^\nu)|^q, \quad \forall f \in X_N.$$

## Question

- For $0 \leq \varepsilon < 1$ and $1 \leq q < \infty$,

$$\mathcal{N}^w(X_N; q, \varepsilon) := \min\Big\{ m \in \mathbb{N} : \quad X_N \in \mathcal{M}^w(m, q, \varepsilon) \Big\}.$$

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- Trivial lower estimate: $\mathcal{N}^w(X_N; q, \varepsilon) \ge N$.
- **Question.** Find conditions on $X_N$ for which $\mathcal{N}(X_N; q, \varepsilon) \le CN \log^\alpha N$ for some $\alpha \ge 0$.

## Exact discretization: $\varepsilon = 0$

**Question.** Find the minimal number $m = \mathcal{N}^w(X_N; q, 0)$ of nodes required for the discretization:

$$\int_\Omega |f(x)|^q \, d\mu(x) = \sum_{j=1}^{m} \lambda_j |f(x_j)|^q, \quad \forall f \in X_N. \tag{4}$$

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- For $q = 2$, setting $Y_N := \text{span}\{f_1 f_2 : \ f_1, f_2 \in X_N\}$, we have

$$(4) \iff \int_\Omega g \, d\mu = \sum_{j=1}^m \lambda_j g(\xi^j), \quad \forall g \in Y_N. \tag{5}$$

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- Note: $\dim Y_N \leq \frac{N(N+1)}{2}$, and by Tchakaloff's theorem,

$$\mathcal{N}_+^w(X_N; 2, 0) \leq \frac{N(N+1)}{2}.$$

## A condition on $m$ for $X_N \in \mathcal{M}^w(m, 2\ell, 0)$.

- A similar argument works equally well for every positive even integer $q = 2\ell$:

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Define

$$\mathcal{M}^w(q) := \sup_{(\Omega, d\mu)} \sup_{X_N \subset L_q(\Omega)} \mathcal{N}^w(X_N; q, 0).$$

Similar definition for $\mathcal{M}_+^w(q)$.

# Optimality of the estimates

- **Theorem.** [DPTT19] For $1 \leq q < \infty$,

$$\mathcal{M}^w(q) = \mathcal{M}_+^w(q) = \begin{cases} \binom{N+q-1}{q}, & \text{if } q \text{ is an even integer,} \\ \infty, & \text{otherwise.} \end{cases}$$

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## Discretization in $L_2$

**Theorem.** ( Rudelson 1999, Temlyakov 2018) Let $X_N$ be an $N$-dimensional real subspace of $L_2(\Omega)$ satisfying

$$\sup_{f\in X_N} \frac{\|f\|_\infty}{\|f\|_2} \le (tN)^{\frac{1}{2}}. \tag{8}$$

Then $\forall \epsilon > 0$, $\exists \{x_1, \cdots, x_m\} \subset \Omega$ with

$$m \le C\frac{t}{\epsilon^2} N \log N$$

such that

$$(1-\epsilon)\|f\|_2^2 \le \frac{1}{m}\sum_{j=1}^m f(x_j)^2 \le (1+\epsilon)\|f\|_2^2, \ \ \forall f \in X_N.$$

## Discretization in $L_1$: conditional result

- Entropy numbers of a compact $A \subset Y$ of $(Y, \|\cdot\|)$:

$$\varepsilon_n(A, Y) := inf\left\{\varepsilon : \exists y_1, \dots, y_{2^n} \in A,\ A \subseteq \cup_{j=1}^{2^n} B(y_j, \varepsilon)\right\},$$

where $B(y, \varepsilon) := \{x \in Y : \ \|x - y\| \leq \varepsilon\}$.

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where $B(y, \varepsilon) := \{x \in Y : \ \ \|x - y\| \le \varepsilon\}$.
- **Theorem.** (Temlyakov 2018) Let $X_N^1 := \{f \in X_N : \|f\|_1 \le 1\}$. If
$$\varepsilon_k(X_N^1, L_\infty) \le KN/k, \quad 1 \le k \le N,$$
then there exist pts $x_1, \cdots, x_m \in \Omega$ with
$$m \le CKN\Big(\log^2 N + \log^2 K\Big)$$
such that
$$\|f\|_1 \sim \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)|, \quad \forall f \in X_N.$$

## More recent progress by Dai, Prymak, Shadrin, Temlyakov, & Tikhonov

**Theorem.** Let $X_N$ be an $N$-dimensional subspace of $L_\infty(\Omega, \mu)$ satisfying

$$\|f\|_\infty \le (KN)^{\frac{1}{2}} \|f\|_2, \quad \forall f \in X_N. \tag{9}$$

Let $1 \le p \le 2$ be fixed and let $N^{-1} < \varepsilon < 1$. Then there exist a constant $C_{\varepsilon,K}$ depending on $K$ and $\varepsilon$ and points $z_1, \cdots, z_m \in \Omega$ with

$$m \le C_{\varepsilon,K} N \log^4 N,$$

such that

$$\sup_{f \in X_N, \|f\|_p \le 1} \left| \|f\|_p - \left( \frac{1}{m} \sum_{j=1}^{m} |f(z_j)|^p \right)^{\frac{1}{p}} \right| \le \varepsilon. \tag{10}$$

## Comments for $p > 2$

- The probabilistic technique used for the case $1 \le p \le 2$ normally does not give good estimates for $L_p$ as $p \to \infty$.
- Reason: Let $x_1, \cdots, x_m \in \Omega$ be the independent uniformly distributed random pts. Given $f \in X_N$, to apply the large deviation inequalities to estimate the probability of

$$\Big| \int_\Omega |f|^p \, d\mu - \frac{1}{m} \sum_{j=1}^m |f(x_j)|^p \Big| > \varepsilon \|f\|_p^p$$

  one need to estimate the following constant for some $\alpha > 0$,

$$L_N := N^{-\alpha} \sup_{f \in X_N} \Big( \frac{\|f\|_\infty}{\|f\|_p} \Big)^p,$$

  which normally goes to $\infty$ exponentially fast as $p \to \infty$.

## Example: Hyperbolic cross polynomials

- Conside the space of hyperbolic polynomials:

$$X := \mathcal{T}(N) := \Big\{ f: \quad f = \sum_{\mathbf{k} \in \Gamma(N)} c_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}, \quad c_k \in \mathbb{R} \Big\}$$

$$\text{with} \quad \Gamma(N) := \Big\{ \mathbf{k} \in \mathbb{Z}^d : \quad \prod_{j=1}^{d} \max\{|k_j|, 1\} \le N \Big\}.$$

- $\dim \mathcal{T}(N) \sim N \log^{d-1} N$.
- Denote by $\mathcal{N}_d(N)$ the minimal number $m$ of pts $\mathbf{x}_1, \cdots, \mathbf{x}_m \in [0, 2\pi)^d)$ required for the inequality

$$\|f\|_\infty \le C(d) \max_{1 \le j \le m} |f(\mathbf{x}_j)|, \quad \forall f \in \mathcal{T}(N).$$

## Historical remarks:

- Trivial estimate:

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- The Kashin -Temlyakov phenomena ( 2003): for $d = 2$, there exist absolute constants $\delta > 0$ and $C_1 > 0$ s.t.

$$\mathcal{N}_2(N) \geq C_1 \Big(\dim \mathcal{T}(N)\Big)^{1+\delta}.$$

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# Recent progress ( [DPTT 2019])

$$\mathcal{N}_d(N) \leq C(d) N^{\alpha_d} (\log N)^{\beta_d},$$

where

$$\alpha_d := \sum_{j=1}^{d} \frac{1}{j} \sim \log d \qquad \text{and} \qquad \beta_d := d - \alpha_d \sim d.$$

## Example: Exact discretization for trigonometric polynomials

For $Q \subset \mathbb{Z}$, define

$$\mathcal{T}(Q) := \left\{ f : \quad f(\theta) := \sum_{k \in Q} c_k e^{ik\theta}, \quad c_k \in \mathbb{C}, \quad k \in Q \right\}.$$

**Theorem.** [DPTT19] Given any $N \in \mathbb{N}$, we have

$$\sup_{\substack{Q \subset \mathbb{Z} \\ |Q|=N}} \min \left\{ m \in \mathbb{N} : \quad \mathcal{T}(Q) \in \mathcal{M}^w(m, 2, 0) \right\} \sim N^2.$$

- The situation is different for the $\mathcal{M}(m, 2)$ problem that is NOT exact (Temlyakov 17 ):

$$\sup_{\substack{Q \subset \mathbb{Z} \\ |Q|=N}} \min \left\{ m \in \mathbb{N} : \quad \mathcal{T}(Q) \in \mathcal{M}(m, 2) \right\} \sim N.$$

# Thank you very much !

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