# PDEs in complex and evolving domains


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## Lecture:- Some geometric notation and concepts

## Tutorial workshop

**Geometry, compatibility and structure preservation in computational differential equations**
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In this lecture we introduce the elementary geometric analysis which is necessary to treat partial differential equations on surfaces. It is our opinion that, in general, numerical methods have to be developed close to the analysis of problems.

- We recall some facts from elementary differential geometry concerning parametrised surfaces.
- We then continue with hypersurfaces in $\mathbb{R}^{n+1}$ and basic analysis concepts on such hypersurfaces .
- We introduce the necessary geometric concepts, for example the notion of curvature.
- The formula for integration by parts is proved and we formulate the coarea formula.
- We introduce global coordinates in a neighbourhood of a hypersurface, the Fermi coordinates.They will be quite useful for the numerical analysis of PDEs on surfaces.
- For theoretical reasons we will introduce the oriented distance function.
- For the treatment of surface PDEs the Poincaré inequality on surfaces is central.

Tangential gradient on the surface $\Gamma$ with normal $\nu$:

$$\nabla_\Gamma f = \nabla f - \nabla f \cdot \nu \, \nu.$$

The tangential gradient only depends on the values of $f$ on the surface.

Laplace-Beltrami-Operator on $\Gamma$:

$$\Delta_\Gamma f = \nabla_\Gamma \cdot \nabla_\Gamma f.$$

Integration by parts on a surface:

$$\int_\Gamma \nabla_\Gamma f = -\int_\Gamma f H \nu + \int_{\partial\Gamma} f \mu.$$

Mean curvature vector satisfies the weak form

$$\int_\Gamma H\nu \cdot \eta = \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \eta$$

where $u(x) = x$.

# Parametrised surface

- Let $n \in \mathbb{N}$. We call $\Gamma \subset \mathbb{R}^{n+1}$ an *$n$-dimensional parametrised $C^k$-surface* ($k \in \mathbb{N} \cup \{\infty\}$), if for every point $x_0 \in \Gamma$ there exists an open set $U \subset \mathbb{R}^{n+1}$ with $x_0 \in U$, an open connected set $V \subset \mathbb{R}^n$ and a map $X : V \to U \cap \Gamma$ with the properties $X \in C^k(V, \mathbb{R}^{n+1})$, $X$ is bijective and rank $\nabla X = n$ on $V$.
- The map $X$ is called a local *parametrization* of $\Gamma$ while $X^{-1}$ is called a local *chart*. A collection $(X_i)_{i \in I}, X_i \in C^k(V_i, \mathbb{R}^{n+1})$ of local parametrizations such that $\cup_{i \in I} X_i(V_i) = \Gamma$ is called a *$C^k$–atlas*. If $X_i(V_i) \cap X_j(V_j) \neq \emptyset$, then the map $X_i^{-1} \circ X_j$ by assumption is a $C^k$-diffeomorphism.
- A function $f : \Gamma \to \mathbb{R}$ is *$k$ times differentiable* if all the functions $f \circ X_i : V_i \to \mathbb{R}$ are $k$ times differentiable. A function $f : \Gamma \to \mathbb{R}$ is $k$ times differentiable if all the functions $f \circ X_i : V_i \to \mathbb{R}$ are $k$ times differentiable.

The linear space

$$T_x\Gamma = \big\{\tau \in \mathbb{R}^{n+1} \,|\, \exists\, \gamma : (-\epsilon,\epsilon) \to \mathbb{R}^{n+1} \text{ differentiable },$$
$$\gamma((-\epsilon,\epsilon)) \subset \Gamma, \gamma(0) = x \text{ and } \gamma'(0) = \tau\big\}$$

is called the *tangent space* to $\Gamma$ at $x \in \Gamma$. The vectors $\{X_{\theta_i}(\theta), i = 1, ..n\}$ form a basis of the tanget space at $T_x\Gamma$ at $x := X(\theta)$. (Exercise.)

- Let $X \in C^2(V, \mathbb{R}^{n+1})$ be a local parametrization of $\Gamma$, $\theta \in V$. We define the *first fundamental form* $G(\theta) = (g_{ij}(\theta))_{ij=1,\ldots,n}$, $\theta \in V$ by

$$g_{ij}(\theta) = \frac{\partial X}{\partial \theta_i}(\theta) \cdot \frac{\partial X}{\partial \theta_j}(\theta), \quad i, j = 1, \ldots, n.$$

- Superscript indices denote the inversion of the matrix $G$ so that $(g^{ij})_{ij=1,\ldots,n} = G^{-1}$, and by $g = \det(G)$ we denote the determinant of the matrix $G$.

- The *Laplace-Beltrami operator* on $\Gamma$ is defined for a twice differentiable function $f : \Gamma \to \mathbb{R}$ as follows. Let $F(\theta) = f(X(\theta))$, $\theta \in V$. Then

$$(\Delta_\Gamma f)(X(\theta)) = \frac{1}{\sqrt{g(\theta)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial \theta_j} \Big( g^{ij}(\theta) \sqrt{g(\theta)} \frac{\partial F}{\partial \theta_i}(\theta) \Big). \tag{0.1}$$

- The *tangential gradient* is given by

$$(\nabla_\Gamma f)(X(\theta)) = \sum_{i,j=1}^{n} g^{ij}(\theta) \frac{\partial F}{\partial \theta_j}(\theta) \frac{\partial X}{\partial \theta_i}(\theta). \tag{0.2}$$

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and let $X \in C^2(\bar{\Omega}, \mathbb{R}^{n+1})$ and $X(\theta) := (\theta, h(\theta)), \theta \in \bar{\Omega}$ be a parametrization of $\Gamma$ where $h \in C^2(\bar{\Omega})$.
Find formulae for

- the first fundamental form
- the tangential gradient
- the Laplace-Beltrami operator.

**Definition**

Let $k \in \mathbb{N} \cup \{\infty\}$. $\Gamma \subset \mathbb{R}^{n+1}$ is called a $C^k$–hypersurface, if for each point $x_0 \in \Gamma$ there exists an open set $U \subset \mathbb{R}^{n+1}$ containing $x_0$ and a function $\phi \in C^k(U)$ with the property that $\nabla\phi \neq 0$ on $\Gamma \cap U$ and such that

$$U \cap \Gamma = \{x \in U \,|\, \phi(x) = 0\}. \tag{0.3}$$

- The linear space

$$T_x\Gamma = \big\{\tau \in \mathbb{R}^{n+1} \,|\, \exists\, \gamma : (-\epsilon, \epsilon) \to \mathbb{R}^{n+1} \text{ differentiable },$$
$$\gamma((-\epsilon, \epsilon)) \subset \Gamma, \gamma(0) = x \text{ and } \gamma'(0) = \tau\big\}$$

  is called the *tangent space* to $\Gamma$ at $x \in \Gamma$.
- Exercise It is easy to show that $T_x\Gamma = [\nabla\phi(x)]^\perp$, the set of all vectors that are orthogonal to $\nabla\phi(x)$, where $\phi$ is as in (0.3). In particular, $T_x\Gamma$ is an $n$–dimensional subspace of $\mathbb{R}^{n+1}$.

A vector $\nu(x) \in \mathbb{R}^{n+1}$ is called a *unit normal vector* at $x \in \Gamma$ if $\nu(x) \perp T_x\Gamma$ and $|\nu(x)| = 1$. In view of the above characterisation of $T_x\Gamma$ we then have

$$\nu(x) = \frac{\nabla\phi(x)}{|\nabla\phi(x)|} \quad \text{or} \quad \nu(x) = -\frac{\nabla\phi(x)}{|\nabla\phi(x)|}. \tag{0.4}$$

A $C^1$–hypersurface is called *orientable* if there exists a continuous vector field $\nu : \Gamma \to \mathbb{R}^{n+1}$ such that $\nu(x)$ is a unit normal vector to $\Gamma$ for all $x \in \Gamma$.

An example of a moving hypersurface $\Gamma(t)$ with boundary $\partial\Gamma(t)$ is given by

$$\begin{aligned}
\tilde{\Gamma}(t) = \quad &= \quad \{x \in \mathbb{R}^{n+1} : \phi(x,t) = 0\} && (0.5)\\
\Gamma(t) \quad &= \quad \{x \in \tilde{\Gamma}(t) : \psi(x,t) < 0\} && (0.6)\\
\partial\Gamma(t) \quad &= \quad \{x \in \tilde{\Gamma}(t) : \psi(x,t) = 0\} && (0.7)\\
\nu \quad &= \quad \nabla\phi/|\nabla\phi|, && (0.8)\\
\nu_{\partial\Gamma} \quad &= \quad \nabla_\Gamma\psi/|\nabla_\Gamma\psi| && (0.9)
\end{aligned}$$

where $\phi : \mathbb{R}^{n+1} \times [0,T], \nabla\phi \neq 0$ on $\tilde{\Gamma}(t)$ and $\psi : \mathbb{R}^{n+1} \times [0,T], \nabla_\Gamma\psi \neq 0$ on $\partial\Gamma(t)$. The conormal velocity of $\Gamma(t)$ is

$$V_{\partial\Gamma} = -\psi_t/|\nabla_\Gamma\psi|.$$

An example of a moving hypersurface without boundary is given by

$$\Gamma(t) = \{x \in \mathbb{R}^{n+1} : \phi(x,t) = 0\} \quad (0.10)$$

$$\nu = \nabla\phi/|(\nabla\phi| \quad (0.11)$$

where $\phi : \mathbb{R}^{n+1} \times [0,T], \nabla\phi(x,t) \neq 0$ on $\Gamma(t)$.
If $\nu$ varies in space then the hypersurface is *flat* otherwise it is *curved*.
An example of a flat hypersurface is

$$\Gamma = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}.$$

The normal velocity of $\Gamma(t)$ is

$$V = -\phi_t/|\nabla\phi|.$$

An example of a flat moving hypersurface with boundary is given by

$$\Gamma(t) = \{x = (y, x_{n+1}) : x_{n+1} = 0, \phi(y,t) < 0\} \tag{0.12}$$

$$\partial\Gamma(t) = \{x = (y, x_{n+1}) : x_{n+1} = 0, \phi(y,t) = 0\} \tag{0.13}$$

$$\nu = e_{n+1}, \quad \nu_{\partial\Gamma} = (\nabla_y, 0)\phi / |(\nabla_y \phi|, 0)| \tag{0.14}$$

where $\phi : \mathbb{R}^n \times [0,T], \phi = \phi(y,t), y \in \mathbb{R}^{n,} |\nabla_y \phi(y,t)| \neq 0$ on $\partial\Gamma(t)$. The normal velocity of $\Gamma(t)$ is zero but $\partial\Gamma(t)$ has the outer conormal velocity

$$V_{\partial\Gamma} = -\phi_t / |\nabla_y \phi(y,t)|.$$

The connection between parametrised surfacesand hypersurfaces is given by the following lemma.

Lemma

*Assume that* $\Gamma$ *is a* $C^k$*-hypersurface in* $\mathbb{R}^{n+1}$*. Then for every* $x \in \Gamma$ *there exists an open set* $U \subset \mathbb{R}^{n+1}$ *with* $x \in U$ *and a parametrised* $C^k$*-surface* $X : V \to U \cap \Gamma$ *such that* $X$ *is a bijective map from* $V$ *onto* $U \cap \Gamma$*. If* $X : V \to U \cap \Gamma$ *is a parametrised* $C^k$*-surface and* $\theta \in V$*, then there is an open set* $\tilde{V} \subset V$ *with* $\theta \in \tilde{V}$ *such that* $X(\tilde{V})$ *is a* $C^k$*-hypersurface.*

This means that locally we always can work with hypersurfaces. And we may use all the previous definitions for hypersurfaces.

# Tangential gradient and Laplace Beltrami

Definition

- Let $\Gamma \subset \mathbb{R}^{n+1}$ be a $C^1$–hypersurface and $f : \Gamma \to \mathbb{R}$ be differentiable at $x \in \Gamma$. We define the tangential gradient of $f$ at $x \in \Gamma$ by

$$\nabla_\Gamma f(x) = \nabla \overline{f}(x) - \nabla \overline{f}(x) \cdot \nu(x)\, \nu(x) = P(x) \nabla \overline{f}(x)$$

  where $P(x)_{ij} = \delta_{ij} - \nu_i(x)\nu_j(x)$ $(i, j = 1, \ldots, n+1)$.
- Here $\overline{f}$ is a smooth extension of $f : \Gamma \to \mathbb{R}$ to an $n+1$-dimensional neighbourhood $U$ of the surface $\Gamma$, so that $\overline{f}\big|_\Gamma = f$. $\nabla$ denotes the gradient in $\mathbb{R}^{n+1}$ and $\nu(x)$ is a unit normal at $x$.
- The *Laplace-Beltrami operator* applied to a twice differentiable function $f \in C^2(\Gamma)$ is given by

$$\Delta_\Gamma f = \nabla_\Gamma \cdot \nabla_\Gamma f = \sum_{i=1}^{n+1} \underline{D}_i \underline{D}_i f. \tag{0.15}$$

We shall use the notation, (as in the above definition),

$$\nabla_\Gamma f(x) = (\underline{D}_1 f(x), \ldots, \underline{D}_{n+1} f(x))$$

for the $n+1$ components of the tangential gradient. Note that $\nabla_\Gamma f(x) \cdot \nu(x) = 0$ and hence $\nabla_\Gamma f(x) \in T_x\Gamma$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and let $\Gamma \subset \mathbb{R}^{n+1}$ be the zero level set of the level set function

$$\phi(x') := x_{n+1} - h(x),\ x \in \Omega$$

where $x' := (x, x_{n+1}) \in \mathbb{R}^{n+1}$ and $h \in C^2(\bar{\Omega})$. That is $\Gamma$ is defined as a graph.
Find formulae for

- the first fundamental form
- the tangential gradient
- the Laplace-Beltrami operator
- the unit normal
- the projection matrix $P$
- the Weingarten map
- the mean curvature

Let us show, that (0.1) and (0.2) are equivalent to the settings in Definition 3. Since the tangential gradient is a tangent vector, $\nabla_\Gamma f \circ X = \sum_{i=1}^n \alpha_i X_{\theta_i}$ with certain scalars $\alpha_j$. We solve this equation for $\alpha_1, \ldots, \alpha_n$ by multiplying it by $X_{\theta_k}$ to get

$$F_{\theta_k} = \nabla_\Gamma f \circ X \cdot X_{\theta_k} = \sum_{i=1}^n \alpha_i X_{\theta_i} \cdot X_{\theta_k} = \sum_{i=1}^n \alpha_i g_{ik}. \tag{0.16}$$

For the first equality on the left we have used that by the chain rule we have from $F(\theta) = f(X(\theta))$ that $F_{\theta_k} = \sum_{l=1}^n \underline{D}_l f \circ X X_{l\theta_k}$ since $X_{\theta_k}$ is a tangent vector. From (0.16) we infer that $\alpha_l = \sum_{k=1}^n F_{\theta_k} g^{kl}$, and this finally gives (0.2). Now it is easy to derive (0.1).

- $\nabla_\Gamma f(x)$ only depends on the values of $f$ on $\Gamma \cap U$, where $U \subset \mathbb{R}^{n+1}$ is a neighbourhood of $x$.
  *Proof*
  It is sufficient to show that $f \equiv 0$ on $\Gamma \cap U$ implies that $\nabla_\Gamma f(x) = 0$. Choose $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ such that $\gamma(0) = x, \gamma((-\epsilon, \epsilon)) \subset \Gamma \cap U$ and $\gamma'(0) = \nabla_\Gamma f(x)$. Since $\tilde{f}(\gamma(t)) = f(\gamma(t)) = 0$ for all $|t| < \epsilon$ we have

  $$0 = \nabla \tilde{f}(x) \cdot \gamma'(0) = \big(\nabla_\Gamma f(x) + \nabla \tilde{f}(x) \cdot \nu(x)\, \nu(x)\big) \cdot \nabla_\Gamma f(x) = |\nabla_\Gamma f(x)|^2,$$

  which implies the result.
- We denote by $C^1(\Gamma)$ the set of functions $f : \Gamma \to \mathbb{R}$, which are differentiable at every point $x \in \Gamma$ and for which $\underline{D}_j f : \Gamma \to \mathbb{R}, j = 1, \ldots, n+1$ are continuous. Similarly one can define $C^l(\Gamma)\,(l \in \mathbb{N})$ provided that $\Gamma$ is a $C^k$–hypersurface with $k \geq l$.

**Definition**

For $\Gamma \in C^2$ we define

$$\mathcal{H}_{ij} = \underline{D}_i \nu_j \qquad (i, j = 1, \ldots, n+1). \tag{0.17}$$

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- It is easily shown that the matrix $\mathcal{H}$ is symmetric and that it possesses an eigenvalue $0$ in the normal direction: $\mathcal{H}\nu = 0$. (Exercise)
- $\mathcal{H}$ is called the *extended Weingarten map*. The restriction of $\mathcal{H}$ to the tangent space is called the Weingarten map.
- For $x \in \Gamma$ the quantity

$$H(x) = \text{trace}\mathcal{H}(x) = \sum_{i=1}^{n+1} \mathcal{H}_{ii}(x) \tag{0.18}$$

is the *mean curvature* of $\Gamma$ at the point $x$. It differs from the common definition by a factor $n$. We note that the eigenvalues $\kappa_1, \ldots, \kappa_n$ of $\mathcal{H}$ (apart from the trivial eigenvalue $0$ in $\nu$-direction) are the *principal curvatures* of $\Gamma$.

## Exercise

- Let us have a look at the most simple example. The sphere of radius $R > 0$, $\Gamma = \{x \in \mathbb{R}^{n+1} | \, |x| = R\}$, is given by the level set function $\phi(x) = |x| - R$ for $0 < |x| < \infty$. We may choose $\nu = \frac{\nabla \phi}{|\nabla \phi|} = \frac{x}{|x|}$ and get for $x \in \Gamma$

$$\mathcal{H}_{ij}(x) = \underline{D}_i \nu_j(x) = \underline{D}_i \frac{x_j}{|x|} = \frac{1}{R} \underline{D}_i x_j = \frac{1}{R}\left(\delta_{ij} - \nu_i \nu_j\right) = \frac{1}{R}\left(\delta_{ij} - \frac{x_i x_j}{R^2}\right).$$

  This matrix has an eigenvalue $0$ with eigenvector $\frac{x}{R}$ and $n$ eigenvalues $\kappa_j = \frac{1}{R}$ $(j = 1, \ldots, n)$. The mean curvature of $\Gamma$ is then given as $H = \frac{n}{R}$.

- The following result on the exchange of tangential derivatives is easily proved.

Lemma

*For $\Gamma \in C^2$ and $u \in C^2(\Gamma)$ we have*

$$\underline{D}_i \underline{D}_j u - \underline{D}_j \underline{D}_i u = \left(\mathcal{H} \nabla_\Gamma u\right)_j \nu_i - \left(\mathcal{H} \nabla_\Gamma u\right)_i \nu_j. \tag{0.19}$$

*for $i, j = 1, \ldots, n+1$.*

It is quite convenient to use global coordinates in a neighbourhood of a hypersurface, the so called Fermi coordinates. This avoids working with charts and atlases when proving results and carrying out the numerical analysis. For this one introduces the *oriented distance function* for $\Gamma$.

**Remark** In the context of surface finite elements we will use the oriented distance function only for our analysis and numerical analysis. We will not use it in defining the computational methods. We will not need the oriented distance function for the implementation of our algorithms. It may be of use in *implicit surface* methods

Assume in the following that $G \subset \mathbb{R}^{n+1}$ is bounded and open with exterior normal $\nu$ and assume that $\Gamma = \partial G$ is a $C^k$-hypersurface ($k \geq 2$). The oriented distance function for $\Gamma$ is defined by

$$d(x) = \begin{cases} \inf_{y\in\Gamma}|x-y|, & x \in \mathbb{R}^{n+1} \setminus \bar{G} \\ -\inf_{y\in\Gamma}|x-y|, & x \in G. \end{cases}$$

One easily verifies that $d$ is globally Lipschitz–continuous with Lipschitz constant $1$.

Since $\partial G$ is a $C^2$–hypersurface, it satisfies both a uniform interior and a uniform exterior sphere condition, which means that for each point $x_0 \in \partial\Omega$ there are balls $B$ and $B'$ such that

$$\bar{B} \cap (\mathbb{R}^{n+1} \setminus \Omega) = \{x_0\}, \quad \bar{B'} \cap \bar{\Omega} = \{x_0\}$$

and the radii of $B$, $B'$ are bounded from below by a positive constant $\delta$ uniformly in $x_0$. With this observation the following Lemma is easily proved.

Lemma

*We define*

$$U_\delta = \left\{x \in \mathbb{R}^{n+1} \,|\, |d(x)| < \delta\right\}.$$

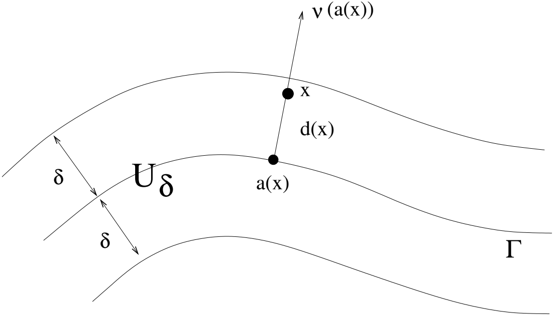
*Then* $d \in C^k(U_\delta)$ *and for every point* $x \in U_\delta$ *there exists a unique point* $a(x) \in \Gamma$*, such that*

$$x = a(x) + d(x)\nu(a(x)). \tag{0.20}$$

*Additionally we have that*

$$\nabla d(x) = \nu(a(x)), \quad |\nabla d(x)| = 1 \qquad \textit{for } x \in U_\delta.$$

We also extend the normal constantly in normal direction: $\nu(x) = \nu(a(x))$ for $x \in U_\delta$.




Strip $U_\delta$ around the hypersurface $\Gamma$ and normal coordinates $x = a(x) + d(x)\nu(a(x))$.

Thus we have introduced a global coordinate system around $\Gamma$. Every point $x \in U_\delta$ can be described by its *Fermi coordinates* (normal coordinates) $d(x)$ and $a(x)$ according to (0.20).

The introduction of global coordinates allows to work with the well known coarea formula.

Theorem

*Let* $\Gamma(r) = \{x \in \mathbb{R}^{n+1} | \, d(x) = r\}$ *be the parallel surface to* $\Gamma = \Gamma(0)$ *for* $|r| < \delta$*. Then*

$$\int_{U_\varepsilon} f(x)\,dx = \int_{-\varepsilon}^{\varepsilon} \int_{\Gamma(r)} f(x)\,dA(x)dr \tag{0.21}$$

*for* $f \in C^0(U_\delta)$ *and* $0 < \varepsilon < \delta$*.*

We note that this formula changes to

$$\int_{U_\varepsilon} f(x)\,dx = \int_{-\varepsilon}^{\varepsilon} \int_{\Gamma(r)} f(x)|\nabla\phi(x)|\,dA(x)dr \tag{0.22}$$

if the surfaces are given by an arbitrary level set function $\phi$ as in (0.3),
$\Gamma(r) = \{x \in \mathbb{R}^{n+1} | \, \phi(x) = r\}$, and the strip around $\Gamma$ is taken to be
$U_\delta = \{x \in \mathbb{R}^{n+1} | \, |\phi(x)| < \delta\}$. In this case one does not work with parallel surfaces to $\Gamma$.
Exercise Write this when the $\Gamma(r)$ are spheres of radius $r$.

With the coarea formula one can prove the formula for *integration by parts* on surfaces $\Gamma$.

Theorem

*Assume that* $\Gamma$ *is a hypersurface in* $\mathbb{R}^{n+1}$ *with smooth boundary* $\partial\Gamma$ *and that* $f \in C^1(\overline{\Gamma})$. *Then*

$$\int_\Gamma \nabla_\Gamma f \, dA = \int_\Gamma f H \nu \, dA + \int_{\partial\Gamma} f \mu \, dA. \qquad (0.23)$$

*Here,* $\mu$ *denotes the conormal vector which is normal to* $\partial\Gamma$ *and tangent to* $\Gamma$. *A compact hypersurface* $\Gamma$ *does not have a boundary,* $\partial\Gamma = \emptyset$, *and the last term on the right hand side vanishes.*

Note that in (0.23) $dA$ in connection with an integral over $\Gamma$ denotes the $n$-dimensional surface measure, while $dA$ in connection with an integral over $\partial\Gamma$ is the $n-1$-dimensional surface measure.

Exercise Formulate and prove this when $\Gamma$ is a graph over a flat domain $\Omega$ using integration by parts on $\Omega$.

The formula for integration by parts on surfaces directly implies Green's formula. One has from Theorem 8, using the summation convention that we sum over doubly appearing indices,

$$\begin{aligned}\int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma g \, dA &= \int_\Gamma \underline{D}_i f \underline{D}_i g \, dA = \int_\Gamma \underline{D}_i \left( f \underline{D}_i g \right) \, dA - \int_\Gamma f \underline{D}_i \underline{D}_i g \, dA \\ &= \int_\Gamma f \underline{D}_i g H \nu_i \, dA + \int_\Gamma f \underline{D}_i g \mu_i \, dA - \int_\Gamma f \Delta_\Gamma g \, dA.\end{aligned}$$

Since $\underline{D}_i g \nu_i = \nabla_\Gamma g \cdot \nu = 0$ we have the following Theorem.

Theorem

$$\int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma g \, dA = -\int_\Gamma f \Delta_\Gamma g \, dA + \int_\Gamma f \nabla_\Gamma g \cdot \mu \, dA \tag{0.24}$$

Let $\Gamma \in C^2$ for the following.

- For $p \in [1, \infty]$ we denote by $L^p(\Gamma)$ the space of functions $f : \Gamma \to \mathbb{R}$ which are measurable with repsect to the surface measure $dA$ (the $n$-dimensional Hausdorff measure) and have finite norm where

$$\|f\|_{L^p(\Gamma)} = \left( \int_\Gamma |f|^p \, dA \right)^{\frac{1}{p}}$$

  for $p < \infty$ and for $p = \infty$ we mean the essential supremum norm.

- $L^p(\Gamma)$ is a Banach space and for $p = 2$ a Hilbert space. For $1 \leq p < \infty$ the spaces $C^0(\Gamma)$ and $C^1(\Gamma)$ are dense in $L^p(\Gamma)$.

The formula for integration by parts on $\Gamma$ leads to the notion of a weak derivative and to the concept of Sobolev spaces on surfaces. Sobolev spaces are the natural spaces for solutions of elliptic partial differential equations.

Definition

A function $f \in L^1(\Gamma)$ has the weak derivative $v_i = \underline{D}_i f \in L^1(\Gamma)$ $(i \in \{1, \ldots, n+1\})$, if for every function $\varphi \in C^1(\Gamma)$ with compact support $\overline{\{x \in \Gamma | \, \varphi(x) \neq 0\}} \subset \Gamma$ we have the relation

$$\int_\Gamma f \underline{D}_i \varphi \, dA = -\int_\Gamma \varphi v_i \, dA + \int_\Gamma f \varphi H \nu_i \, dA$$

Definition

The Sobolev space $H^{1,p}(\Gamma)$ is defined by

$$H^{1,p}(\Gamma) = \{f \in L^p(\Gamma) |\, \underline{D}_i f \in L^p(\Gamma), i = 1, \dots, n+1\}$$

with norm

$$\|f\|_{H^{1,p}(\Gamma)} = \left(\|f\|_{L^p(\Gamma)} + \|\nabla_\Gamma f\|_{L^p(\Gamma)}\right)^{\frac{1}{p}}.$$

For $k \in \mathbb{N}$ we define

$$H^{k,p}(\Gamma) = \left\{f \in H^{k-1,p}(\Gamma) |\, \underline{D}_i v^{(k-1)} \in L^p(\Gamma), i = 1, \dots, n+1\right\}$$

where $H^{0,p}(\Gamma) = L^p(\Gamma)$. For $p = 2$ we use the notation $H^k(\Gamma) = H^{k,2}(\Gamma)$. If we denote by $v^{(l)}$ *all* weak derivatives of order $l$, then

$$\|v\|_{H^{k,p}(\Gamma)} = \left(\sum_{l=0}^{k} \|v^{(l)}\|^p_{L^p(\Gamma)}\right)^{\frac{1}{p}}.$$

Note that for the previous definition we only have assumed that $\Gamma \in C^2$. This was done because in the formulation of the weak derivative we used the mean curvature of $\Gamma$.

Poincaré's inequality for a function with mean value zero on a compact $n$-dimensional hypersurface can be deduced from the Poincaré inequality in $\mathbb{R}^{n+1}$ with the use of global coordinates.

Theorem

*Assume that $\Gamma \in C^3$ and $1 \leq p < \infty$. Then there is a constant $c$ such that for every function $u \in H^{1,p}(\Gamma)$ with $\int_\Gamma u \, dA = 0$ one has the inequality*

$$\|u\|_{L^p(\Gamma)} \leq c \|\nabla_\Gamma u\|_{L^p(\Gamma)}. \tag{0.25}$$