Finite permutation groups: applications to transformation semigroups and synchronization

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The dungeon

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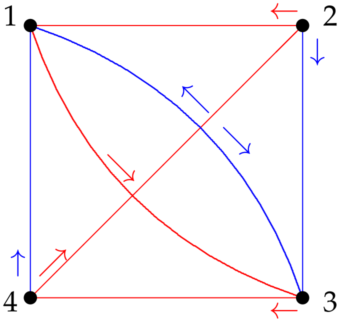
Can you escape? In other words, is there a sequence of colours such that, if you use the doors in this sequence from any starting point, you will end in a known place?


1
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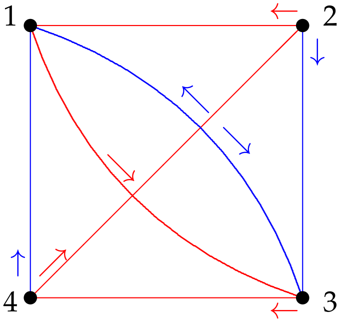

You can check that (Blue, Red, Blue) takes you to room 1 no matter where you start.

Automata

The diagram on the last page shows a finite-state deterministic automaton. This is a machine with a finite set of states, and a finite set of transitions, each transition being a map from the set of states to itself. The machine starts in an arbitrary state, and reads a word over an alphabet consisting of labels for the transitions (Red and Blue in the example); each time it reads a letter, it undergoes the corresponding transition.

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A reset word is a word with the property that, if the automaton reads this word, it arrives at the same state, independent of its start state. An automaton which possesses a reset word is called synchronizing.

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A reset word is a word with the property that, if the automaton reads this word, it arrives at the same state, independent of its start state. An automaton which possesses a reset word is called synchronizing.

Not every finite automaton has a reset word. For example, if every transition is a permutation, then every word in the transitions evaluates to a permutation. How do we recognise when an automaton is synchronizing?

Automata and transformation monoids

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So an automaton is a transformation monoid with a distinguished generating set. It is synchronizing if it contains a map with rank 1.

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Proposition

- *A homomorphism from K_m to Γ is an embedding of K_m into Γ; such a homomorphism exists if and only if $\omega(\Gamma) \geq m$.*
- *A homomorphism from Γ to K_m is a proper colouring of Γ with m colours; such a homomorphism exists if and only if $\chi(\Gamma) \leq m$.*
- *There are homomorphisms in both directions between Γ and K_m if and only if $\omega(\Gamma) = \chi(\Gamma) = m$.*

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Theorem

Let S be a transformation monoid on Ω. Then S fails to be synchronizing if and only if there exists a non-null graph Γ on the vertex set Ω for which $S \leq \mathrm{End}(\Gamma)$. Moreover, we may assume that $\omega(\Gamma) = \chi(\Gamma)$.

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Proof.

Given a transformation monoid S, we define a graph $\mathrm{Gr}(S)$ in which x and y are joined if and only if there is no element $s \in S$ with $xs = ys$. Show that $S \leq \mathrm{End}(\mathrm{Gr}(S))$, that $\mathrm{Gr}(S)$ has equal clique and chromatic number, and that S is synchronizing if and only if $\mathrm{Gr}(S)$ is null. □

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Theorem

A permutation group G on Ω is non-synchronizing if and only if there exists a G-invariant graph Γ, not complete or null, which has clique number equal to chromatic number.

So the definition of "synchronizing" exactly matches our previous template for permutation group properties: G is synchronizing if and only if there is no non-trivial G-invariant graph with clique number equal to chromatic number.

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- *If G is synchronizing, then it is primitive and basic.*
- *If G is 2-homogeneous, then it is synchronizing.*
- *None of these implications reverses.*

The O'Nan–Scott Theorem

Recall that, by the O'Nan–Scott Theorem, a basic group is affine, diagonal or almost simple.

Affine groups have abelian normal subgroups. They have the form

$$\{x \mapsto xA + c : c \in V, A \in H\},$$

where V is a finite vector space and H an irreducible linear group on V. They may or may not be synchronizing.

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Diagonal groups are considered below.

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is primitive. One can enlarge the group by adjoining automorphisms of S (the inner automorphisms are already included as the "diagonal" subgroup $\{(g,g) : g \in S\}$) and the map $x \mapsto x^{-1}$. The result is the 2-factor diagonal group $D(T,2)$.

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They noted that $|A_5| = 59 + 1$, $|\mathrm{PSL}(2,7)| = 167 + 1$, $|A_6| = 359 + 1$, $|\mathrm{PSL}(2,8)| = 503 + 1$, $|\mathrm{PSL}(2,11)| = 659 + 1$, $\ldots$. (It is not known whether there are infinitely many counterexamples.)

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b	**c**	**e**	**a**
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This is not just any old Latin square: it is the Cayley table, or multiplication table, of the Klein group of order 4.

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If L is the Cayley table of a group T, the graph admits T^3 (acting on rows, columns and symbols), as well as automorphisms of T and the symmetric group permuting the three types of object. If T is simple, the group generated by all of these is primitive, and is a three-factor diagonal group $D(T, 3)$.

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Latin square graphs are strongly regular, but almost all have only the trivial group of automorphisms.

Transversals and orthogonal mates

A *transversal* is a set of cells, one in each row, one in each column, and one containing each letter.

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Regarding the colours as an alphabet we see a second Latin square which is orthogonal to the first square, in the sense that each combination of letter and colour occurs precisely once.

Not all Latin squares have transversals. Consider the following square:

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

Given a set of cells, one from each row and one from each column, the sum of the row indices is $0+1+2+3 = 2 \pmod 4$. Similarly for the columns. Since each entry is the sum of its row and column indices, the entries sum to $2+2 = 0 \pmod 4$. Thus the entries cannot be $\{0, 1, 2, 3\}$.

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More generally, the Cayley table of a cyclic group of even order has no transversal.

Complete mappings

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The Hall–Paige conjecture

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Hall was a well known group theorist and combinatorialist. Paige was much less well known: he was a student of Richard Bruck, had 6 students at UCLA, and has 18 papers (including his thesis on neofields) listed on MathSciNet.

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The papers of Wilcox and Evans were published in the *Journal of Algebra* in 2009. But Bray's work has was not published at the time.

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Also, the chromatic number is n if and only if A has an orthogonal mate:

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By Burnside's Transfer Theorem, a non-abelian simple group cannot have cyclic Sylow 2-subgroups. So by Hall–Paige, the Latin square graph of its Cayley table has clique number equal to chromatic number. □

With a little more effort, we have:

Theorem

The diagonal group $D(T, r)$ is non-synchronizing for $r \geq 3$.

Other properties

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- S is regular if all its elements are regular;
- an element $e \in S$ is an idempotent if $e^2 = e$;
- S is idempotent-generated if all its elements are products of idempotents.

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Theorem

Let S be a transformation monoid, G its normaliser in the symmetric group. If SG is regular, then S is regular.

The first breach in the wall

Things began with a paper of Araújo, Mitchell and Schneider in 2011. Which permutation groups G on Ω have the property that, for any transformation a on Ω which is not a permutation, one of the semigroups $\langle G, a \rangle$, $\langle G, a \rangle \setminus G$, or $\langle a^g : g \in G \rangle$ is regular, or idempotent generated?

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For "regular", the necessary and sufficient condition is that G is the symmetric or alternating group or one of nine specific groups with degrees from 5 to 9 inclusive. For "idempotent-generated", (excluding the semigroup $\langle G, a \rangle$), the condition is that G is the symmetric or alternating group or one of three specific groups of degrees 5 or 6.

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These theorems are significant extensions of earlier results of Howie, Symons, Levi and McFadden.

Regularity

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If b is a quasi-inverse of a, then b must map the image of a to a transversal for the kernel of a. If a is regular in $\langle G, a \rangle$, it can be shown that there must be an element of G mapping the image of a to a transversal for the kernel of a.

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Accordingly, we say that a permutation group G on Ω has the k-universal transversal property, or k-ut for short, if for every k-subset A and k-partition P of Ω, there is an element of G mapping A to a transversal for P.

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It follows from what is said above that every rank-k map a is regular in $\langle G, a\rangle$ if and only if G has the k-ut. But much more is true …

The k-ut property and regularity

João Araújo and I showed the following result.

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Theorem

Let $n \geq 5$ and $2 \leq k \leq n/2$. If G is a permutation group of degree n with the k-ut property, then G has the $(k-1)$-ut property.

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The earlier analysis shows that elements of rank k in $\langle G, a \rangle$ are regular. But by the theorem, for $l < k$, G also has the l-ut property, and so elements of smaller rank are also regular.

The case $k = 2$

For $2 < k < n/2$, the k-ut property implies $(k-1)$-homogeneity with known exceptions; these groups are known, and in principle a classification of groups with k-ut can be done (though significant difficulties remain).

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Theorem

For $n > 2$, the 2-ut property is equivalent to primitivity.

For a G-orbit on 2-sets is the edge set of an (undirected) orbital graph; it contains a transversal to every 2-partition if and only if it is connected. Now G is primitive if and only if all orbital graphs are connected.

Idempotent generation

Idempotent generation of $\langle G, a \rangle \setminus G$ is stronger than regularity of $\langle G, a \rangle$. The case that has received most attention is $k = 2$.

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A primitive permutation group G on Ω has the road closure property if, given any orbit O of G on 2-sets, and any proper block of imprimitivity for G in its action on O, the graph with vertex set Ω and edge set $O \setminus B$ is connected. In other words, if workmen dig up the edges in the block B, the orbital graph remains connected.

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Theorem

The permutation group G on Ω has the property that $\langle G, a \rangle \setminus G$ is idempotent-generated for any rank 2 map a if and only if G has the road closure property.

The road closure property

Primitive groups with the road closure property must be basic (the property fails for the Hamming graphs).

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Primitive groups with the road closure property must be basic (the property fails for the Hamming graphs).
A primitive group which has an imprimitive normal subgroup of index 2 fails the road closure property. This accounts for most of the basic groups which fail to have the property.
But there are others, known examples related to triality, and potential examples of almost simple groups with “novelty” maximal subgroups of certain types. Work to classify these is in progress. I hope it will be complete by the end of this INI programme.

References

J. Araújo and P. J. Cameron, Two generalizations of homogeneity in groups with applications to regular semigroups, *Trans. Amer. Math. Soc.* **368** (2016), 1159–1188.

J. Araújo, P. J. Cameron and B. Steinberg, Between primitive and 2-transitive: Synchronization and its friends, *Europ. Math. Soc. Surveys* **4** (2017), 101–184.

J. Araújo, J. D. Mitchell and C. Schneider, Groups that together with any transformation generate regular semigroups or idempotent generated semigroups, *J. Algebra* **343** (2011), 93–106.

J. N. Bray, Q. Cai, P. J. Cameron, P. Spiga and H. Zhang, The Hall–Paige conjecture, and synchronization for affine and diagonal groups, *J. Algebra*, in press.