Diffraction by elastic wedge

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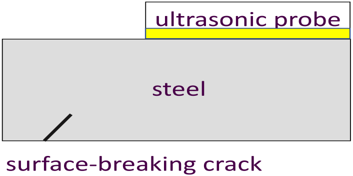
Abstract

Diffraction of the elastic plane wave by an infinite straight-edged 2D or 3D wedge made of an isotropic solid is a canonical problem that has no analytical solution. We review three major semi-analytical approaches to this problem and discuss their application in non-destructive evaluation as well as testing, cross-validation and experimental validation. We draw attention to high sensitivity of the backscatter diffraction coefficients to the Poisson ratio.

Many worked on the wedge problems in solids throughout the second half of XX century. For years the problem was a diffractionist's analogue of Fermat's Last Theorem!

Outline

Problem statement



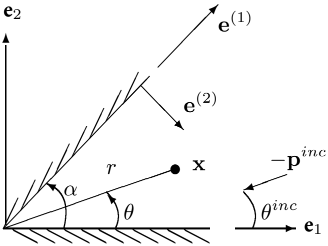


Figure. Wedge geometry.

Elastic solids can be characterised by C - stiffness tensor and ρ - mass density. For zero forcing, waves in elastic solids are governed by the elastodynamic equation (Newton's 2nd law)

$$\nabla \cdot \sigma = \rho \ddot{\boldsymbol{u}} \quad \text{inside the solid},$$

where $\boldsymbol{u}$ is the displacement vector field & σ - stress tensor related by Hooke's law $\sigma = C : \epsilon$ to strain tensor $\epsilon = \frac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]$.

- The boundary condition: The boundary is usually stress-free, so given $\boldsymbol{n}$, the boundary normal, is

$$\sigma \cdot \boldsymbol{n}\big|_{\text{on the boundary}} = \mathbf{0}, \tag{1}$$

- The radiation condition: the scattered waves are outgoing at infinity,
- The tip condition: The mean elastic energy carried by the scattered waves is bounded.

Existence and uniqueness - Kamotski and Lebeau, 2006.

Consider the incident plane wave $\boldsymbol{u}_n^{inc}(t, \boldsymbol{x}) = \boldsymbol{A}_n^{inc} e^{-\mathrm{i}\omega t + \mathrm{i}k_n \boldsymbol{p}^{inc}\cdot\boldsymbol{x}}$, with t - time, $\boldsymbol{x}$ - point in space, ω - circular frequency, $k_n = \omega/c_n$ - wave no, $\kappa_n = c_2/c_n$, c_n - transverse (n=1) or longitudinal (n=2)speed. Scaling $\boldsymbol{x}$ by ω/c_2,

- the total displacement field becomes $\boldsymbol{u}(\boldsymbol{x})e^{-\mathrm{i}\omega t}$ and $\boldsymbol{u}$ satisfies the reduced elastodynamic equation

$$L\boldsymbol{u} - \boldsymbol{u} = \boldsymbol{0}, \quad \text{inside the solid,}$$

$$B\boldsymbol{u}|_{\text{on the boundary}} = \boldsymbol{0},$$

where $L = -\frac{1}{\rho c_2^2}\boldsymbol{\nabla}^T C\boldsymbol{\nabla}$, $B = \frac{1}{\rho c_2^2}\boldsymbol{n}^T C\boldsymbol{\nabla}$.

- $\boldsymbol{u} = \nabla^\perp \cdot \psi_1 + \nabla \cdot \psi_2$, where ψ_1, ψ_2 are the elastodynamic potentials, which satisfy the Helmholtz equations,

$$\triangle\psi_n + \kappa_n^2\psi_n = 0, \quad n = 1, 2.$$

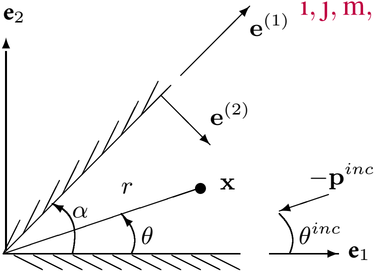
$$\Longrightarrow \psi_1 = -\kappa_1^{-2}\nabla^\perp \cdot \boldsymbol{u}, \quad \psi_2 = -\nabla \cdot \boldsymbol{u}.$$

Gautesen's approach: Extinction theorem (eqn+bcds), $LHS = 0 \implies$

4 unknowns: $\mathrm{u}_i(\mathrm{y}) = u_{\mathrm{i}}(\mathrm{y}, 0), \mathrm{u}^{(\mathrm{i})}(\mathrm{y}) = \boldsymbol{u}(\mathrm{y}\cos\theta, \mathrm{y}\sin\alpha) \cdot \mathbf{e}^{(\mathrm{i})}$

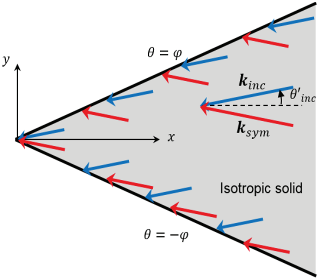
$$H[\alpha - \theta]u_{\mathrm{k}}(\boldsymbol{x}) = u_{\mathrm{k}}^{inc}(\boldsymbol{x}) -$$
$$\int_0^\infty \left[\sigma_{\mathrm{i2k}}^{G}(\mathrm{x}_1 - \mathrm{y}, \mathrm{x}_2)\mathrm{u}_{\mathrm{i}}(\mathrm{y}) + \sigma_{\mathrm{jmk}}^{\mathrm{G}}(\mathrm{x}_1 - \mathrm{y}\cos\alpha, \mathrm{x}_2 - \mathrm{y}\sin\alpha)\mathrm{e}_{\mathrm{j}}^{(\mathrm{i})}\mathrm{e}_{\mathrm{m}}^{(2)}\mathrm{u}^{(\mathrm{i})}(\mathrm{y})\right]\mathrm{dy},$$

$\mathrm{i}, \mathrm{j}, \mathrm{m}, \mathrm{k} = 1, 2$ & summation convention applies

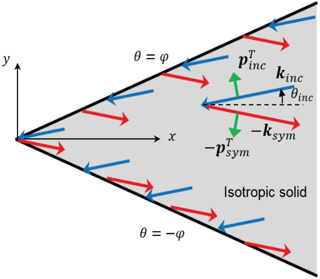


Gautesen's approach: Symmetrise, separate transverse and longitudinal components $\implies$ 2 unknowns, $\mathrm{u}_1, \mathrm{u}_2$

$\ell = 1, n = 2 \text{ or } \ell = 2, n = 1$ $\qquad$ $\ell = 1, n = 1 \text{ or } \ell = 2, n = 2$



(a) Symmetric problem for an incident longitudinal wave or antisymmetric problem for an incident transversal wave



(b) Antisymmetric problem for an incident longitudinal wave or symmetric problem for an incident transversal wave

Gautesen's approach: LT, functional equations, radiation cdtn

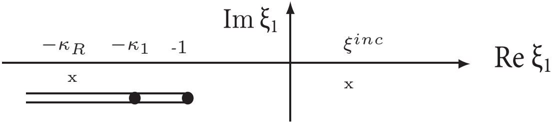
$$\hat{\mathrm{u}}_{\mathrm{i}}(\xi_1) = \int_0^\infty \mathrm{u}_{\mathrm{i}}(x_1)\mathrm{e}^{-i\xi_1 x_1}\,dx_1,\ \hat{\mathrm{u}}^{(i)}(\xi^{(1)}) = \int_0^\infty \mathrm{u}^{(i)}(x^{(1)})\mathrm{e}^{-i\xi^{(1)}x^{(1)}}\,dx^{(1)},\ \mathrm{i}=1,2$$

$$\begin{cases} \hat{u}_1(\xi_1) = \frac{1}{R(\xi_1)}\{a(\xi_1)[f_1(\xi_1) - \kappa_1^2\hat{\psi}_1(\xi^{(1)})] + b_1(\xi_1)[f_2(\xi_1) - \hat{\psi}_2(\xi^{(1)})]\}, \\ \hat{u}_2(\xi_1) = \frac{1}{R(\xi_1)}\{-b_2(\xi_1)[f_1(\xi_1) - \kappa_1^2\hat{\psi}_1(\xi^{(1)})] + a(\xi_1)[f_2(\xi_1) - \hat{\psi}_2(\xi^{(1)})]\} \end{cases}$$

where for $j = 1, 2$, $a(\xi_1) = \kappa_1^2 - 2\xi_1^2$, $b_j(\xi_1) = 2\xi_1\sqrt{\kappa_j^2 - \xi_1^2}$,

$$f_j(\xi_1) = -2\pi\kappa_1(-1)^{-1}[p_2^{inc}\delta(\xi_1 - \kappa_n p_1^{inc}) + p_2^{sym}\delta(\xi_1 - \kappa_n p_1^{sym})]\delta_{\mathrm{nj}}$$

$$\hat{\boldsymbol{u}}(\xi_1) = \hat{\boldsymbol{v}}(\xi_1) + \hat{\boldsymbol{u}}^{ge}(\xi_1) + D_n^R\hat{\boldsymbol{d}}^R(\xi_1) + \hat{\boldsymbol{u}}^{ge(E)}(\xi_1) + D_n^{R(E)}\hat{\boldsymbol{d}}^{R(E)}(\xi_1).$$



Lebeau & Croiselle's approach: Eq of motion, radiation cdtn: 4 unknowns (spectral fns $\mathbf{U}^\ell$ on each wedge face $\ell = 1$ or $\ell = 2$)

- For each wedge face ℓ, introduce the tempered distribution $\boldsymbol{U}^{n,\ell}(x_1)\delta(x_2)$ and Fourier Transform

$$\tilde{f}(\xi_1, x_2) = \int_{-\infty}^{+\infty} f(x_1,\, x_2)\, e^{-\mathrm{i}\xi_1 x_1}\, \mathrm{d}x_1 .$$

- Apply to the equation of motion and seek the outgoing solution

$$\widetilde{\boldsymbol{u}}^{n,\ell}(\xi_1, x_2) = \widetilde{M}^n \widetilde{\boldsymbol{U}}^\ell(\xi_1) e^{\mathrm{i}\zeta_2^n |x_2|},\ \widetilde{M}^n(\xi_1) = \operatorname{Res}\Big|_{\xi_2 = \xi_2^n} [\widetilde{L}(\xi_1, \xi_2) - \mathbb{I}_2]^{-1},$$

$$\zeta_2^n = \sqrt{\kappa_n^2 - \xi_1^2}, \quad \boldsymbol{\xi}^n = (\xi_1, \operatorname{sgn}(x_2)\zeta_2^n).$$

- Take FT of the bdry cndts, substitute $\widetilde{\boldsymbol{u}}^{n,\ell}(\xi_1, 0) = \widetilde{M}^n \widetilde{\boldsymbol{U}}^\ell(\xi_1)$

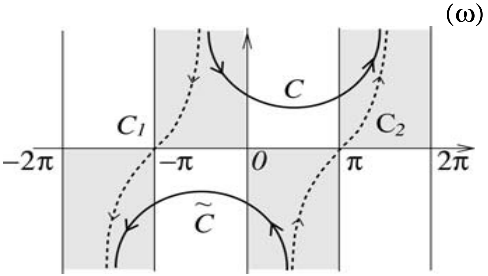
$\Longrightarrow$ functional eqns for spectral functions $\widetilde{\boldsymbol{U}}^\ell(\xi_1)$ rather than Gautesen's $\hat{\boldsymbol{u}}^{n,\ell}(\xi_1)$ $\Longrightarrow$ $\tilde{\boldsymbol{U}}(\xi_1) = \hat{\boldsymbol{U}}^{ge}(\xi_1) + \hat{\boldsymbol{V}}(\xi_1)$.

Budaev*et al.* approach: the Helmholtz eqns, Sommerfeld Integrals

$$\psi_i(kr,\theta) = \int_{C \cup \widetilde{C}} \Psi_i(\omega+\theta) \mathrm{e}^{\mathrm{i}r\cos\omega} \mathrm{d}\omega,$$

. can be rewritten as

$$\psi_i(kr,\theta) = \int_C \Big[\Psi_i(\omega+\theta) - \Psi_i(-\omega+\theta)\Big] \mathrm{e}^{\mathrm{i}r\cos\omega} \, \mathrm{d}\omega.$$



Approaches to solving the functional equations

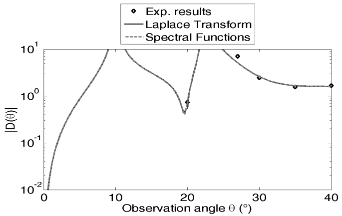
- In the LT approach the problem is reformulated as a regular integral equation.
- In the SI approach the problem is reformulated as a singular integral equation.
- In the SF approach the functional equations are solved directly using the Galerkin collocation method (now for 3D as well).

$$H(\alpha-\theta)\boldsymbol{v}(\boldsymbol{x}) \approx D_n^T(\theta;\theta^{inc})\frac{e^{i\kappa_1 r}}{\sqrt{\kappa_1 r}}e_\theta + D_n^L(\theta;\theta^{inc})\frac{e^{i\kappa_2 r}}{\sqrt{\kappa_2 r}}e_r, \quad \kappa_n r \to \infty,$$

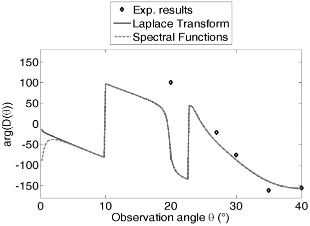
where D_n^T and D_n^L are known as diffraction coefficients and

$$e_\theta = (-\sin\theta, \cos\theta),$$
$$e_r = (\cos\theta, \sin\theta).$$

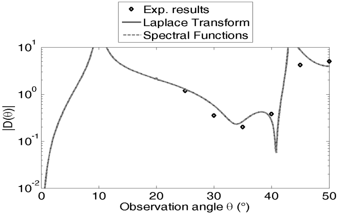
Cross-validation and experimental validation



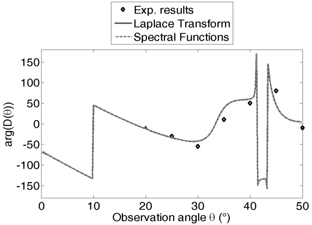
(a) Incident and diffracted T wave, $\alpha = 80^o$.



(b) Incident and diffracted T wave, $\varphi = 80^o$



(c) Incident and diffracted T wave, $\alpha = 100^o$.



(d) Incident and diffracted T wave, $\alpha = 100^o$.

Numerical considerations: tip cdtn, internal checks

- SI Functional eqns contain not only unknown functions but also an unknown constant c_1 (related to the constant term in the tip asymptotics of $\boldsymbol{u}$). One imposes the regularisation condition that the RHS of the integral eqn is in the range of the operator in the LHS. Internal tests are offered.
- LT Functional eqns contain not only unknown functions but also unknown constants, D_n^R and $D_n^{R(E)}$. One imposes the regularisation condition that the corresponding terms have zero residues. They are related to the constant c_1 and thus assure the correct tip behavior. Internal tests are offered.
- SF It is not clear how the tip cdtn is satisfied or internal consistency checked.

SI involves extra technical complications, such as various integral transforms and their inverses as well as the fctn $cos^{-1}(\kappa_1^{-1} cos\,\omega)$, with multiple cuts in its domain.

Sensitivity of backscatter to the Poisson ratio

Comparison of the backscatter diffraction coefficients computed using the LT *approach for the* 100^{o} *wedges with the Poisson ratios* 0.29 *(steel) and* 0.33 *(aluminum).*

	$\nu = 0.29$		$\nu = 0.33$	
Angle θ	Magnitude $\lvert D_1^S \rvert$	Phase $\arg(D_1^S)$	Magnitude $\lvert D_1^S \rvert$	Phases $\arg(D_1^S)$
25^0	1.27	-34^0	1.36	-36^0
30^0	0.57	-43^0	0.72	-51^0
35^0	0.26	34^0	0.27	-63^0
40^0	0.27	61^0	5.17	-131^0
45^0	8.59	41^0	5.25	-25^0
50^0	3.94	5^0	3.52	7^0

	$\nu = 0.29$		$\nu = 0.33$	
Angle θ	Magnitude $\lvert D_2^P \rvert$	Phase $\arg(D_2^P)$	Magnitude $\lvert D_2^P \rvert$	Phases $\arg(D_2^P)$
25^0	0.81	45^0	1.26	45^0
30^0	0.65	45^0	1.04	45^0
35^0	0.60	45^0	0.94	45^0
40^0	0.59	45^0	0.89	45^0
45^0	0.60	45^0	0.88	45^0
50^0	0.61	45^0	0.87	45^0

References

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