Unipotent Classes and Representations

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7th January 2019

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Classes and Characters

Setup:

- G a finite group
- $\mathrm{Irr}(G)$ = characters of irreducible $\mathbb{C}$-representations of G
- $\mathrm{Class}(G) = \{f : G \to \mathbb{C} \mid f(gxg^{-1}) = f(x) \text{ all } g, x \in G\}$

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An Interlocutor:

Conjugacy Classes

Irreducible Characters

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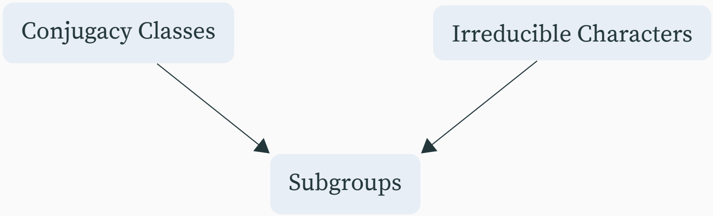

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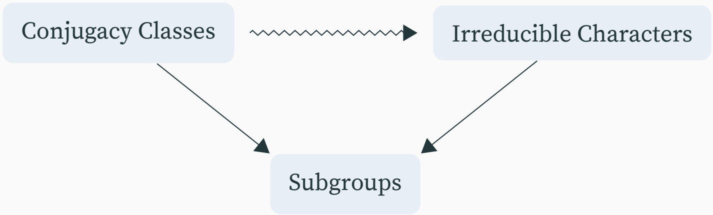

The Symmetric Group

$(W, \mathbb{S}) = (\mathfrak{S}_n, \{s_1, \ldots, s_{n-1}\})$ with $s_i = (i, i+1)$

$$W = \langle s_i \in \mathbb{S} \mid s_i^2 = (s_i s_j)^2 = (s_i s_{i+1})^3 = 1 \text{ if } |i-j| > 1 \rangle$$

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Subset $\mathbb{I} \subseteq \mathbb{S}$:

- $W_{\mathbb{I}} \leqslant W$ generated by $\mathbb{I}$
- $W_{\mathbb{I}} \cong \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k}$ with $n_1 + \cdots + n_k = n$
- $\mathbb{I} \to \lambda_{\mathbb{I}} = (n_1, \ldots, n_k) \vdash n$

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For any $\mathbb{I}, \mathbb{J} \subseteq \mathbb{S}$:

- $W_{\mathbb{I}}, W_{\mathbb{J}}$ are W-conjugate if and only if $\lambda_{\mathbb{I}} = \lambda_{\mathbb{J}}$
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$$\mathcal{O} \subseteq W \quad \rightsquigarrow \quad W_{\mathbb{I}} = \bigcap_{\mathcal{O} \cap W_{\mathbb{J}} \neq \emptyset} W_{\mathbb{J}} \quad \rightsquigarrow \quad \lambda_{\mathbb{I}} \vdash n$$

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Example

Consider $W = \mathfrak{S}_4$

$w \in \mathcal{O}$	$\mathbb{I}$	$W_{\mathbb{I}}$	$\lambda_{\mathbb{I}}$
$(1,2,3,4)$	$\mathbb{S}$	W	(4)
$(1,2,3)$	$\{(1,2),(2,3)\}$	$\mathfrak{S}_3$	(31)
$(1,2)(3,4)$	$\{(1,2),(3,4)\}$	$\mathfrak{S}_2 \times \mathfrak{S}_2$	(22)
$(1,2)$	$\{(1,2)\}$	$\mathfrak{S}_2$	(211)
$()$	$\emptyset$	$\{1\}$	(1111)

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Observation

$(1^n) = \lambda_1 \leqslant \cdots \leqslant \lambda_r = (n)$ the lexicographic order

$$(\langle \Theta_{\lambda_i}, \rho_{\lambda_j} \rangle) = \begin{bmatrix} 1 & * & \dots & * \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathsf{GL}_r(\mathbb{Z}) \quad \rightsquigarrow \quad \rho_{\lambda_j} = \sum_{j \leqslant i} m_{ji} \cdot \Theta_{\lambda_i}$$

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Corollary

$\rho(w) \in \mathbb{Z}$ for all $w \in W$ and $\rho \in \mathsf{Irr}(W)$

Duality

Tensoring with Sign

We have an involution

$$\mathsf{Irr}(W) \to \mathsf{Irr}(W)$$
$$\rho_\lambda \mapsto \rho_{\lambda^\star} := \rho_\lambda \otimes \mathsf{sgn}_W$$

where $\lambda^\star = (\lambda_1^\star, \ldots, \lambda_n^\star)$ is the transpose partition.

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$\lambda = (5, 4, 4, 2) \quad \rightsquigarrow \quad \lambda^\star = (4, 4, 3, 3, 1)$

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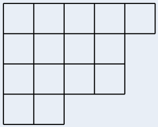

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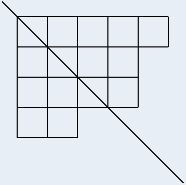

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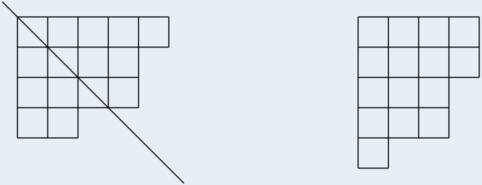

General Linear Group

- $\mathbb{F} = \overline{\mathbb{F}_p} \quad \rightsquigarrow \quad \mathbf{G} = \mathrm{GL}_n(\mathbb{F})$
- Frobenius: $F : \mathbf{G} \to \mathbf{G}$ given by $F(a_{ij}) = (a_{ij}^q) \quad \rightsquigarrow \quad \mathbf{G}^F = \mathrm{GL}_n(\mathbb{F}_q)$

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Special Subgroups

Torus and Borel

$$\mathbf{T} = \left\{ \begin{bmatrix} * & 0 & \dots & 0 \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & * \end{bmatrix} \right\} \quad \leqslant \quad \mathbf{B} = \left\{ \begin{bmatrix} * & * & \dots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{bmatrix} \right\}$$

and quotient $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T} \cong \mathfrak{S}_n$

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Parabolics

Subset $\mathbb{I} \subseteq \mathbb{S}$:

- $\mathbf{P}_{\mathbb{I}} = \mathbf{L}_{\mathbb{I}} \ltimes \mathbf{U}_{\mathbb{I}}$ a subgroup of $\mathbf{G}$ containing $\mathbf{B}$
- $\mathbf{L}_{\mathbb{I}} \cong \mathsf{GL}_{n_1}(\mathbb{F}) \times \cdots \times \mathsf{GL}_{n_k}(\mathbb{F})$ with $n_1 + \cdots + n_k = n$
- $W_{\mathbb{I}} = \mathsf{N}_{\mathbf{L}_{\mathbb{I}}}(\mathbf{T})/\mathbf{T} \leqslant \mathsf{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T} = W$

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Example

Consider $\mathbf{G} = \mathrm{GL}_4(\mathbb{F})$

$$\begin{bmatrix} 1 & 1 & . & . \\ . & 1 & 1 & . \\ . & . & 1 & 1 \\ . & . & . & 1 \end{bmatrix} \in \mathcal{O}_{(4)} \qquad \begin{bmatrix} 1 & 1 & . & . \\ . & 1 & . & . \\ . & . & 1 & 1 \\ . & . & . & 1 \end{bmatrix} \in \mathcal{O}_{(2^2)}$$

$$\mathbb{I} \subseteq \mathbb{S} \quad \rightsquigarrow \quad \mathbf{P} = \mathbf{P}_{\mathbb{I}} \quad \rightsquigarrow \quad \mathcal{R}_\lambda = \mathrm{Ind}_{\mathbf{P}^F}^{\mathbf{G}^F}(1)$$

Harish-Chandra Induction

If $\mathbf{L} = \mathbf{L}_{\mathbb{I}}$ then

$$\underbrace{R_{\mathbf{L}}^{\mathbf{G}}}_{\text{Induction}} = \mathrm{Ind}_{\mathbf{P}^F}^{\mathbf{G}^F} \circ \mathrm{Inf}_{\mathbf{L}^F}^{\mathbf{P}^F} \qquad \text{and} \qquad \underbrace{{}^*R_{\mathbf{L}}^{\mathbf{G}}}_{\text{Restriction}} = \mathrm{Def}_{\mathbf{L}^F}^{\mathbf{P}^F} \circ \mathrm{Res}_{\mathbf{P}^F}^{\mathbf{G}^F}$$

Proposition

For each $\lambda \vdash n$ there is a unique (unipotent) character $\chi_\lambda \in \mathrm{Irr}(G)$ such that

$$\mathcal{R}_\lambda = \sum_{\mu \vdash n} \langle \Theta_\lambda, \rho_\mu \rangle \cdot \chi_\mu$$

Example

- $\chi_{(n)} = 1_{\mathbf{G}^F}$
- $\chi_{(1^n)} = \mathrm{St}_{\mathbf{G}^F} \quad \rightsquigarrow \quad \mathrm{St}_{\mathbf{G}^F}(1) = q^{n(n-1)/2}$

Some Observations

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For any $x \in \mathbf{G}^F$ we have

$$\mathcal{R}_\lambda(x) = \#\{g\mathbf{P}^F \in \mathbf{G}^F/\mathbf{P}^F \mid x^g \in \mathbf{P}^F\} > 0$$

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$\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n \quad \rightsquigarrow \quad \mathbf{P}_\lambda = \mathbf{L}_\lambda \ltimes \mathbf{U}_\lambda$ with

$$\mathbf{U}_\lambda := \left\{ \begin{bmatrix} I_{\lambda_1} & \star & \ldots & \star \\ 0 & I_{\lambda_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \star \\ 0 & \cdots & 0 & I_{\lambda_r} \end{bmatrix} \right\}$$

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Question

For which $x \in \mathbf{G}^F$ is $x^g \in \mathbf{U}_\lambda^F$ for some $g \in \mathbf{G}^F$?

$$\mathbb{I} \subseteq \mathbb{S} \quad \rightsquigarrow \quad \mathbf{U} = \mathbf{U}_{\mathbb{I}} \quad \rightsquigarrow \quad \mathrm{Ind}_{\mathbf{U}^F}^{\mathbf{G}^F}(1)?$$

$\lambda = (\lambda_1, \dots, \lambda_r) \vdash n \quad \rightsquigarrow \quad \mathbf{P}_\lambda = \mathbf{L}_\lambda \ltimes \mathbf{U}_\lambda$ with

$$\mathbf{U}_\lambda := \left\{ \begin{bmatrix} I_{\lambda_1} & \star & \dots & \star \\ 0 & I_{\lambda_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \star \\ 0 & \cdots & 0 & I_{\lambda_r} \end{bmatrix} \right\}$$

Question

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Theorem

For any $A \in \mathcal{O}_\mu$ and $B \in \mathcal{O}_\lambda$:

- $A \leqslant_{\mathbf{G}} B$ if and only if $\mu \lhd \lambda$
- $A^{\mathbf{G}} \cap \mathbf{U}_\lambda \neq \emptyset$ if and only if $\mu \lhd \lambda^*$

Characterising a Jordan Block

$$N = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \qquad \rightsquigarrow \qquad \mathrm{rank}_{\mathbb{F}}(N^k) = n - k$$

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$\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n}) \vdash n$ then

$$A \in \mathcal{O}_\lambda \quad \Leftrightarrow \quad r_k(A) = n - \sum_{i=1}^{k} \lambda_i^\star \quad \text{for all } 1 \leqslant k \leqslant n$$

where $\lambda^\star = (\lambda_1^\star, \dots, \lambda_n^\star)$ and $\lambda_k^\star = m_k + \cdots + m_n$.

Duality Between Unipotent Classes

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Given $A \in \mathcal{O}_\mu$ and $B \in \mathcal{O}_\lambda$ we have

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Duality between unipotent classes:

Bala–Carter: $\mathbf{L}_\lambda$ ⟷ Richardson: $\mathbf{U}_\lambda$

Alvis–Curtis Duality

Definition

$$\mathbb{D}_{\mathbf{G}} = \sum_{\mathbb{I} \subseteq \mathbb{S}} (-1)^{r(\mathbf{P}_{\mathbb{I}})} \mathrm{R}_{\mathbf{L}_{\mathbb{I}}}^{\mathbf{G}} \circ {}^{*}\mathrm{R}_{\mathbf{L}_{\mathbb{I}}}^{\mathbf{G}}$$

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Example

If $\lambda \vdash n$ then $\mathbb{D}_{\mathbf{G}}(\chi_\lambda) = \chi_{\lambda^\star}$

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$$\mathrm{Ind}_{\mathbf{U}_\lambda^F}^{\mathbf{G}^F}(1)$$

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- $\Gamma_\lambda(1)_p = q^{d_\lambda}$ where $d_\lambda = \frac{1}{2}(\dim(\mathcal{O}_{(n)}) - \dim(\mathcal{O}_\lambda))$
- $\Gamma_\lambda(u) \neq 0$ implies $u^{\mathbf{G}} \leqslant_{\mathbf{G}} \mathcal{O}_\lambda$
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Which linear character?

$$\mathbf{V}^F/[\mathbf{V}^F, \mathbf{V}^F] \cong \underbrace{\mathbb{F}_q^+ \oplus \cdots \oplus \mathbb{F}_q^+}_{n-1 \text{ copies}}$$

Characterising Unipotents

Assume $0 \neq \langle \Gamma_\lambda, \chi_{\mu^\star} \rangle$ then

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Multiplicities

$(1^n) = \lambda_1 \leqslant \cdots \leqslant \lambda_r = (n)$ in lexicographic ordering

$$(\langle \Gamma_{\lambda_i^\star}, \chi_{\lambda_j} \rangle) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ * & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & 1 \end{bmatrix} \in \mathsf{GL}_r(\mathbb{Z})$$

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- $F : \mathbf{G} \to \mathbf{G} \quad \rightsquigarrow \quad F^* : \mathscr{D}_{\mathbf{G}}(\mathbf{G}) \to \mathscr{D}_{\mathbf{G}}(\mathbf{G})$
- $\phi : F^*A \to A$ an isomorphism. If $x \in \mathbf{G}^F$ then

$$\mathscr{H}^i_x(F^*A) = \mathscr{H}^i_{F(x)}(A) = \mathscr{H}^i_x(A).$$

We get an isomorphism $\phi : \mathscr{H}^i_x(A) \to \mathscr{H}^i_x(A)$ and the pair (A, ϕ) gives a characteristic function $\chi_{A,\phi} \in \mathrm{Class}(\mathbf{G}^F)$

$$\chi_{A,\phi}(x) = \sum_{i \in \mathbb{Z}} (-1)^i \,\mathrm{Tr}(\phi, \mathscr{H}^i_x(A))$$

Intersection Cohomology Complexes

Local Systems

$\mathcal{O} \subseteq \mathbf{G}$ a conjugacy class and $x \in \mathcal{O}$:

$$\{\text{Irreducible } \mathbf{G}\text{-equivariant Local Systems on } \mathcal{O}\} \longleftrightarrow \mathsf{Irr}(A_{\mathbf{G}}(x))$$

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- if $F(\mathcal{O}) = \mathcal{O}$ and $\phi : F^*\mathscr{L} \to \mathscr{L}$ an isomorphism $\quad \rightsquigarrow \quad \chi_{C_\iota,\phi} \in \mathrm{Class}(\mathbf{G}^F)$

$$\chi_{C_\iota,\phi}(u) \neq 0 \quad \Rightarrow \quad u \in \overline{\mathcal{O}}$$

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- Fix $u \in \mathcal{O}^F$ then $\mathscr{L} \rightsquigarrow \psi \in \mathsf{Irr}(A_{\mathbf{G}}(u))$. If $v = {}^g u \in \mathcal{O}^F$ then $g^{-1}F(g) \in C_{\mathbf{G}}(u)$ and $\mathsf{Tr}(\phi, \mathscr{L}_v) = \psi(\overline{g^{-1}F(g)})$

Inducing From a Torus

- $\mathbf{T} \leqslant \mathbf{G}$ an F-stable maximal torus

$$\mathsf{K} = \mathsf{ind}_{\mathbf{T}}^{\mathbf{G}}(\overline{\mathbb{Q}_\ell})[\dim(\mathbf{T})] \in \mathscr{D}_{\mathbf{G}}(\mathbf{G})$$

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- $\mathsf{End}_{\mathscr{D}_{\mathbf{G}}(\mathbf{G})}(\mathsf{K}) \cong \overline{\mathbb{Q}_\ell}[W] \quad \rightsquigarrow \quad \mathsf{K} \cong \bigoplus_{\rho \in \mathsf{Irr}(W)} \mathsf{K}_\rho^{\oplus \rho(1)}$ of character sheaves

Inducing From a Torus

- $\mathbf{T} \leqslant \mathbf{G}$ an F-stable maximal torus

$$\mathsf{K} = \mathsf{ind}_{\mathbf{T}}^{\mathbf{G}}(\overline{\mathbb{Q}_\ell})[\dim(\mathbf{T})] \in \mathscr{D}_{\mathbf{G}}(\mathbf{G})$$

- $\mathsf{End}_{\mathscr{D}_{\mathbf{G}}(\mathbf{G})}(\mathsf{K}) \cong \overline{\mathbb{Q}_\ell}[W] \quad \rightsquigarrow \quad \mathsf{K} \cong \bigoplus_{\rho \in \mathsf{Irr}(W)} \mathsf{K}_\rho^{\oplus \rho(1)}$ of character sheaves

Theorem (Lusztig)

For each $\rho \in \mathsf{Irr}(W)$ there is a unique $\iota \in \mathcal{N}_{\mathbf{G}}$ such that

$$\mathsf{K}_\rho|_{\mathbf{G}_{\mathsf{uni}}} \cong \mathsf{K}_\iota[\dim(\mathbf{T})].$$

This gives an injection $\mathsf{Irr}(W) \hookrightarrow \mathcal{N}_{\mathbf{G}}$ called the Springer correspondence.

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Corollary

Let $A = K_\rho$ and assume F acts trivially on W then there is an isomorphism $\phi : F^*A \to A$

$$\chi_{K_\rho,\phi} = \chi_{K_\iota[\dim(\mathbf{T})],\phi} \quad \rightsquigarrow \quad \chi_{K_\rho,\phi}(u) \neq 0 \Rightarrow u \in \overline{\mathcal{O}_\iota}$$

Unipotent Supports for $\mathrm{GL}_n(q)$

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Let $A = K_\rho$ and assume F acts trivially on W then there is an isomorphism $\phi : F^*A \to A$

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Theorem

Assume $\mathbf{G} = \mathrm{GL}_n(\mathbb{F})$ and F is as before then

$$\mathrm{UCh}(\mathbf{G}^F) = \{\chi_{K_\rho,\phi} \mid \rho \in \mathrm{Irr}(W)\}$$