PDEs in complex and evolving domains

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Lecture:-Surface elliptic equations

Tutorial workshop

Geometry, compatibility and structure preservation in computational differential equations
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Content

- Poisson equation
- Wellposdness for Poisson equation
- Regularity
- General second order elliptic equations
- Some fourth order surface equations
- A class of saddle point problems

The Poisson equation

- We assume that Γ is a compact and connected C^2 hypersurface. (Hence Γ is without boundary.)
- We begin with the model case of the Poisson equation

$$-\Delta_\Gamma u = f \tag{0.1}$$

  on a compact hypersurface Γ in $\mathbb{R}^{n+1}$.
- Here f is a given right hand side or source term which is taken to be from $L^2(\Gamma)$ or more generally from $H^{-1}(\Gamma)$.
- Integrating the equation over Γ yields a necessary condition for existence, $\int_\Gamma f = 0$.
- Clearly adding a constant to a possible solution yields another solution so we need to fix a condition on u for uniqueness. This is analogous to the pure Neumann problem for the Laplace operator.

Poisson equation

- A *weak solution* of (0.1) is a function $u \in H^1(\Gamma)$ which satisfies the relation

$$\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi \, dA = \int_\Gamma f \varphi \, dA \tag{0.2}$$

  for every test function $\varphi \in H^1(\Gamma)$.
- Since $\varphi = 1$ is allowed as a test function, again we see that we have to impose the condition $\int_\Gamma f = 0$ on the right hand side.
- If the right hand side f is a functional from $H^{-1}(\Gamma)$ only then the weak form of the equation reads

$$\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi \, dA = \langle \varphi, f \rangle$$

  where the brackets stand for the evaluation of the functional f at the function φ.
- Obviously there is no uniqueness of weak solutions in this case, since every constant is a solution. We will fix the free constant by choosing the mean value of u to vanish.

Existence for Poisson equation

The following theorem can easily be proved.

Theorem

Let $\Gamma \in C^2$ *be a compact hypersurface in* $\mathbb{R}^{n+1}$ *and assume that* $f \in H^{-1}(\Gamma)$ *with the property* $\langle 1, f \rangle = 0$. *Then there exists a unique solution* $u \in H^1(\Gamma)$ *of (0.2) with* $\int_\Gamma u \, dA = 0$.

Exercise

The *proof* is an application of the Lax-Milgram theorem or of the Riesz representation theorem. The bilinear form

$$a(u, v) = \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v \, dA$$

is a scalar product on the Hilbert space $X = \{u \in H^1(\Gamma) | \int_\Gamma u \, dA = 0\}$ because of Poincaré's inequality from the last lecture. The right hand side f was chosen to be in the space $H^{-1}(\Gamma)$ of linear functionals.

Besides the existence of weak solutions the most important ingredient for suitable numerics is the proof of regularity and of a priori estimates for solutions of the Poisson equation. We shortly show how one proves an a priori estimate in the $H^2(\Gamma)$ norm. For this we use the following little lemma.

Lemma

Let $\Gamma \in C^2$ *and* $u \in H^2(\Gamma)$. *Then*

$$|u|_{H^2(\Gamma)} \leq \|\Delta_\Gamma u\|_{L^2(\Gamma)} + c|u|_{H^1(\Gamma)} \tag{0.3}$$

with the constant $c = \sqrt{\|H\mathcal{H}\|_{L^\infty(\Gamma)}}$.

Regularity theorem

With the help of the previous Lemma and standard arguments we arrive at the regularity estimate for the solution of the Poisson equation on a compact hypersurface.

Theorem

Assume that $\Gamma \in C^2$ *and that* $f \in L^2(\Gamma)$ *with* $\int_\Gamma f\, dA = 0$. *Then the weak solution from Theorem 1 satisfies* $u \in H^2(\Gamma)$ *and*

$$\|u\|_{H^2(\Gamma)} \leq c\|f\|_{L^2(\Gamma)}.$$

For the *proof* we use the basic estimate (choose $\varphi = u$ in (0.2))

$$|u|_{H^1(\Gamma)} \leq c\|f\|_{L^2(\Gamma)}$$

– for which we use Poincaré's inequality again – together with the PDE pointwise almost everywhere to obtain

$$|u|_{H^2(\Gamma)} \leq c\|f\|_{L^2(\Gamma)},$$

if the solution has square integrable second derivatives.

In the previous section we have shown how the Poisson equation is solved on a compact surface. The methods are easily extended to general linear elliptic PDEs in divergence form and to boundary value problems (on surfaces with a boundary).

$$-\sum_{i,j=1}^{n+1} \underline{D}_i \left(a_{ij} \underline{D}_j u\right) - \sum_{i=1}^{n+1} \underline{D}_i \left(a_i u\right) + \sum_{i=1}^{n+1} b_i \underline{D}_i u + cu = f - \sum_{i=1}^{n+1} \underline{D}_i g_i. \tag{0.4}$$

We assume for the given coefficients that

$$a_{ij}, a_i, b_i, c \in L^\infty(\Gamma), g_i \in L^2(\Gamma) \quad (i, j = 1, \ldots, n+1).$$

We also assume that the coefficient vectors $a(x) = (a_1(x), \ldots, a_{n+1}(x))$ and $g(x) = (g_1(x), \ldots, g_{n+1}(x))$ are tangent vectors at $x \in \Gamma$, i. e. lie in $T_x\Gamma$, and that the matrix $\mathcal{A}(x) = (a_{ij}(x))_{i,j=1,\ldots,n+1}$ is symmetric and maps the tangent space $T_x\Gamma$ into itself. We emphasize that the latter condition implies that in general constant coefficients a_{ij} are not admissible. They have to depend on the x-variable. Nevertheless $a_{ij} = \delta_{ij}$ obviously is allowed.

As ellipticity condition we assume a so called Ladyzenskaja condition, which says that there exists a number $c_0 > 0$, such that

$$\sum_{i,j=1}^{n+1} a_{ij}\xi_i\xi_j + \sum_{i=1}^{n+1} a_i\xi_0\xi_i + \sum_{i=1}^{n+1} b_i\xi_i\xi_0 + c\xi_0^2 \geq c_0 \sum_{i=1}^{n+1} \xi_i^2$$

almost everywhere on Γ for all $\xi = (\xi_1, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+1}$ with $\xi \cdot \nu = 0$ and all $\xi_0 \in \mathbb{R}$. With the PDE (0.4) we associate the bilinear form a,

$$a(u, \varphi) = \int_\Gamma \sum_{i,j=1}^{n+1} a_{ij}\underline{D}_j u \underline{D}_i \varphi + \sum_{i=1}^{n+1} a_i \underline{D}_i \varphi + \sum_{i=1}^{n+1} b_i \underline{D}_i u\varphi + cu\varphi \, dA,$$

and the functional F,

$$\langle \varphi, F \rangle = \int_\Gamma f\varphi + \sum_{i=1}^{n+1} g_i \underline{D}_i \varphi \, dA,$$

for $u, \varphi \in H^1(\Gamma)$, for which we assume $\langle F, 1 \rangle = 0$.

Theorem

Let $\Gamma \in C^2$ *be a compact hypersurface. Assume that the coefficients satisfy the above conditions. Then there exists a unique weak solution of (0.4) with* $\int_\Gamma u\, dA = 0$*, i. e. there exists a unique* $u \in H^1(\Gamma)$ *such that*

$$a(u, \varphi) = \langle \varphi, F \rangle$$

for every $\varphi \in H^1(\Gamma)$*.*

Proof Exercise.