Quantitative results on continuity of the spectral factorization mapping

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"... the theory of computations is just as inconceivable without Banach spaces as without computers"

S.L. Sobolev

Notation

$\mathbb{T}$ – unit circle, $\mathbb{D}$ – unit disk:

$$\mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}, \quad \mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}.$$

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Hardy spaces:

$$H^p(\mathbb{T}) := \{f \in L^p(\mathbb{T}) \big| \ f_n = 0 \ \text{ for } n < 0\}, \quad 1 \le p \le \infty,$$

where f_n is the n-th Fourier coefficient of f;

$$H^p(\mathbb{D}) := \{f \ \text{ analytic in } \ \mathbb{D} \big| \ \|f\|_{H^p(\mathbb{D})} := \sup_{0<r<1} \|f(r\cdot)\|_{L^p(\mathbb{T})} < \infty\},$$
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$$H^p(\mathbb{T}) \rightleftarrows \text{ boundary values of functions in } H^p(\mathbb{D}).$$

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$$F(z) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |f(e^{it})| \, dt \right) \quad \text{(outer function)};$$

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$$S(z) = \exp\left(- \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t) \right) \qquad \text{(singular inner function)},$$

μ is a bounded nondecreasing singular function: $\mu'(t) = 0$ a.e..

Note that

$$f \in H^p(\mathbb{D}) \setminus \{0\} \implies \log|f| \in L^1(\mathbb{T}).$$

Let $\mathcal{S}_N(\mathbb{T})$ denote the class of matrix functions F such that $F \in \left(L^1(\mathbb{T})\right)^{N\times N}$ is a.e. positive definite and

$$\log\det F \in L^1(\mathbb{T}).$$

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If $F \in \mathcal{S}_N(\mathbb{T})$, then it admits a unique spectral factorisation:

$$F(\zeta) = F^+(\zeta)\left(F^+(\zeta)\right)^*, \quad \zeta \in \mathbb{T},$$

where $F^+ \in \left(H^2(\mathbb{T})\right)^{N\times N}$ is an outer matrix function and $F^+(0)$ is positive definite (N. Wiener, 1955; H. Helson and D. Lowdenslager, 1958).

A matrix function $A \in \left(H^2(\mathbb{T})\right)^{N\times N}$ is called outer if $\det A \in H^{2/N}(\mathbb{T})$ is an outer function or, equivalently, if the set of products $A\mathcal{P}^+(\mathbb{C}^N)$ is dense in $\left(H^2(\mathbb{T})\right)^N$.

Here $\mathcal{P}^+(\mathbb{C}^N)$ is the set of $\mathbb{C}^N$–valued analytic polynomials

$$c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0, \quad c_0, \ldots, c_n \in \mathbb{C}^N, \quad n \in \mathbb{N}.$$

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but

$$\|F_n - F\|_{L^1} \to 0 \quad \text{and}$$

$$\int_{-\pi}^{\pi} \log\det F_n(e^{i\theta})\, d\theta \to \int_{-\pi}^{\pi} \log\det F(e^{i\theta})\, d\theta$$

$$\Downarrow$$

$$\|F_n^+ - F^+\|_{H^2} \to 0 \quad \text{as} \quad n \to \infty$$

(G. Janashia, E. Lagvilava, and L. Ephremidze, 1999, 2011; S. Barclay, 2004)

Quantitative results?

Question:

Does there exist

$$\Pi : [0, \infty) \times [0, \infty) \to [0, \infty)$$

such that $\Pi(\alpha, \beta) \to 0$ as $\alpha, \beta \to 0$, and

$$\|F_n^+ - F^+\|_{H^2} \leq \Pi\left(\|F_n - F\|_{L^1}, \|\log\det F_n - \log\det F\|_{L^1}\right)?$$

Scalar case

Suppose $f \in L^1(\mathbb{T})$, $f > 0$ a.e., and

$$\log f \in L^1(\mathbb{T}).$$

Then there exists a unique outer function $f^+ \in H^2(\mathbb{T})$ such that $f^+(0) > 0$ and

$$f(\zeta) = |f^+(\zeta)|^2, \quad \zeta \in \mathbb{T}.$$

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$$f^+(z) = \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log f(e^{i\theta})\, d\theta\right), \quad |z| < 1,$$

$$f^+(e^{i\vartheta}) = \exp\left(\frac{1}{2}\left(\log f(e^{i\vartheta}) + i\,(\log f)^{\sim}(e^{i\vartheta})\right)\right), \quad \vartheta \in (-\pi, \pi],$$

where " $\sim$ " denotes the harmonic conjugate:

$$\tilde{h}(e^{i\vartheta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{i\theta}) \cot\left(\frac{\vartheta - \theta}{2}\right) d\theta.$$

Negative results

Theorem.

There exist functions $f_n, g_n \geq 0$, $n \in \mathbb{N}$ such that

$$\|f_n\|_{L_1}, \|g_n\|_{L_1} \leq 1, \quad \|f_n - g_n\|_{L_1} \leq \frac{1}{n}, \quad \|\log f_n - \log g_n\|_{L_1} \leq \frac{1}{n},$$

but $\|f_n^+ - g_n^+\|_{H^2} \geq 2 - 1/n$.

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but $\|f_n^+ - g_n^+\|_{H^2} \geq 2 - 1/n$.

Remark. The norms $\|\log f_n\|_{L_1}$ and $\|\log g_n\|_{L_1}$ might not be bounded in the above theorem. One can change the estimates $\|f_n\|_{L_1}, \|g_n\|_{L_1} \leq 1$ in the theorem for $\|f_n\|_{L_1} = 2\pi + 1$, $\|g_n\|_{L_1} \leq 2\pi + 1$, $\|\log f_n\|_{L_1} \leq 1$.

Positive results?

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Every function $f \in L^1(\mathbb{T})$ belongs to an Orlicz space ...

A crash course on Orlicz spaces

Let (Ω, Σ, μ) be a measure space and let
$\Psi : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function.

We want to define a Banach space of functions $f : \Omega \to \mathbb{C}$ (or $\mathbb{R}$) such that

$$\int_\Omega \Psi\big(|f(x)|\big) d\mu(x) < \infty. \qquad (*)$$

If $\Psi(t) \equiv t^p$, then this is just $L_p(\Omega)$.

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Difficulties:

- If Ψ is rapidly increasing, e.g. exponentially increasing, then the set of functions satisfying $(*)$ is not a linear space: $(*)$ does not imply that the same integral for $2f$ is finite.

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Difficulties:

- If Ψ is rapidly increasing, e.g. exponentially increasing, then the set of functions satisfying $(*)$ is not a linear space: $(*)$ does not imply that the same integral for $2f$ is finite.
- How do we define a norm? (What is a Ψ-analogue of taking the p-th power root of the integral?)

A crash course on Orlicz spaces

Definition

A continuous convex function $\Psi : [0, +\infty) \to [0, +\infty)$ is called an *N*-function if

$$\lim_{t \to 0+} \frac{\Psi(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to +\infty} \frac{\Psi(t)}{t} = +\infty.$$

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The function $\Phi : [0, +\infty) \to [0, +\infty)$ defined by

$$\Phi(t) := \sup_{s\geq 0} \big(st - \Psi(s)\big)$$

is called complementary to Ψ.

Examples of complementary N-functions:

$$\Psi(t) = \frac{t^p}{p}, \quad 1 < p < \infty, \qquad\qquad \Phi(t) = \frac{t^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

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$$\mathcal{A}(t) = e^t - 1 - t, \qquad \mathcal{B}(t) = (1+t)\ln(1+t) - t.$$

A crash course on Orlicz spaces

Let Φ and Ψ be mutually complementary N-functions.

Norms:

$$\|f\|_\Psi = \|f\|_{\Psi,\Omega} := \sup\left\{\left|\int_\Omega fg\,d\mu\right| : \int_\Omega \Phi(|g|)d\mu \leq 1\right\},$$

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$$\|f\|_{(\Psi)} = \|f\|_{(\Psi,\Omega)} := \inf\left\{ \kappa > 0 : \int_{\Omega} \Psi\left(\frac{|f|}{\kappa}\right) d\mu \leq 1 \right\}.$$

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These norms are equivalent:

$$\|f\|_{(\Psi)} \leq \|f\|_\Psi \leq 2\|f\|_{(\Psi)}\,.$$

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The Orlicz space:

$$L_\Psi(\Omega) := \{f \mid \|f\|_{\Psi,\Omega} < \infty\}.$$

Notation

Let K be the best constant in Kolmogorov's weak type (1, 1) estimate

$$m\{\vartheta \in [-\pi,\pi) : |\widetilde{\psi}(\vartheta)| \geq \lambda\} \leq \frac{K}{\lambda}\int_{-\pi}^{\pi} |\psi(\vartheta)|\, d\vartheta,$$
$$\lambda > 0, \quad \psi \in L_1[-\pi,\pi),$$

where m stands for the Lebesgue measure on the real line

(B. Davis, 1974:

$$K = \frac{1 + 3^{-2} + 5^{-2} + \cdots}{1 - 3^{-2} + 5^{-2} - \cdots} \approx 1.347),$$

and let

$$K_0 := \frac{K}{2}\int_0^{\pi} \frac{\sin\lambda}{\lambda}\, d\lambda\,.$$

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Let Φ be an N-function and let

$$\Lambda_{\Phi}(s) := \inf\left\{t > 0 :\ \frac{1}{t}\Phi'\left(\frac{1}{t}\right) \le \frac{1}{s}\right\}, \quad s > 0.$$

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Then

$$\Lambda_\Phi(s)\to 0\ \text{ as } s\to 0+\,.$$

and

$$\Phi(\tau)\equiv\frac{\tau^q}{q},\ 1<q<\infty\quad\Longrightarrow\quad\Lambda_\Phi(s)\equiv s^{1/q}.$$

Positive results

Theorem.

For every pair Φ and Ψ of mutually complementary N-functions, the following estimate holds

$$\|f^+ - g^+\|^2_{H^2} \leq 2\|f - g\|_{L_1} + 4\|f\|_\Psi \Lambda_\Phi \left(\frac{K_0}{2} \| \log f - \log g\|_{L_1} \right) .$$

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For every pair Φ and Ψ of mutually complementary N-functions, the following estimate holds

$$\|f^+ - g^+\|_{H^2}^2 \leq 2\|f-g\|_{L_1} + 4\|f\|_\Psi \Lambda_\Phi \left(\frac{K_0}{2} \|\log f - \log g\|_{L_1} \right) .$$

Corollary.

For every $p \in (1,\infty)$, there exists a constant $C(p)$ such that

$$\|f^+ - g^+\|_{H^2}^2 \leq 2\|f-g\|_{L_1} + C(p)\|f\|_{L_p} \|\log f - \log g\|_{L_1}^{\frac{p-1}{p}} .$$

One can take $C(p) = 2^{\frac{p+1}{p}} K_0^{\frac{p-1}{p}} \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}}$.

Positive results

Corollary.

There exists a constant C such that

$$\|f^+ - g^+\|_{H^2}^2 \leq 2\|f-g\|_{L_1} + C\|f\|_{L_\infty}\|\log f - \log g\|_{L_1} .$$

One can take $C = 2K_0 < 2.5$.

The power $\frac{p-1}{p}$ in the estimate

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is optimal.

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is optimal.

Theorem.

Let $p > 1$. For every $\gamma > \frac{p-1}{p}$, there exist functions $f, f_k \in \mathcal{S}_1(\mathbb{T}) \cap L_p(\mathbb{T})$, $k = 1, 2, \ldots$, such that

$$\|f - f_k\|_{L_1} \to 0 \quad \textit{and} \quad \|\log f - \log f_k\|_{L_1} \to 0,$$

while

$$\frac{\|f^+ - f_k^+\|_{H_2}^2}{\|\log f - \log f_k\|_{L_1}^{\gamma}} \to \infty.$$

Back to matrix functions

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The negative result we had in the scalar case is of course correct in the matrix case too.

For positive results, we will need the following notation. For $F \in \mathcal{S}_n(\mathbb{T})$, let

$$\ell_F := \log \det F - n \log_+ \|F\| \in L_1(\mathbb{T}),$$

where $\log_+ x = \max(0, \log x)$ (note that $\ell_F \leq 0$ a.e.), and

$$Q_F := \frac{(\max\{1, \|F\|\})^n}{\det F} = e^{-\ell_F} = e^{|\ell_F|} .$$

Below, we will use conditions like $\ell_F \in L_{p_1}(\mathbb{T})$ and $Q_F \in L_{p_1}(\mathbb{T})$.

In the scalar case $n = 1$, one has $\ell_F = \log_+ \frac{1}{F}$ and $Q_F = \max\left\{1, \frac{1}{F}\right\}$. So, these conditions measure not how large F is but how strong its zeros are.

In the matrix case, these conditions control comparative sizes of the eigenvalues of F. More precisely, they control how small the determinant of F is compared to the maximum between 1 and the n-th power of the norm of F.

Theorem.

Suppose $F \in \mathcal{S}_n(\mathbb{T}) \cap L_{p_0}(\mathbb{T})^{n\times n}$, $G \in \mathcal{S}_n(\mathbb{T})$, *and* $\ell_F \in L_{p_1}(\mathbb{T})$, $p_0, p_1 \in (1, \infty)$, $\|G - F\|_{L_1} \le 1$. *Then for any* $\alpha \in (0,1)$, *the following estimate holds*

$$\|G^+ - F^+\|^2_{H_2} \le 4\|G - F\|_{L_1} + 2q_0^{\frac{1}{q_0}} \left(\|F\|_{L_{p_0}} + 1\right)$$

$$\times \Big[c(p_0) \big\| \log_+ \|G\| - \log_+ \|F\| \big\|^{\frac{1}{q_0}}_{L_1} + 2(2p_0)^{\frac{1}{p_0}} \Big(3n \|G - F\|^{1-\alpha}_{L_1}$$

$$+ 2(n+1)\alpha^{1-p_1} p_1 \|\ell_F\|^{p_1}_{L_{p_1}} \left| \log \|G - F\|_{L_1} \right|^{1-p_1}$$

$$+ \left\| \log \frac{\det G}{\det F} \right\|_{L_1} \Big)^{\frac{1}{q_0}} \Big],$$

where

$$c(p_0) := 2^{\frac{p_0+1}{p_0}} K_0^{\frac{1}{q_0}}, \quad q_0 := \frac{p_0}{p_0 - 1}.$$

Theorem (continued)

Furthermore, if in addition $F \in L_\infty(\mathbb{T})^{n\times n}$, then the above inequality can be modified as

$$\|G^+ - F^+\|_{H_2}^2 \leq 4\|G - F\|_{L_1} + 4\max\{\|F\|_{L_\infty}, 1\}$$

$$\times\Big(K_0\Big\|\log_+\|G\| - \log_+\|F\|\Big\|_{L_1} + 3n\,\|G - F\|_{L_1}^{1-\alpha}$$

$$+2(n+1)\alpha^{1-p_1}p_1\,\|\ell_F\|_{L_{p_1}}^{p_1}\,\Big|\log\|G - F\|_{L_1}\Big|^{1-p_1} + \Big\|\log\frac{\det G}{\det F}\Big\|_{L_1}\Big).$$

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The exponent $1 - p_1$ in the above theorem cannot be improved beyond $-p_1$ even for 2×2 matrix functions.

Theorem.

Suppose $F \in \mathcal{S}_n(\mathbb{T}) \cap L_{p_0}(\mathbb{T})^{n\times n}$, $G \in \mathcal{S}_n(\mathbb{T})$, *and* $Q_F \in L_{p_1}(\mathbb{T})$, $p_0, p_1 \in (1, \infty)$. *If* $\|F - G\|_{L_1} \le e^{-4}$, *then*

$$\begin{aligned}\|G^+ - F^+\|_{H_2}^2 &\le 4\|G - F\|_{L_1} \\ &+ 2q_0^{\frac{1}{q_0}} \left(\|F\|_{L_{p_0}} + 1\right) \left[c(p_0)\Big\| \log_+ \|G\| - \log_+ \|F\|\Big\|_{L_1}^{\frac{1}{q_0}} \right. \\ &+ 2(2p_0)^{\frac{1}{p_0}} \left(\left(3n + \frac{4(n+1)\,\|Q_F\|_{L_{p_1}}^{2p_1}}{p_1 + 1} \left|\log \|G - F\|_{L_1}\right|\right) \|G - F\|_{L_1}^{\frac{p_1}{p_1+1}} \right. \\ &\left.\left. + \left\|\log \frac{\det G}{\det F}\right\|_{L_1}\right)^{\frac{1}{q_0}}\right],\end{aligned}$$

where $c(p_0)$ *and* q_0 *are as in the previous theorem.*

Theorem (continued)

Furthermore, if in addition $F \in L_\infty(\mathbb{T})^{n\times n}$, then the above inequality can be modified as

$$\|G^+ - F^+\|_{H_2}^2 \le 4\|G-F\|_{L_1}$$
$$+4\max\{\|F\|_{L_\infty}, 1\}\Big[K_0\Big\|\log_+\|G\| - \log_+\|F\|\Big\|_{L_1}$$
$$+\left(3n + \frac{4(n+1)\,\|Q_F\|_{L_{p_1}}^{2p_1}}{p_1+1}\Big|\log\|G-F\|_{L_1}\Big|\right)\|G-F\|_{L_1}^{\frac{p_1}{p_1+1}}$$
$$+\Big\|\log\frac{\det G}{\det F}\Big\|_{L_1}\Big].$$

Theorem (continued)

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$$\left.+\left\|\log\frac{\det G}{\det F}\right\|_{L_1}\right].$$

The exponent $\frac{p_1}{p_1+1} = 1 - \frac{1}{p_1+1}$ in the above theorem cannot be improved beyond $\frac{2p_1}{2p_1+1} = 1 - \frac{1}{2p_1+1}$ even for 2×2 matrix functions.

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$$R_\Psi(\tau) \to 0 \text{ as } \tau \to 0+$$

and

$$\frac{1}{2}\Lambda_\Phi(\tau) < R_\Psi(\tau) \leq 8\Lambda_\Phi(\tau), \quad \forall \tau > 0.$$

Theorem.

Let $F, G \in \mathcal{S}_n(\mathbb{T})$, *and let* (Φ_0, Ψ_0) *and* (Φ_1, Ψ_1) *be two pairs of mutually complementary N-functions such that*

$$F \in L_{\Psi_0} \quad \text{and} \quad \ell_F \in L_{\Psi_1}.$$

Then for any nondecreasing function $\nu : [0, \infty) \to [0, 1]$ *satisfying*

$$\nu(\tau) \to 0 \text{ and } \tau/\nu(\tau) \to 0 \text{ as } \tau \to 0+$$

the following estimate holds

$$\|G^+ - F^+\|_{H_2}^2 \leq 4\|G - F\|_{L_1}$$
$$+2\left(\|F\|_{\Psi_0} + \Phi_0^{-1}(1)\right)\left[4\Lambda_{\Phi_0}\left(\frac{K_0}{2}\left\|\log_+\|G\| - \log_+\|F\|\right\|_{L_1}\right)\right.$$
$$\left.+R_{\Psi_0}\left(\frac{6n\|G-F\|_{L_1}}{\nu\left(\|G-F\|_{L_1}\right)} + \frac{4(n+1)|\log\nu\left(\|G-F\|_{L_1}\right)|}{\Psi_1\left(\frac{|\log\nu\left(\|G-F\|_{L_1}\right)|}{\|\ell_F\|_{(\Psi_1)}}\right)} + 2\left\|\log\frac{\det G}{\det F}\right\|_{L_1}\right)\right].$$

Theorem.

Let $F, G \in \mathcal{S}_n(\mathbb{T})$, and let (Φ_0, Ψ_0) be a pair of mutually complementary N-functions. Suppose $F \in L_{\Psi_0}$ and $\ell_F \in L_\infty$. Then

$$\|G^+ - F^+\|_{H_2}^2 \leq 4\|G - F\|_{L_1}$$
$$+2\left(\|F\|_{\Psi_0} + \Phi_0^{-1}(1)\right)\left[4\Lambda_{\Phi_0}\left(\frac{K_0}{2}\left\|\log_+\|G\| - \log_+\|F\|\right\|_{L_1}\right)\right.$$
$$\left.+R_{\Psi_0}\left(\left(2e^{\|\ell_F\|_{L_\infty}} + 1\right) n\|G - F\|_{L_1} + \left\|\log\frac{\det G}{\det F}\right\|_{L_1}\right)\right].$$

Theorem.

Let $F \in \mathcal{S}_n(\mathbb{T}) \cap L_\infty(\mathbb{T})^{n\times n}$ *and* $\ell_F \in L_\infty(\mathbb{T})$*. Then*

$$\|G^+ - F^+\|_{H_2}^2 \leq \|F\|_{L_\infty} \left(n\, e^{\|\ell_F\|_{L_\infty}} \|G - F\|_{L_1} + \left\| \log \frac{\det G}{\det F} \right\|_{L_1} \right).$$