Modules for algebraic groups with finitely many orbits on singular 1-spaces

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Groups, representations and applications: new perspectives

Linear algebraic groups

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- P denotes a parabolic subgroup, B a Borel subgroup.
- We use λ_i to denote a fundamental dominant weight with respect to a root system $\Phi = \langle \alpha_i \rangle_i$.
- If G has root system Φ, $V_G(\lambda_i)$ is then the irreducible rational G-module of highest weight λ_i.

A double coset problem

We now describe the motivating problem behind our research.

Let G be a simple connected algebraic group over k.

Throughout let H, K be closed connected subgroups of G.

The set of (H, K)-double cosets in G is given by

$$H\backslash G/K := \{HgK : g \in G\}.$$

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A double coset problem: some examples

- $|B\backslash G/B| < \infty$ (or $|P\backslash G/P| < \infty$) for any Borel (or parabolic) subgroup of G (Bruhat decomposition).
- $G = SL(V)$.
- $G = HK$ (classified in [Liebeck et al., 1996]) is equivalent to $|H\backslash G/K| = 1$. For example $SO_7 = G_2 P_1$ or $SO_8 = B_3 P_1$, where $V \downarrow B_3 = V_{B_3}(\lambda_3)$.
- If H, K are both reductive and $|H\backslash G/K| < \infty$ then by [Brundan, 2000] there is actually only a single double coset. In characteristic 0 this is a consequence of Luna's slice theorem.

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- [Guralnick et al., 1997] finds all irreducible modules for semisimple algebraic groups with finitely many orbits on m-spaces.
- We call modules with finitely many orbits on $P_1(V)$ finite orbit modules.

Finite orbit modules

Theorem

[Guralnick et al., 1997] Let G be a connected simple algebraic group over $k = \overline{k}$. If V is an irreducible rational finite orbit kG-module then either V is an internal module for G or V is one of the following:

G	V	$\dim V$	p
A_n	$\lambda_1 + p^i\lambda_1, \lambda_1 + p^i\lambda_n$	$(n+1)^2$	$\neq 0$
A_2	$\lambda_1 + \lambda_2$	7	$p = 3$
A_3	$\lambda_1 + \lambda_2$	16	$p = 3$
B_4	λ_4	16	*any*
B_5	λ_5	32	*any*
C_3	λ_2	13	$p = 3$
G_2	λ_1	$7 - \delta_{p,2}$	*any*
F_4	λ_4	25	$p = 3$

Table: *Finite orbit modules*

Internal modules

One way to obtain finite orbit modules is the following:

- Let H be a simple algebraic group and $P = QG$ be a proper parabolic subgroup of H with unipotent radical Q and Levi subgroup G.
- Let $1 = Q_0 < Q_1 < \cdots < Q_r = Q$ be a G-invariant composition series for Q.
- Then each factor Q_{i+1}/Q_i has the structure of a rational irreducible kG-module.
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Internal module: an example

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- D_5 has 6 orbits on 2-spaces in the half-spin module. (Upcoming work).

Our setting and theorem

We consider the double coset problem when $G = SO_n$ and $K = P_1$ is a maximal parabolic subgroup of G stabilizing a singular 1-space.

Theorem

[AR, 2019] Let H be a simple irreducible closed connected subgroup of $G = SO(V)$ such that H has finitely many orbits on singular 1-spaces. Then either V is a finite orbit module or a composition factor of the adjoint module, or up to field or graph twists (H, V) is as follows.

H	V	$\dim V$	p	Stabilizer of dense orbit
A_1	$4\lambda_1$	5	≥ 5	Alt_4
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C_3	λ_2	14	$\neq 3$	$(A_1)^3.(2^3.3)$
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The A_1 case: setup

Let $p > 5$ and let $H = SL_2(k)$.

Let $W = \langle e, f \rangle$ be the natural 2-dimensional H-module, where e, f is a hyperbolic pair for an alternating form on W stabilized by H.

Then $V = S^4(W)$ is an irreducible 5-dimensional module for $H = A_1$.

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- A point in a dense H-orbit on singular 1-spaces in V must have a finite stabilizer.

The A_1 case: the orbits

We can easily find two orbits:

- $\langle e \otimes e \otimes e \otimes e \rangle$ is fixed by a P_1 parabolic stabilising $\langle e \rangle$.
- $\langle e \otimes e \otimes e \otimes f + e \otimes e \otimes f \otimes e + e \otimes f \otimes e \otimes e + f \otimes e \otimes e \otimes e \rangle$ is fixed only by a T_1.

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Note that both 1-spaces are singular.

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We can assume that $k = \overline{\mathbb{F}_q}$. By standard theory there is a subgroup of H isomorphic to Alt_4.

- Alt_4 fixes two singular 1-spaces in V. This can be seen using standard character theory.
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- The dense orbit splits into 4 orbits if $q \equiv 1 \mod 3$ and into 2-orbits otherwise. The stabilizers have size $3, 3, 4, 12$ and $2, 2$ respectively.
- Adding up the sizes of the orbits we have found gives us the number of singular 1-spaces, therefore 6 is a uniform bound on the number of A_1-orbits.

A spin module

Our theorem had the following list:

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C_4	λ_2	26	2	$(A_1)^4.(2^4.Alt_4)$
F_4	λ_4	26	$p \neq 3$	$D_4.3$

Table: Finite singular orbit modules for simple groups

We now consider the spin module for B_6.

- $V_{B_6}(\lambda_6)$ is the unique irreducible rational module for B_6 of dimension 64.

A spin module

Our theorem had the following list:

H	V	$\dim V$	p	Stabilizer of dense orbit
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- p must be 2 for the module to be orthogonal.

Spin module for D_7

Using previous work by [Popov, 1980] and [Igusa, 1970] we have the following list of spinor representatives for the D_7-orbits on the spin module:

Orbit type	Spinor x	$(D_7)^\circ_x$
2	1	$U_{21}SL_7$
3	$1 + f_1f_2f_3f_4$	$U_{27}(SL_3 \times Spin_7)$
4	$1 + f_1f_2f_3f_4f_5f_6$	$U_{12}SL_6$
5	$1 + f_1f_2f_3f_7 + f_1f_2f_3f_4f_5f_6$	$U_{21}(SL_3 \times SL_3)$
6	$\lambda(1 + f_1f_2f_3f_7 + f_4f_5f_6f_7 + f_1f_2f_3f_4f_5f_6) : \lambda \in k^*$	$G_2 \times G_2$
7	$1 + f_1f_2f_3f_7 + f_1f_5f_6f_7 + f_1f_2f_3f_4f_5f_6$	$U_{19}(SL_2 \times Sp_4)$
8	$1 + f_1f_2f_3f_7 + f_1f_5f_6f_7 + f_2f_4f_6f_7 + f_1f_2f_3f_4f_5f_6$	$U_{14}G_2$
9	$1 + f_1f_2f_3f_4 + f_3f_4f_5f_6$	$U_{26}(Sp_6 \times T_1)$
10	$1 + f_1f_2f_3f_4 + f_3f_4f_5f_6 + f_1f_3f_6f_7$	$U_{26}SL_4$

Table: Subgroups of stabilizers if $p = 2$

Spin module: restriction to B_6

To restrict to B_6 we need to find the orbits of the stabilizers listed on non-singular 1-spaces.

S'	Rep v	S'_v
$U_{21}.SL_7$	$e_1 + f_1$	$U_{21}.SL_6$
$U_{26}.(Sp_6 T_1)$	$e_1 + f_1$	$U_{25}.(Sp_4 T_1)$
	$e_7 + f_7$	$U_{14}.Sp_6$
$U_{27}.(Spin_7 SL_3)$	$e_5 + f_5$	$U_{23}.(Spin_7 SL_2)$
	$e_1 + f_1$	$U_{21}.(G_2 SL_3)$
$U_{12}.SL_6$	$e_1 + f_1$	$U_{11}.SL_5$
	$\alpha e_7 + \alpha^{-1} f_7 : \alpha \neq 0$	SL_6
$U_{21}.(SL_3 SL_3)$	$e_1 + f_1$	$U_{16}.(SL_2 SL_3)$
	$e_4 + f_4$	$U_{16}.(SL_2 SL_3)$
	$\alpha e_7 + \alpha^{-1} f_7 : \alpha \neq 0$	$U_{15}.(SL_3 SL_3)$
	$e_1 + f_1 + f_4$	$U_{18}.(SL_2 SL_2)$
$G_2 G_2$	$\alpha e_7 + \alpha^{-1} f_7 : \alpha \neq 0, 1$	$(SL_3.2)(SL_3.2)$
	$e_7 + f_7$	$G_2 \times G_2$
	$f_4 + e_4$	$G_2 P'_1$
	$e_1 + f_1$	$P'_1 G_2$
	$e_1 + f_1 + e_4$	$P'_1 P'_1$
$U_{19}.(SL_2 Sp_4)$	$e_1 + f_1$	$U_{10}.Sp_4$
	$e_5 + f_5$	$U_{16}.(SL_2 SL_2)$
	$\alpha e_4 + \alpha^{-1} f_4 : \alpha \neq 0$	$U_9.(Sp_4 T_1.2)$
$U_{14}.G_2$	$e_1 + f_1$	$U_{14}.SL_2$
	$\alpha e_7 + \alpha^{-1} f_7 : \alpha \neq 0$	$U_8.(SL_3.2)$
$U_{26}.SL_4$	$e_1 + f_1$	$U_{22}.Sp_4$
	$e_7 + f_7$	$U_{20}.SL_3$

Adjoint modules

One of the possibilities in our theorem is that we have a composition factor of the adjoint module. The adjoint module for a connected linear algebraic group H is the module $Lie(H)$ on which H acts by conjugation.

- For example if $H = SL_n$ then the adjoint module for H is the Lie-algebra sl_n of trace-0 matrices.
- If $p \mid n$ then this is not an irreducible module, because scalar matrices are in sl_n.
- If $p \mid n$ the non-trivial composition factor $sl_n/\langle I \rangle$ is an irreducible module of highest weight $\lambda_1 + \lambda_{n-1}$.

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Adjoint modules: the zero-space

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This implies the following:

- If $\dim V_0 \geq 3$, then V is not a finite singular orbit module.

Adjoint modules, an easy example

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Since there is only 1 orbit on semisimple singular 1-spaces and finitely many on nilpotent 1-spaces, this concludes.

Semisimple case

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H	V	dim V	p	Stabilizer of dense orbit
C_2C_2	$\lambda_1 \otimes \lambda_1$	16	$p \neq 2$	$(A_1A_1).2$
			$p = 2$	U_3A_1
$C_2C_n, n > 2$	$\lambda_1 \otimes \lambda_1$	$8n$	$p \neq 2$	$(A_1A_1).2(C_{n-2})$
			$p = 2$	$U_3A_1(C_{n-2})$

Table: Finite singular orbit modules for maximal semisimple groups

Upcoming work

Eventually we want to achieve a characterization of finite singular orbit modules analogous to the one for finite orbit modules. In order to do this we need to:

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Eventually we want to achieve a characterization of finite singular orbit modules analogous to the one for finite orbit modules. In order to do this we need to:

- Determine modules with finitely many orbits on totally singular m-spaces.
- Remove maximality requirement on the semisimple case.
- G exceptional.

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