# *Structure preserving methods and equivariance*

*Some ideas and examples*

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## Section 1

## *Warm-up examples*

$$I : F \to \mathbb{R}, \qquad I[\alpha, \beta, f] = \int_\alpha^\beta f(x)\,dx$$

Action by $\mathbf{a} = (a, b) \in \mathcal{A}(1)$ on $F$ and $\mathbb{R}$

$$\mathbf{a} \cdot (\alpha, \beta, f) = (a\alpha + b, a\beta + b, f \circ \mathbf{a}^{-1})$$
$$\mathbf{a} \cdot v = T\mathbf{a}(v) = av$$

$$\int_{a\alpha+b}^{a\beta+b} f(\frac{y-b}{a})\,dy = a\int_\alpha^\beta f(x)\,dx$$

or $\qquad I[\mathbf{a} \cdot (\alpha, \beta, f)] = \mathbf{a} \cdot I[(\alpha, \beta, f)]$

We can use a standard linear quadrature formula with $n+1$ points

$$Q_n[\alpha, \beta, f] = h \sum_{k=0}^{n} w_k f(\alpha + kh), \quad h = \frac{\beta - \alpha}{n}$$

One easily sees that

$$Q_n[\mathbf{a} \cdot (\alpha, \beta, f)] = \mathbf{a} \cdot Q_n[\alpha, \beta, f]$$

Let

$$\dot{y} = f(y), \quad y(0) = y^0, \quad y \in \mathbb{R}^m$$

Affine transformation, $(A, b) \in \mathcal{A}(m)$

$$y = Az + b, \quad A \in \mathbb{R}^{m \times m},\ b \in \mathbb{R}^m$$

$$\begin{array}{ccc} \dot{y} = f(y) & \xrightarrow{\text{Euler}} & y^n = y^{n-1} + hf(y^{n-1}) \\ {\scriptstyle y=Az+b}\Big\uparrow & & {\scriptstyle y^n=Az^n+b}\Big\uparrow \\ \dot{z} = A^{-1}f(Az+b) & \xrightarrow{\text{Euler}} & z^n = z^{n-1} + hA^{-1}f(Az^{n-1}+b) \end{array}$$

Euler commutes with affine transformations

## Section 2

## *Equivariance*

The category of $G$-spaces is the set of all pairs $(G, X)$ where $G$ is fixed Lie group and $X$ is any space acted upon by $G$. In most cases of interest to us, $X$ is a manifold.
Equivariant maps are the maps $\psi$ that makes the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow{\psi} & & \downarrow{\psi} \\ Y & \xrightarrow{g} & Y \end{array}$$

$X$ is the Problem/Data space and $Y$ is the Computation space

| Problem space | Computation space |
|---|---|
| Manifold $M$ | Manifold $N$ |
| Functions on $M$ | Functions on $N$ |
| Vector fields on $M$ | Diffeomorphisms on $M$ |
| Vector fields on $M$ | Functions on $M$ |
| Geometric domain/Graph $\Omega$ | Linear operator on $\Omega$ |

### *Two situations*

1. There is an "exact solution", a map from the problem space to the computation space which is $G$-equivariant. Our goal is to preserve this "structure" in numerical methods.
2. No underlying "exact solution", our motivation for obtaining $G$-equivariance has a different origin.

# Section 3

*Affine equivariance in numerical integrators*

## Subsection 1

*The case of B-series methods*

McLachlan, Modin, Munthe-Kaas, Verdier (2016)
Munthe-Kaas and Verdier (2016)
We have seen that the Euler method is an affine equivariant map

$$\Psi_E : \mathcal{X}(\mathbb{R}^m) \to \mathrm{Diff}(\mathbb{R}^m)$$

This is an example of a method belonging to the very rich class of B-series methods, that can be formally expanded as

$$y \mapsto y + \sum_{t \in T} \frac{h^{|t|}}{\alpha}(t) F(t)[f](y)$$

| | |
|---|---|
| $T$ | the set of non-planar rooted trees |
| $F(t)$ | Elementary differential operators |
| $\alpha(t)$ | Coefficients that determine the map |

It is an easy exercise to prove inductively that for any B-series map $\Psi_h(f)$ with a formal expansion

$$\Psi_h(\mathbf{a} \cdot f) = \mathbf{a} \cdot \Psi_h(f)$$

for affine transformations $\mathbf{a}$.
Interesting conjecture was resolved in 2016: Are "all equivariant maps" B-series maps?
The answer turned out to be no. For instance, the method

$$\Psi_h(y) = y + h \,\mathrm{div}(f)(y)\, f(y)$$

is affine equivariant, but not a B-series map.
This example is an instance of a map with an aromatic series

Note that most integrators, including those with a B-series, are defined on linear spaces of arbitrary dimension.
One can generalise affine equiavariance to full affine equivariance.
Replace the affine group $\mathcal{A}(m)$ by the family of affine maps $\mathbf{A} : \mathrm{R}^m \to \mathbb{R}^p$

$$\mathbf{A}(x) = Ax + b,\ \ A \in \mathbb{R}^{p\times m},\ \ b \in \mathbb{R}^p$$

One must replace pullbacks/pushforwards with relatedness, $f \rightsquigarrow \tilde{f}$

$$f \xrightarrow{\mathbf{a}} \tilde{f} \quad \text{means} \quad T\mathbf{a} \circ \tilde{f} = f \circ \mathbf{a}$$

Similarly for diffeomorphisms

$$\varphi \xrightarrow{\mathbf{a}} \tilde{\varphi} \quad \text{means} \quad a \circ \tilde{f} = f \circ \mathbf{a}$$

Full affine equivariance of the map $\Psi$

$$f \xrightarrow{\mathbf{a}} \tilde{f} \quad \Leftrightarrow \quad \Psi(f) \xrightarrow{\mathbf{a}} \Psi(\tilde{f})$$

In this sense, B-series maps are precisely the full affine equivariant maps

Let $(G, M)$ be a $G$-space where $M$ is a smooth manifold.
We have

| | |
|---|---|
| Problem space | Vector fields on $M$ |
| Computation space | Diffeomorphisms on $M$ |
| Group action on $M$ | $x \mapsto \phi_g(x)$ |
| Action on v.f.'s ($\Gamma(TM)$) | $F \mapsto \phi_{g*}F = T\phi_g \circ F \circ \phi_g^{-1}$ |
| Action on $\mathrm{Diff}(M)$ | $\psi \mapsto \phi_g \circ \psi \circ \phi_g^{-1}$ |

## *The Lie-Euler method*

$$\Psi_{LE}(y) = \exp(hF|_y)(y)$$

is and equivariant map between the $G$-sets

$$(G, \Gamma(TM)) \to (G, Diff(M))$$

## Subsection 2

*Where equivariance seems irrelevant*

In situations where "extra information" is required for the method to work, it is not so clear how to think about equivariance.

*Example (Splitting methods.)*

$$\dot{y} = f(y) = a(y) + b(y)$$

One adds information (e.g. $a(y)$) that affects the method, say

$$y^{n+1} = \exp(ha)\exp(hb)\,y^n$$

.

How should the problem transform under a group action?

When the method for instance depends on

1. An arbitrary partitioning of phase space
2. The vector field has a definition depending on extra structure that is used in the method

*Example (Lie-Poisson systems)*

$$\dot{y} = B(y) \cdot \nabla H(y)$$

## Section 4

## *Searching for invariant measures of (Kahan) maps*

The method is applied to quadratic vector fields in $\mathbb{R}^n$.

$$\dot{x} = Q(x, x) + Bx + c,$$

where $Q$ is a bilinear, symmetric, $\mathbb{R}^n$-valued form, $B$ is an $n \times n$-matrix and $c$ is an $n$-vector. The method reads

$$\frac{x^{k+1} - x^k}{h} = Q(x^k, x^{k+1}) + B\frac{x^k + x^{k+1}}{2} + c$$

Kahan's method is linearly implicit and generates a birational map.

Important remark. The Kahan method is the restriction of a Runge–Kutta method to quadratic systems, and therefore has a B-series and is equivariant w.r.t. affine transformations.

We always consider quadratic vector fields which preserve a measure

$$f(x) = Q(x, x) + Bx + c$$

can be

1. Standard volume

$$\nabla \cdot f = 0$$

2. Hamiltonian (canonical), i.e. for a cubic function $H : \mathbb{R}^n \to \mathbb{R}$

$$f = J\,\nabla H, \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

3. Poisson systems

$$f = J(x)\nabla H(x), \quad \text{e.g. Nambu system} \quad f = \nabla H_1 \times \nabla H_2$$

The perturbed Hamiltonian

$$\tilde{H}(x) = H(x) + \frac{h}{3}\nabla H^T \left( I - \frac{h}{2} f'(x) \right)^{-1} f(x)$$

is exactly preserved by Kahan's method, i.e.

$$\tilde{H}(x^k) = \tilde{H}(x^0), \quad \text{for all } k.$$

Kahan's method also preserves the perturbed measure

$$\frac{dx_1 \wedge \cdots \wedge dx_n}{\left| I - \frac{h}{2} f'(x) \right|}$$

Homogeneous Nambu systems

$$\dot{x} = \nabla H_1 \times \nabla H_2, \quad H_1 = x^T A x,\ \ H_2 = x^T B x$$

The ODE system preserves the measure $dx_1 \wedge dx_2 \wedge dx_3$ and has first integrals $H_1(x)$ and $H_2(x)$.
Kahan's method preserves the measure

$$\frac{dx_1 \wedge dx_2 \wedge dx_3}{(1 + 4h^2 H_3(x))^2}, \qquad H_3(x) = x^T C x, \quad C = A\,\mathrm{adj}(B)\,A$$

## Section 5

## *A search technique for finding preserved measures and integrals*

Simplified notation:

$$x' := x^{k+1}, \quad x := x^k, \quad \text{Kahan map: } x' = \phi(x)$$

Suppose that $\phi$ preserves the measure

$$\frac{dx_1 \wedge \cdots \wedge dx_n}{\mu(x)}$$

This means that

$$|D\phi(x)| = \frac{\mu(x')}{\mu(x)}$$

Ansatz: Since $D\phi$ is rational we assume that $g(x)$ is polynomial.

1. Basis $e_1, e_2, \ldots$ (e.g. monomials) for the multivariate polynomials up to some given degree. Set

$$\mu(x) = \sum_i \mu_i e_i \quad \text{for unknown parameters } g_i$$

2. Now, determine (if possible) $\mu_1, \mu_2, \ldots$ such that

$$|D\phi(x)|\mu(x) - (\mu \circ \phi)(x) = 0, \quad \forall x \in \mathbb{R}^n$$

A linear, but usually overdetermined homogeneous system of equations. Can in many interesting cases be solved by a CAS.

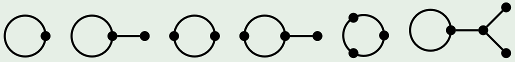
## Aromatic Kahan trees

- Aromas, are finite, connected, directed graphs where each node has precisely one outgoing edge.
- An aroma has to contain exactly one cycle.
- Multisets of such graphs represent scalar functions depending on a vector field $f$ and its derivatives.

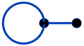
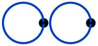
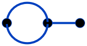
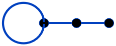
$$F : \mathcal{X}(\mathbb{R}^n) \times \mathcal{A} \to \mathcal{F}(\mathbb{R}^n), \text{ e.g. } F($$  $$) = f^i_j f^j_i$$

- Aromatic series are series indexed by multisets of aromas
- Kahan aromas are particular aromas with at most two incoming edges for each node

## Example



Quadratic vector field: $F = f^i \partial_i$

| $t$ | $F_t$ |
|---|---|
|  | $f^i_i = \nabla \cdot f$ |
|  | $f^i_{ij} f^j$ |
|  | $f^i_i f^j_j$ |
|  | $f^i_{jk} f^j_i f^k$ |
|  | $f^i_{ij} f^j_k f^k$ |

**Ansatz:** There exist preserved measure(s) where the density can be expressed as a finite aromatic series

$$\mu(x) = \sum_{|t| \leq p} \mu_t F(t)(x)$$

Recall condition

$$\mu \circ \phi = |\det D\phi| \cdot \mu$$

- $\mu \circ \phi$ is a composition of an aromatic series with a B-series and is therefore an aromatic series
- $|D\phi|$ has an aromatic series thanks to the Faddeev-LeVerrier formula
- The product of two aromatic series is again an aromatic series

We consider the affine right actions on $\mathcal{X}(\mathbb{R}^m)$ and $\mathcal{F}(\mathbb{R}^m)$.
$\varphi_{\mathbf{g}}(x) = Ax + b$, $\mathbf{g} = (A, b)$.

$$[f \cdot \mathbf{g}](x) = (\varphi_{\mathbf{g}}^* f)(x) = A^{-1} f(Ax + b),$$
$$[\mu \cdot \mathbf{g}](x) = (\varphi_{\mathbf{g}}^* \mu)(x) = \mu(Ax + b)$$

### *Equivariance of measures under the affine action*

Suppose that $\mu_f(x)$ is the density function of a preserved measure for the Kahan map, i.e.

$$|D\phi_f(x)| = \frac{\mu_f \circ \phi(x)}{\mu_f(x)}$$

Then

$$|D\phi_{f\cdot\mathbf{g}}(x)| = \frac{\mu_{f\cdot\mathbf{g}} \circ \phi(x)}{\mu_{f\cdot\mathbf{g}}(x)}$$

We now consider

### *Affine equivariance of aromatic functions*

Let $F(f,\tau)$ be the function corresponding to the aroma $\tau$. Then it hold that

$$F(f\cdot\mathbf{g},\tau) = F(f,\tau)\cdot\mathbf{g} \quad \text{for all aromas } \tau$$

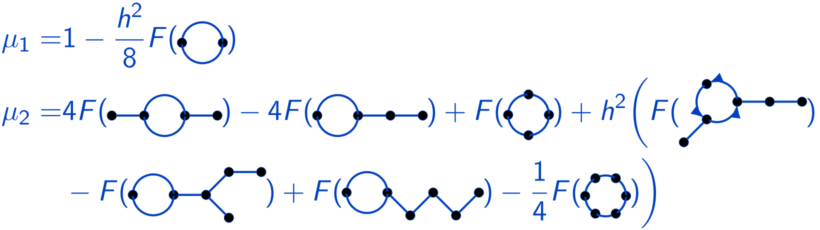
**3D periodic Volterra chain**

$$\dot{x} = x(y - z)$$
$$\dot{y} = y(z - x)$$
$$\dot{z} = z(x - y)$$

**Dressing chain**

$$\dot{x} = -y^2 + z^2$$
$$\dot{y} = x^2 - z^2$$
$$\dot{z} = -x^2 + y^2$$

Both yield the same aromatic densities with Kahan's method



Why? They are related by an affine transformation

## *Equivariance benefit 2. Independence of parameters due to affine invariance*

- Often one studies classes of vector fields (with parameters) and finds that the complexity gets too high for Maple to handle
- One can try values for parameter, but usually this fails to give the general answer
- With aromas, the answer is often represented easily and the aromatic expression does not depend on the parameters.
- One can then sometimes verify that the found expression holds in general

# Example: family of inhomogeneous Nambu problems

Two functions

$$H(x,y,z) = \alpha_{11}x^2 + \alpha_{22}y^2 + \alpha_{33}z^2 + \alpha_{12}xy + \alpha_{13}xz + \alpha_{23}yz$$
$$K(x,y,z) = \beta_{11}x^2 + \beta_{22}y^2 + \beta_{12}xy + \beta_1 x + \beta_2 y$$

ODE

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \nabla H \times \nabla K$$

We apply Kahans method to this problem and use the monomial basis to search for preserved measures. Here is what happens

- With a full set of parameters, Maple chokes and cannot return an answer
- So we reduce our ambitions and select random integer values for the $\alpha$'s and the $\beta$'s
- And we get an answer, which I give on the next slide
- As you will see, there is no reason for celebration

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> subs(rsol, Q)
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$$
\begin{aligned}
& r_0 - \frac{h^2 r_0 \left(17784 x^2 - 53856 x y - 12960 x z + 7056 y^2 - 10080 y z - 7200 z^2 - 35532 x - 5076 y + 21762\right)}{12} - \frac{1}{72}\Big(h^4 r_0 \big(-124874784 x^4 + 10513152 x^3 y - 153031680 x^3 z + 192357504 x^2 y^2 \\
& + 171175680 x^2 y z - 85017600 x^2 z^2 + 38216448 x y^3 + 157127040 x y^2 z + 161222400 x y z^2 - 36098784 y^4 - 53343360 y^3 z - 38102400 y^2 z^2 + 306444384 x^3 + 333650016 x^2 y + 395642880 x^2 z \\
& + 155491488 x y^2 + 484548480 x y z + 219801600 x z^2 + 69177888 y^3 + 137531520 y^2 z + 98236800 y z^2 + 308100024 x^2 + 636982704 x y + 848206080 x z + 224549496 y^2 + 659715840 y z \\
& + 471225600 z^2\big)\Big) + \frac{1}{36}\Big(h^4 r_0 \big(-249749568 x^4 + 21026304 x^3 y - 306063360 x^3 z + 384715008 x^2 y^2 + 342351360 x^2 y z - 170035200 x^2 z^2 + 76432896 x y^3 + 314254080 x y^2 z + 322444800 x y z^2 \\
& - 72197568 y^4 - 106686720 y^3 z - 76204800 y^2 z^2 - 54000432 x^3 + 711195552 x^2 y + 543192480 x^2 z + 445867416 x y^2 + 560234880 x y z + 301773600 x z^2 - 131792184 y^3 + 107140320 y^2 z \\
& + 76528800 y z^2 + 712526220 x^2 - 371284560 x y + 212576400 x z + 101588580 y^2 + 165337200 y z + 118098000 z^2 - 146257326 x + 42965802 y\big)\Big) \\
& - \frac{27 h^4 r_0 \left(988 x^2 - 2992 x y - 720 x z + 392 y^2 - 560 y z - 400 z^2 - 1974 x - 282 y + 1209\right)^2}{16} - \frac{1}{384}\Big(h^6 r_0 \big(-310203240192 x^6 + 2165815563264 x^5 y + 268637783040 x^5 z + 459695969280 x^4 y^2 \\
& + 3651447859200 x^4 y z + 645065856000 x^4 z^2 - 2818847278080 x^3 y^3 - 3027965068800 x^3 y^2 z + 1743534720000 x^3 y z^2 + 550914048000 x^3 z^3 - 472020151680 x^2 y^4 - 3196318579200 x^2 y^3 z \\
& - 4110160320000 x^2 y^2 z^2 - 616232448000 x^2 y z^3 + 153031680000 x^2 z^4 + 652910782464 x y^5 + 1130815468800 x y^4 z + 466466688000 x y^3 z^2 - 565657344000 x y^2 z^3 - 290200320000 x y z^4 \\
& - 17562158208 y^6 + 128631525120 y^5 z + 226304928000 y^4 z^2 + 192036096000 y^3 z^3 + 68584320000 y^2 z^4 + 3529171487808 x^5 - 1750219125120 x^4 y + 4475433409920 x^4 z - 6313246958880 x^3 y^2 \\
& - 6341598293760 x^3 y z + 1204468963200 x^3 z^2 - 2837975654880 x^2 y^3 - 11137805233920 x^2 y^2 z - 8023896547200 x^2 y z^2 - 1424314368000 x^2 z^3 + 8076970080 x y^4 - 1380734864640 x y^3 z \\
& - 3610047657600 x y^2 z^2 - 1744374528000 x y z^3 - 395642880000 x z^4 + 485027914464 y^5 + 1042223656320 y^4 z + 397866038400 y^3 z^2 - 495113472000 y^2 z^3 - 176826240000 y z^4 - 3937795898256 x^4 \\
& - 5997125708352 x^3 y - 5809921914240 x^3 z - 6625238091984 x^2 y^2 - 14986619170560 x^2 y z - 5975922096000 x^2 z^2 - 1760755239648 x y^3 - 8274332191680 x y^2 z - 10090398096000 x y z^2 \\
& - 3053541888000 x z^3 + 29198444544 y^4 - 103224300480 y^3 z - 1736215560000 y^2 z^2 - 2374977024000 y z^3 - 848206080000 z^4 - 643155479424 x^3 - 3695938425936 x^2 y - 4434372047520 x^2 z \\
& - 1893947738688 x y^2 - 3841724790720 x y z - 2463540026400 x z^2 + 15617314512 y^3 - 305486808480 y^2 z - 218204863200 y z^2 + 1676218180572 x^2 + 3465504401112 x y + 4614665178240 x z \\
& + 1221661532988 y^2 + 3589184027520 y z + 2563702876800 z^2\big)\Big) - \frac{729 h^6 r_0 \left(588 x^2 + 68 x y - 328 y^2 + 3212 x - 2454 y - 893\right)\left(294 x^2 + 34 x y - 164 y^2 - 337 x + 99 y\right)\left(9 x + 7 y + 10 z\right)^2}{8} \\
& + \frac{1}{384}\Big(h^6 r_0 \big(179776858752 x^6 + 3041335558656 x^5 y + 1357482447360 x^5 z + 392301377280 x^4 y^2 + 4750168665600 x^4 y z + 1249979558400 x^4 z^2 - 3653650022400 x^3 y^3 - 4032291456000 x^3 y^2 z \\
& + 1883446732800 x^3 y z^2 + 550914048000 x^3 z^3 - 744614830080 x^2 y^4 - 4270293043200 x^2 y^3 z - 4776939878400 x^2 y^2 z^2 - 616232448000 x^2 y z^3 + 153031680000 x^2 z^4 + 851836663296 x y^5 \\
& + 1360362988800 x y^4 z + 388420531200 x y^3 z^2 - 565657344000 x y^2 z^3 - 290200320000 x y z^4 + 74670035328 y^6 + 392152078080 y^5 z + 414533894400 y^4 z^2 + 192036096000 y^3 z^3 + 68584320000 y^2 z^4 \\
& + 5644085588352 x^5 - 95694488640 x^4 y + 9175242522240 x^4 z - 8974873548000 x^3 y^2 - 6320283966720 x^3 y z + 3815474025600 x^3 z^2 - 4951942452000 x^2 y^3 - 17069108797440 x^2 y^2 z \\
& - 10042836969600 x^2 y z^2 - 1424314368000 x^2 z^3 + 794141457120 x y^4 - 1465202753280 x y^3 z - 5334929308800 x y^2 z^2 - 1744374528000 x y z^3 - 395642880000 x z^4 + 1119405440736 y^5 \\
& + 2854730874240 y^4 z + 1692514051200 y^3 z^2 - 495113472000 y^2 z^3 - 176826240000 y z^4 - 616369954896 x^4 - 7407860380992 x^3 y + 1571024626560 x^3 z - 11798603154864 x^2 y^2 \\
& - 23862321308160 x^2 y z - 1875396240000 x^2 z^2 - 996965959968 x y^3 - 12867375113280 x y^2 z - 18210641616000 x y z^2 - 3053541888000 x z^3 + 1873585124064 y^4 + 5166451926720 y^3 z \\
& + 2027838888000 y^2 z^2 - 2374977024000 y z^3 - 848206080000 z^4 - 3068770301424 x^3 - 5482222114176 x^2 y - 9824627207520 x^2 z - 270569539728 x y^2 - 3618823417920 x y z - 5458126226400 x z^2 \\
& + 1217566095552 y^3 + 3128652565920 y^2 z + 2234751832800 y z^2 + 2052926708580 x^2 + 4051495444680 x y + 5451795240480 x z + 1449546938820 y^2 + 4240285187040 y z + 3028775133600 z^2\big)\Big)
\end{aligned}
$$

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>
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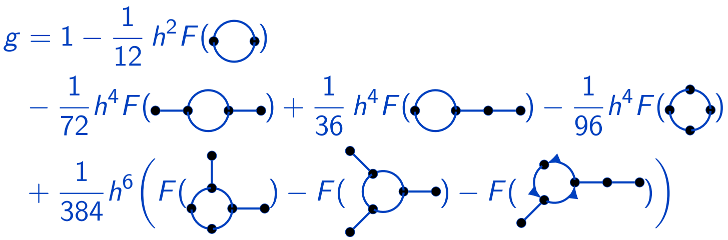
Here is the preserved measure we get for the inhomogeneous Nambu system



Section 6

## *Group Convolutional Neural Networks*

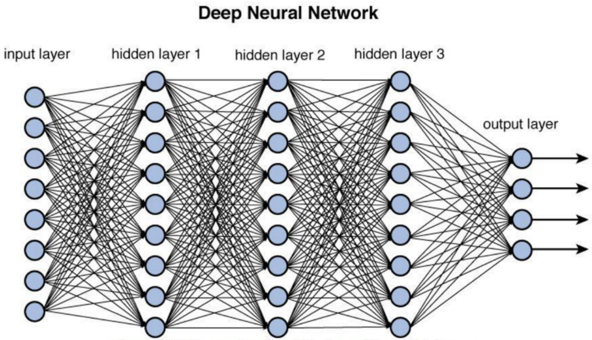



Figure 12.2 Deep network architecture with multiple layers.

"Data/problem space" and "Computational space" are consecutive layers, say $\mathcal{I}_1$ and $\mathcal{I}_2$.
Cohen & Welling ++ take $\mathcal{I}_i$ to consist of sections of certain vector bundles with an induced group action.

We omit the details of the construction of the bundles, the main point is that there is a group $G$ with two actions

$$\pi_i : G \times \mathcal{I}_i \to \mathcal{I}_i, \quad (u, f_i) \mapsto \pi_i(u) f_i$$

The map between layers is convolution

$$f_2(g) = [\mathcal{K} \cdot f_1](g) = \int_G \kappa(g, g') f_1(g') \, dg'$$

where $\kappa : G \times G \to \mathrm{Hom}(\mathcal{I}_1, \mathcal{I}_2)$ is the convolution kernel.

The G-CNN is equivariant if

$$[\mathcal{K} \cdot (\pi_1(u) f_1)](g) = \pi_2(u)[\mathcal{K} \cdot f_1](g)$$

There is an induced left action by $G$ on the sections of $P$ realised as

$$[\pi_G(g)f](k) = f(g^{-1}k)$$

Convolution maps $\mathcal{I}_1$ to $\mathcal{I}_2$ (layer to layer)
Let $f_1 \in \mathcal{I}_1$

$$f_2(g) = [\mathcal{K} \cdot f_1](g) = \int_G \kappa(g, g') f_1(g')\, dg'$$

$$\mathcal{K} \cdot \pi_G[u]f = \pi_G(u)[\mathcal{K} \cdot f]$$

For notational simplicity we assume in this condition that $\mathcal{I}_1 = \mathcal{I}_2$

Imposing equivariance implies

1. $\kappa(g, g')$ is a cross-correlation, meaning that it is translation invariant

$$\kappa(ug, ug') = \kappa(g, g') = \kappa(e, g^{-1}g') =: \bar{\kappa}(g^{-1}g')$$

2. If the bundles $\mathcal{I}_i$ have homogeneous base manifolds $G/H_1$, $G/H_2$ the convolution kernel $\bar{\kappa}$ can only depend on the double coset space $H \backslash G / H$.

**Example ($G = SO(3), H = SO(2)$)**

The double coset space $SO(2)\backslash SO(3)/SO(2)$ consists of the latitude angles $\phi \in [0, \pi]$ of the sphere.

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