Classifying 2-blocks with an elementary abelian defect group

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Introductory Workshop GRAW01 (Isaac Newton Institute, Cambridge)

Blocks of finite groups

Ingredients

- A finite group G
- For a prime p, a *p-modular system* $(K, \mathcal{O}, k)$ where
  1. k is an algebraically closed field of characteristic p.
  2. $\mathcal{O}$ is a complete discrete valuation ring such that $k = \mathcal{O}/J(\mathcal{O})$.
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We can form the group algebra kG (or $\mathcal{O}G$) and decompose it as follows: We decompose 1 into the sum of primitive orthogonal central idempotents $e_i \in Z(kG)$. Then

$$kG = kGe_1 \times \cdots \times kGe_n$$

$B_i = kGe_i$ is called a p-**block** of kG.

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- ...and many more.

...however, the proof does not produce an explicit list.

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- $n = 6$: ≥ 97 classes.

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Blocks of quasisimple groups with an abelian defect 2-group D were classified by Eaton, Kessar, Külshammer, Sambale in 2013.

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If D is a defect group of B, then $D \cap N$ is a defect group of b.

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$F^*(G)$ is normal in G and *self-centralising*, so:

$$G/F^*(G) \text{ can be embedded in } \mathrm{Out}(F^*(G))$$

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Using Schreier's conjecture, in several situations we can show that $G/F^*(G)$ is a solvable group. Whenever this holds then we obtain a chain of normal subgroups with prime indices:

$$\begin{matrix} F^*(G) & \lhd & N_1 & \lhd & \dots & \lhd & N_k & \lhd & G \\ b & & b_1 & & & & b_k & & B \end{matrix}$$

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The problem reduces to the following question:

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Question

Let $N \lhd H$ with $[H : N] = q$ prime and let c be a block of $\mathcal{O}N$ covered by a block C of $\mathcal{O}H$. If we know the Morita equivalence class of c, what can we say about the one of C?

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Remark

We can determine all the possibilities that can occur for the blocks of

$$F^*(G) = Z(G)(O_2(G) * E(G))$$

covered by B, since their defect groups are of the form $D \cap F^*(G)$ and blocks of central or direct products are well-understood.

Even extensions ($q = 2$)

Theorem (Koshitani-Külshammer 1996)

In the situation above, if the defect group D of B has a decomposition $D = (D \cap N) \times Q$ then B is Morita equivalent to $b \otimes kQ$.

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- "D is elementary abelian" is a crucial hypothesis in this case.

Odd extensions (q odd) - Crossed products

Definition

A G-graded algebra A is an $\mathcal{O}$-algebra that admits a decomposition $A = \bigoplus_{g \in G} A_g$ such that $A_g A_h \subseteq A_{gh} \quad \forall g, h \in G$.

A **crossed product** of a group G and a k-algebra R is a G-graded algebra A such that $A_1 \simeq R$ and A_g contains a unit for any $g \in G$.

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Equivalence classes of crossed products of a ring R with a group X are parametrized by pairs (ω, ζ), where:

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Then there is a one-to-one correspondence:

$$\{\text{Equivalence classes for } fBf\} \longleftrightarrow \{\omega : G/N \to \mathrm{Out}(fbf)\}$$

So we can bound the possibilities for the Morita equivalence class of B by counting the possibilities for ω.

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So if we can compute Picard groups for all blocks that appear in our chain, we can list all possibilities for B (but, in principle, some might not occur as blocks of finite groups).

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For example, for $D = (C_2)^5$, this information can be found in:

- Eaton, Kessar, Külshammer, Sambale (2013).
- Follows from Schreier's conjecture and a theorem of Puig (2001).
- Usami, Puig (1994), Watanabe (2005) + two new situations.
- Boltje, Kessar, Linckelmann (2017), Eaton, Livesey (2018).

The list of blocks with defect group $(C_2)^n$ for $n \leq 5$, and many more lists of blocks are available on

https://wiki.manchester.ac.uk/blocks/

maintained by Charles Eaton.