Computational geometric optics: Monge-Ampère

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Outline

An accurate control of light is needed in non-imaging optics for the design of projection displays, laser weapons, medical illuminators ...

There is therefore a critical need for efficient and robust numerical methods backed up theoretically to solve computational geometric optics problems.

No mathematical justification for the (efficient) design of a typical near-field problem. Example : given a point light source, find a mirror (reflector) or a lens (refractor) which redirects the light onto a close target.

Structure preservation

Discretizations set up in such a way that proofs at the discrete level mimic the proofs at the continuous level

- density functions are discretized with a linear combination of Dirac masses
- convexity of solutions replaced by discrete convexity
- Some apparently non structure preserving discretizations suggest focus on alternate analysis of problems at the continuous level.

Parallel far field reflector problem
The position and shape $u(x)$ of the (convex) reflector solves the second boundary value problem for the Monge-Ampère equation

$$g(Du(x)) \det D^2 u(x) = f(x), x \in X$$
$$Du(X) = Y,$$

where $f \in L^1(X), f \geq 0$ is the incoming light distribution density and $g \in L^1(Y), g > 0$ is the prescribed irradiance density.

Conservation of energy : $\int_X f(x)dx = \int_Y g(y)dy$.

The ray tracing map associates to an incoming light direction $x \in \mathcal{S}^2$ the reflected ray t. With $T(x)$ the stereographic projection of $t(x)$, it can be shown that $T(x) = Du(x)$.

Conservation of energy : $\int_B f(x) dx = \int_{T(B)} g(T(x) dT(x))$ gives $f(x) = g(T(x)) | \det DT(x)|$.

convexity/concavity of the surfaces make it easier to build them

far field problems fall in the class of optimal transport problems

Monge principle Let $\mathcal{T}$ be the set of mappings that satisfy $f(x) = g(T(x))|\det DT(x)|$. A mapping $\tilde{T}$ is a critical point of

$$\int_X f(x)c(x, T(x))dx,$$

for a given convex cost function $c(x, y)$ if and only if $\tilde{T}$ solves

$$D_x c(x, y)|_{y=\tilde{T}(x)} = Du(x),$$

for a potential function $u(x)$ which defines the position of the reflector/refractor.

Methods for solving optimal transport problems can be used for far field problems, e.g. Kantorovich duality with Dirac approximations of $g(x)dx$.

Near-field reflector problem
Graf and Oliker : Not all solutions of the reflector problem can be realized as an extremum of their variational principle

Their variational principle is more complicated than the one from Kantorovich duality for the usual optimal transport problems.

Similar studies have not been done for other near field problems, e.g. parallel near field refractors.

Geometric optics problems can be studied under the umbrella of generated Jacobian equations.

$$\det DT_u(x) = \psi(x, u(x), T_u(x)), \quad T_u(x) = T(x, u(x), Du(x))$$
$$T_u(X) = Y.$$

The transformation T and the "potential" u are now related through a generating function $G : \overline{\Omega} \times \overline{\Omega^*} \times \mathbb{R}^+ \mapsto \mathbb{R}$ and $T(x, u, p)$ is obtained by solving the system

$$D_x G(x, T, Z) = p \qquad G(x, T, Z) = u.$$

When $G(x, y, z) = x \cdot y + \log z$, we obtain $T(x, y, p) = p$, i.e. $T_u(x) = Du(x)$ and ψ does not depend on $u(x)$

One dimensional Monge-Ampère equation
Find u convex, $u''(x) = f(x)$ in $(-1,1)$, $u'(0,1) = (-1,1)$.

$$\int_B u''(x)\,dx = \int_B f(x)\,dx.$$

Monge-Ampère measure $M[u](B) = \int_B u''(x)\,dx$.
Change of variable $x \to \gamma(x) = u'(x) = p$ (gradient mapping)
Monge-Ampère measure $M[u](B) = \int_{\gamma(B)} dp$
Replace $\gamma(x)$ by subgradient mapping for non smooth solutions

$$\partial u(x_0) = \{\, p \in \mathbb{R} : u(x) \geq u(x_0) + p(x - x_0), \text{ for all}\, x \in \Omega \,\}.$$

$M[u](\{x_0\}) = |\partial u(x_0)|$.

$M[u](B) = \int_B f(x)\,dx, \partial u(0,1) = (-1,1)$

Normal mapping : $\partial u : X \to \mathcal{P}(\mathbb{R}^d)$ defined by

$$\partial u(x_0) = \{ p \in \mathbb{R}^d : u(x) \geq u(x_0) + p \cdot (x - x_0), \text{ for all } x \in X \}.$$

For $E \subset \Omega$, define $\partial u(E) = \cup_{x \in E} \partial u(x)$. Monge-Ampère measure associated with u : $M[u](E) = |\partial u(E)|$.

$M[u](B) = \int_B f(x)\, dx$ B Borel set in X, $\partial u(X) = Y$.

Weak formulation of type A needs explicit discretization of $\partial u(X) = Y$

Above weak formulation not often used for discretization. An iterative method proposed by Oliker and Prussner (coordinate descent) is used for different weak formulations.

+ a damped Newton's method.

Given $y_0 \in Y$ let $(\partial u)^{-1}(y_0) = \{ x \in X, \partial u(x) = y_0 \}$. Define for a Borel set $F \subset Y$

$$\eta_u(F) = \int_{(\partial u)^{-1}(F)} f(x)dx$$

Brenier formulation or weak formulation of type B : find a convex function u such that

$$\eta_u(F) = \int_F g(p)dp, F \subset Y, F \text{ Borel.} \qquad (1)$$

Let $y_1, \ldots, y_N$ be distinct points of Y and let $g_1, \ldots, g_N$ be positive numbers such that $\int_\Omega f(x)dx = \sum_{i=1}^N g_i$.
Discrete problem : find a convex function u_N such that

$$\eta_{u_N}(y_i) = g_i, i = 1, \ldots, N.$$

Now $\partial u_N(X) \subset Y$ by construction.

There exists constants $b_1, \dots, b_N > 0$ such that

$$\phi_b(x) = \max_{1 \leq i \leq N} G(x, y_i, b_i),$$

solves

$$\eta_{\phi_b}(y_i) = f_i, 1 \leq i \leq N.$$

In addition, if $\inf_\Omega f > 0$, and $G(x, y, b) = x \cdot y + b$ the solution is unique up to a constant.

Coordinate ascent consists in iteratively increasing the values b_i starting from an initial guess till the equation is solved.

Convergence analysis of coordinate ascent for generated Jacobians were studied by Abedin and Gutiérrez 2017.

And applied to the far field refractor problem by de Leo, Gutiérrez and Mawi 2017.

G. Awanou, R. Awi and J. Kitagawa (2019) Convergence of a damped Newton's method for generated Jacobian equations (In preparation)

The damped Newton's method was successful for a dual formulation of optimal transport. Merigot, Kitagawa and Thiebert 2019.

Find u convex, $u''(x) = f(x)$ in $(-1, 1)$, $u'(0, 1) = (-1, 1)$.

$M[u](B) = \int_B f(x)\, dx, \partial u(0, 1) = (-1, 1)$

Discrete problem : find a piecewise linear convex function such that

$$\frac{z_{i+1} - 2z_i + z_{i-1}}{h} = hf_i,$$

at interior grid points.

Here discrete convexity ($z_{i+1} - 2z_i + z_{i-1} \geq 0$) is enforced as $f_i \geq 0$.

Main Ingredient For convex approximations, solution is determined by values at mesh points where the discrete Hessian is prescribed and the behaviour at infinity of the convex approximation (as determined by the 2nd boundary condition).

Formulation in terms of the geometry of convex hypersurfaces : find a convex function u with asympotic cone a polyhedral cone K associated with $K^* \subset \Omega^*$ and such that

$$R(Du) \det D^2 u = \sum_{x \in \Omega_h} c_x \delta_x.$$

$K^* \to \Omega^*$, R-curvature of convex functions, $\sum_{x \in \Omega_h} c_x \delta_x \to \mu_f$

A set $K \subset E^d$ is a cone with vertex A if for $X \in K$, we have $A + \lambda \, \overline{AX} \in K$ for all $\lambda > 0$. The set K is the union of rays with a common vertex.

The asymptotic cone of the convex set M is the set of points lying on the rays starting from the point $A \in M$ and contained in M. As M is convex, it is independent (up to a parallel) translation of the point A.

Cone K_{Ω^*} associated with the target domain Ω^*
For each $p \in \overline{\Omega^*}$ one associates the half-space $Q(p) = \{\, (x, z), z \geq p \cdot x \,\}$. The cone K_{Ω^*} is the intersection of the half-spaces $Q(p), p \in \overline{\Omega^*}$.

Second b.v.p. of the geometry of convex hypersurfaces

Find a convex function u on $\mathbb{R}^d$ whose epigraph has asymptotic cone K_{Ω^*} and solves $R(Du) \det D^2 u = f(x)$.

For $y \in \mathbb{R}^d$, the normal image of the point y (with respect to v) or the subdifferential of v at y is defined as

$$\chi_v(y) = \{ q \in \mathbb{R}^d : v(x) \geq v(y) + q \cdot (x - y), \text{ for all } x \in \mathbb{R}^d \}.$$

For $y \in \Omega$, the local normal image of the point y (with respect to v) is defined as

$$\partial v(y) = \{ q \in \mathbb{R}^d : v(x) \geq v(y) + q \cdot (x - y), \text{ for all } x \in \Omega \}.$$

R-curvature of convex functions

Let v be a convex function on $\mathbb{R}^d$.

$$\chi_v(E) = \cup_{x \in E} \chi_v(x).$$

R-curvature as the set function

$$\omega(R, v, E) = \int_{\chi_v(E)} R(p)dp.$$

Extend f and R by 0 to $\mathbb{R}^d$ with equation in measures

$$\omega(R, u, E) = \int_E f(x)dx \text{ for all Borel sets } E \subset \overline{\Omega}$$
$$\chi_u(\overline{\Omega}) = \overline{\Omega^*}.$$

Theorem (Bakelman)

Under the compatibility condition $\int_\Omega f(x)dx = \int_{\Omega^*} R(p)dp$ *there exists a convex function* v *on* $\mathbb{R}^d$ *with asymptotic cone* K_{Ω^*} *such that*

$$\omega(R, v, E) = \int_E f(x)dx \text{ for all Borel sets } E \subset \overline{\Omega}.$$

Such a function is unique up to an additive constant.

Moreover [Oliker 2003] $\chi_v(\overline{\Omega}) = \overline{\Omega^*}$. We also have $\partial v(\Omega) = \Omega^*$ up to a set of measure 0.

If u solves

$$\omega(R,u,E) = \int_E f(x)dx \text{ for all Borel sets } E \subset \Omega$$
$$\chi_u(\Omega) = \Omega^*,$$

extend u to $\mathbb{R}^d$ using

$$\tilde{u}(x) = \inf\{\, y \in \Omega, u_0(y) + \sup_{z \in \partial\Omega^*} (x-y)\cdot z \,\},$$

Chou and Wang (1995). Topol. Methods Nonlinear Anal. Then $\chi_{\tilde{u}}(\overline{\Omega}) = \overline{\Omega^*}$.

Sequences of polygons $K^* \subset \Omega^*$ $K^* \to \Omega^*$. To K^* one associates a cone K which is the epigraph of

$$\max_{j=1,\ldots,N} x \cdot a_j^*,$$

where a_j^* is a vertex of K^*.

Find a piecewise linear convex function u_h with asymptotic cone K such that

$$\omega(R, u_h, x) = \sum_{x \in \Omega_h} c_x \delta_x,$$

where $\sum_{x \in \Omega_h} c_x \delta_x \to \mu_f$.

The unknown are the (finite set of) mesh values $\{ u_h(x), x \in \Omega_h \}$ and the second boundary condition is enforced implicitly using the extension formula

$$u_h(x) = \min_{y \in \partial \Omega_h} \max_{1 \leq j \leq N} (x - y) \cdot a_j^* + u_h(y).$$

Computable with a fast discrete Legendre transform.

The min and the max are over a finite number of points.

Theorem

Let K_m^* be bounded convex polygonal domains increasing to Ω^*. Then the convex solution u_m of

$$\omega(R, u, E) = \int_E f_{K_m^*}(x)dx \text{ for all Borel sets } E \subset \overline{\Omega}$$
$$\chi_u(\overline{\Omega}) = K_m^*$$
$$u(x^0) = \alpha,$$

for $x^0 \in \Omega$ and $\alpha \in \mathbb{R}$ converges uniformly on compact subsets of Ω to the solution u with $u(x^0) = \alpha$.

Find a piecewise linear convex function such that
$\partial u(0,1) = (-1,1)$ and
$\partial u(1/4) = 1$, $\partial u(1/2) = 2/3$ and $\partial u(3/4) = 1/3$.

Claim 1 : The iterative method of coordinate descent may be more efficient than other approaches.

Claim 2 : It is now possible to prove convergence rates without regularity assumptions on the domains and the data.

G. Awanou (2019) The second boundary value problem for the Oliker-Prussner discretization of the Monge-Ampère equation. (In preparation)

Orthogonal lattice with mesh length h : $\mathbb{Z}^d_h = \{ mh, m \in \mathbb{Z}^d \}$.
Put $\Omega_h = \Omega \cap \mathbb{Z}^d_h$ and

$$\partial\Omega_h = \{ x \in \Omega_h \text{ such that for some } i = 1, \ldots, d, x + hr_i \notin \Omega_h \text{ or } x - hr_i \notin \Omega_h \}.$$

For a function v_h on $\mathbb{Z}^d_h$, $e \in \mathbb{Z}^d$ and $x \in \Omega_h$

$$\Delta_{he} v_h(x) = v_h(x + he) - 2v_h(x) + v_h(x - he).$$

Definition

A mesh function v_h on Ω_h which is extended to $\mathbb{Z}^d$ using the extension formula, and which is discrete convex ($\Delta_{he} v_h(x) \geq 0$) is said to have asymptotic cone K associated with K^*.

First relaxation : for a discrete convex function

$$\partial_h v_h(y) = \{ p \in \mathbb{R}^d, p \cdot (he) \geq v_h(y) - v_h(y - he) \, \forall e \in \mathbb{Z}^d \},$$

and consider discrete version of the R-curvature

$$\omega_h(R, v_h, E) = \int_{\partial_h v_h(E)} R(p) dp.$$

Weak convergence results hold and used as a tool in
G. Awanou (2019) : Discrete Aleksandrov solutions of the Monge-Ampère equation.

Second relaxation : localization idea due to Mirebeau.

Let V be a (finite) subset of $\mathbb{Z}^d \setminus \{0\}$ of vectors with co-prime coordinates which span $\mathbb{R}^d$ and which is symmetric with respect to the origin. Furthermore, assume that V contains the elements of the canonical basis of $\mathbb{R}^d$ and that V contains a normal to each side of the target domain K^*.

Now let

$$D_V v_h(x) = \{\, p \in \mathbb{R}^d, 2p \cdot e \leq \frac{\Delta_{he} v_h(x)}{h} \forall e \in V \,\}.$$

and define the symmetric version of the discrete R-curvature

$$\omega_s(R, v_h, x) = \int_{D_V v_h(x)} R(p) dp, x \in \Omega_h.$$

Localization for $R = 1$ and the Dirichlet problem was shown to be very efficient by Mirebeau (2015) in 3D.

Intimately related with Lattice Basis Reduction Monge-Ampère operator MALBR

$$\omega_s(R, v_h, x) \leq MALBR[v_h](x),$$

with equality for quadratic polynomials.

Differences with Benamou-Duval

No discretization of the gradient

No ”boundary condition” is imposed. The only unknowns are mesh values at interior grid points.

Only the discrete Monge-Ampère operator is enforced at grid points

Need of the symmetrization in Mirebeau's scheme was motivated by enforcing Dirichlet b.c.

For the second boundary value problem, this does not seem necessary.

$$\partial_V v_h(y) = \{ p \in \mathbb{R}^d, p \cdot (he) \geq v_h(y) - v_h(y - he) \, \forall e \in V \},$$

and consider the asymmetrical version of the discrete R-curvature :

$$\omega_a(R, v_h, E) = \int_{\partial_V v_h(E)} R(p) dp.$$

Additional motivation : existence, stability, uniqueness and convergence are essentially the same as for convex polyhedral approximations.

The non symmetric discrete R-curvature suggested a new Lattice basis reduction Monge-Ampère operator for which one can prove existence and uniqueness results without restricted assumptions.

Effect of numerical integration in evaluating $\int_{\partial_V v_h(E)} R(p)dp$.

G. Awanou (2019) Convergence of a lattice basis reduction scheme for asymptotic cones of Monge-Ampère functions. (In preparation) find $u_h \in \mathcal{C}_h$ with asymptotic cone K such that

$$\omega_a(R, u_h, \{ x \}) = \int_{C_x} \tilde{f}(t) dt, x \in \Omega_h,$$

where C_x with $C_x \cap \Omega_h = \{ x \}$ form a partition of Ω.

Discrete convex mesh functions with asymptotic cone K are Lipschitz continuous with a uniform Lipschitz bound, i.e.

$$|v_h(x) - v_h(y)| \leq C||x - y||_1.$$

Proof similar to Benamou-Duval 2018.

Convergence

Require $u_h(x^1) = \alpha$

The mesh functions u_h are uniformly Lipschitz for having asymptotic cone K, the family u_h is a uniformly bounded sequence of discrete convex functions.

There is a subsequence u_{h_k} which converges uniformly on compact subsets to a convex function $v \in C(\overline{\Omega})$. We have

$$\omega(R, u_h, \{\, x \,\}) \leq \omega_a(R, u_h, \{\, x \,\}) = \int_{C_x} \tilde{f}(t) dt, x \in \Omega_h.$$

$$\omega(R, v, E) \leq \int_E \tilde{f}(t) dt,$$

Since u_h is extended to $\mathbb{Z}_h^d$ using the extension formula, the limit function v has asymptotic cone K and thus

$$\omega(R, v, \overline{\Omega}) = \int_{K^*} R(p)dp = \omega(R, u, \overline{\Omega}).$$

We have $\omega(R, v, E) \leq \omega_a(R, u, E)$ for all Borel sets $E \subset \overline{\Omega}$. Thus, it is not possible to have $\omega(R, v, E) < \omega_a(R, u, E)$ for a Borel set E.

Arguments do not require regularity of the exact solution.

$G : \mathbb{U} \to \mathbb{R}^M$ with $G(v_h) = (G_i(v_h))_{i=1,\ldots,M}$.

$$\mathbb{U}_\epsilon = \{ v_h \in \mathbb{U}, G_i(v_h) > -\epsilon, i = 1, \ldots, M \},$$

for a parameter $\epsilon > 0$

1. Choose $v_h^0 \in \mathbb{U}_\epsilon$ and set $k = 0$
2. **If** $G(v_h^k) = 0$ **stop**
3. Let i_k be the smallest non-negative integer i such that $p^k(\rho^i) \in \mathbb{U}_\epsilon$ and

   $$||G(p^k(\rho^i))|| \leq (1 - \delta\rho^i)||G(v_h^k)||.$$

   Set $v_h^{k+1} = p^k(\rho^{i_k})$
4. $k \leftarrow k + 1$ and **go to** 2

Basic convergence result

Theorem

Let $G \in C^1(\mathbb{U}_{\epsilon/2}, \mathbb{R}^M)$ and assume that G is a proper map with a unique zero u_h in $\overline{\mathbb{U}_\epsilon}$ and $\det G'(x) \neq 0$ for all $x \in \mathbb{U}_{\epsilon/2}$. Assume that there exists $\check{\tau}_k$ in $(0, 1]$ such that for all $0 < \tau \leq \check{\tau}_k, p_k(\tau) \in \mathbb{U}$.

$$||v_h^{k+1} - u_h|| \leq C\, o(||v_h^k - u_h||),$$

for $k \geq k_0$ if $i_k = 0$ for $k \geq k_0$ and k_0 sufficiently large. For k sufficiently large, with step 3 replaced with $i_k = 0$, i.e. a full Newton's step, the convergence is guaranteed to be at least linear.

Mirebeau 2015 and Butikofer ph.d thesis 2008.

Improvements possible using results by Kitagawa, Merigot and Thiebert. May not hold for more general equations.

$$G_i(v_h) = \omega_a(R, v_h, \{ x^i \}) - \int_{C_{x^i}} f(t)dt.$$

We now make the assumption that

there exists a constant $\epsilon > 0$ such that $f > 2\epsilon/h^d$ on Ω.

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there exists a constant $\epsilon > 0$ such that $f > 2\epsilon/h^d$ on Ω.

C^1 continuity of G uses some lemma from Merigot, Meyron and Thiebert 2018

Invertibility of the Jacobian matrix is done as in Mirebeau 2015 by showing that it is a weakly chained diagonally dominant matrix.

G is proper as a consequence of the uniform boundedness of discrete convex mesh functions with asymptotic cone K.

$$g(Du(x)) \det D^2 u(x) = f(x), Du(\Omega) = \Omega^*$$

Prins, Beltman, Thije Boonkkamp, Ijzerman, and Tukker 2015 : use of standard discretizations and a least squares approach

Lindsey and Rubinstein 2017, SIAM J. Math. Anal. : second boundary condition enforced as global constraints in an optimization framework

" our working proof involves new ideas that take us too far afield from the analysis in this paper and which are applicable more widely in the numerical analysis for nonlinear elliptic PDEs. "

For the Dirichlet problem

$$\det D^2 u(x) = f(x), x \in \Omega, u = g \text{ on } \partial\Omega,$$

solutions can be computed safely with a time marching iterative method

G. Awanou (2015). Pseudo transient continuation and time marching methods for Monge-Ampere type Equations, Advances in Comp. Math.

generalizing earlier work

C.J.Budd and J.F. Williams, (2008). Moving mesh generation using the Parabolic Monge-Ampere equation

P. Marcellini and K. Miller (1997). Elliptic versus parabolic regularization for the equation of prescribed mean curvature.

Theory was supported (with some limitations) by approximation by smooth functions.

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Thank you

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