Zeta functions of arithmetic schemes
Cambridge, 17 January 2020

Matthias Flach

Joint work with B. Morin, Bordeaux

- $\mathcal{X}$ is a regular connected scheme of dimension d, proper over $\mathrm{Spec}(\mathbb{Z})$
- Zeta function
$$\zeta(\mathcal{X}, s) = \prod_{x \in \mathcal{X} \text{ closed}} \frac{1}{1 - Nx^{-s}}$$
Converges for $\Re(s) > d$
- Aim: For any $n \in \mathbb{Z}$ describe
$$\mathrm{ord}_{s=n}\, \zeta(\mathcal{X}, s)$$
and
$$\zeta^*(\mathcal{X}, n) \in \mathbb{R}^\times$$

Weil-étale cohomology

(Lichtenbaum)

- $\mathcal{X} \to \mathrm{Spec}(\mathbb{F}_p)$ smooth, proper
- $\mathbb{Z}(n)$ on $\mathcal{X}_{et}$ (Higher Chow or Suslin-Voevodsky complex)
  $\mathbb{Z}(0) = \mathbb{Z}, \quad \mathbb{Z}(1) = \mathbb{G}_m[-1], \ldots$
- $W_{\mathbb{F}_p} \cong \mathbb{Z} \subseteq \hat{\mathbb{Z}} \cong G_{\mathbb{F}_p}$
- $\mathcal{X} = \mathrm{Spec}(\mathbb{F}_p)$, $n = 0$

$$H^i(G_{\mathbb{F}_p}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Q}/\mathbb{Z} & i = 2 \end{cases} \qquad H^i(W_{\mathbb{F}_p}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & \text{else} \end{cases}$$

- $\mathcal{X}$ a smooth, proper curve, $n = 1$

$$H^i(\mathcal{X}_{et}, \mathbb{Z}(1)) = \begin{cases} \mathcal{O}(\mathcal{X})^\times & i = 1 \\ \mathrm{Pic}(\mathcal{X}) = \text{finite} \oplus \mathbb{Z} & i = 2 \\ 0 & i = 3 \\ \mathbb{Q}/\mathbb{Z} & i = 4 \end{cases}$$

$$H^i(\mathcal{X}_W, \mathbb{Z}(1)) = \begin{cases} \mathcal{O}(\mathcal{X})^\times & i = 1 \\ \mathrm{Pic}(\mathcal{X}) = \text{finite} \oplus \mathbb{Z} & i = 2 \\ \mathbb{Z} & i = 3 \end{cases}$$

Weil-étale cohomology
(Special values of $\zeta(\mathcal{X}, s)$)

- **Conjecture:** $R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) := R\Gamma(W_{\mathbb{F}_p}, R\Gamma(X_{\overline{\mathbb{F}}_p}, \mathbb{Z}(n))$ is a perfect complex of abelian groups.
- Known for $d \leq 2$
- **Theorem:** The conjecture implies (Milne, Lichtenbaum, Geisser)
  - There is a long exact sequence (concentrated in $2n, 2n+1$)

$$\cdots \to H^i(\mathcal{X}_W, \mathbb{Z}(n))_{\mathbb{R}} \xrightarrow{\cup\theta} H^{i+1}(\mathcal{X}_W, \mathbb{Z}(n))_{\mathbb{R}} \to \cdots$$

  -
$$\operatorname{ord}_{s=n} \zeta(\mathcal{X}, s) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^i(\mathcal{X}_W, \mathbb{Z}(n))_{\mathbb{R}}$$

  -
$$\zeta^*(\mathcal{X}, n) = \pm\chi(R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)), \cup\theta) \cdot p^{\chi(\mathcal{X}, \mathcal{O}, n)}$$

    where

$$\chi(\mathcal{X}, \mathcal{O}, n) = \sum_{i \leq n, j} (-1)^{i+j} (n-i) \dim_{\mathbb{F}_p} H^j(\mathcal{X}, \Omega^i)$$

Proofs

- Grothendieck's formula: $l \neq p$ prime

$$\zeta(\mathcal{X}, s) = Z(\mathcal{X}, p^{-s})$$

$$Z(\mathcal{X}, T) = \prod_{i=0}^{2\dim(\mathcal{X})} \det(1 - \mathrm{Frob}^{-1} \cdot T | H^i(\mathcal{X}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l))^{(-1)^{i+1}}$$

- $\mathbb{Z}(n)/l^\nu \cong \mu_{l^\nu}^{\otimes n}$

$$R\Gamma(G_{\mathbb{F}_p}, R\Gamma(\mathcal{X}_{\overline{\mathbb{F}}_p}, \mathbb{Z}_l(n))) \cong R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}_l$$

- For $l = p$ one has

$$Z(\mathcal{X}, T) = \prod_{i=0}^{2\dim(\mathcal{X})} \det(1 - \mathrm{Frob} \cdot T | H^i_{cris}(\mathcal{X}/\mathbb{F}))^{(-1)^{i+1}}$$

Weil-Arakelov cohomology
(Assumptions)

- $\mathcal{X}$ regular of dimension d, $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ flat, proper
- $\mathbb{Z}(n)$ on $\mathcal{X}_{et}$ (Higher Chow complex) For $n < 0$ define $\mathbb{Z}(n)$ by pushforward under $f : \mathbb{P}^N_{\mathcal{X}} \to \mathcal{X}$

$$Rf_{et,*}\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}(-1)[-2] \oplus \cdots \oplus \mathbb{Z}(-N)[-2N].$$

- **Assumptions:**

FG $H^i(\mathcal{X}_{et}, \mathbb{Z}(n))$ is finitely generated for $i \leq 2n+1$

B Beilinson conjectures. There is a perfect duality

$$H^i_c(\mathcal{X}, \mathbb{R}(n)) \times H^{2d-i}(\mathcal{X}, \mathbb{R}(d-n)) \to H^{2d}_c(\mathcal{X}, \mathbb{R}(d)) \to \mathbb{R}$$

where

$$R\Gamma_c(\mathcal{X}, \mathbb{R}(n)) \to R\Gamma(\mathcal{X}, \mathbb{R}(n)) \to R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n))$$

Known for $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$

AV Artin-Verdier duality. There is a perfect duality for any integer m

$$\hat{H}^i_c(\mathcal{X}_{et}, \mathbb{Z}(n)/m) \times H^{2d+1}(\mathcal{X}_{et}, \mathbb{Z}(d-n)/m) \to \hat{H}^{2d+1}_c(\mathcal{X}_{et}, \mathbb{Z}(d)/m) \to \mathbb{Z}/m$$

Known for $d \leq 2$ or $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ smooth, or $n \geq d$ or $n \leq 0$.

Weil-Arakelov cohomology ($\mathbb{Z}(n)$-coefficients)

- If $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ is flat then $\mathcal{X}$ is not "compact". One has a diagram with exact rows and columns

$$\begin{array}{ccccc} R\Gamma_{ar,c}(\mathcal{X},\mathbb{Z}(n)) & \to & R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n)) & \to & R\Gamma_{ar,\mathcal{D}}(\mathcal{X}_{/\mathbb{R}},\mathbb{Z}(n)) \\ \| & & \uparrow & & \uparrow \\ R\Gamma_{ar,c}(\mathcal{X},\mathbb{Z}(n)) & \to & R\Gamma_{ar}(\overline{\mathcal{X}},\mathbb{Z}(n)) & \to & R\Gamma_{ar}(\mathcal{X}_\infty,\mathbb{Z}(n)) \\ & & \uparrow & & \uparrow \\ & & R\Gamma_{ar,\mathcal{X}_\infty}(\overline{\mathcal{X}},\mathbb{Z}(n)) & = & R\Gamma_{ar,\mathcal{X}_\infty}(\overline{\mathcal{X}},\mathbb{Z}(n)) \end{array}$$

in D^b(l.c.a.grps). In general, only $R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n))$ is a perfect complex of abelian groups.

- $$H^i_{ar}(\mathrm{Spec}(\mathcal{O}_F),\mathbb{Z}(1)) = \begin{cases} \mathcal{O}_F^\times & i=1 \\ \mathrm{Pic}(\mathcal{O}_F) \oplus (\bigoplus_{v|\infty}\mathbb{Z})^{\Sigma=0} & i=2 \end{cases}$$

$$H^i_{ar}(\overline{\mathrm{Spec}(\mathcal{O}_F)},\mathbb{Z}(1)) = \begin{cases} \mu_F^\times & i=1 \\ \mathrm{Pic}(\mathcal{O}_F) \oplus (\bigoplus_{v|\infty}\mathbb{R})/\log(\mathcal{O}_F^\times) & i=2 \\ \mathbb{Z} & i=3 \end{cases}$$

Weil-Arakelov cohomology ($\tilde{\mathbb{R}}(n)$- and $\tilde{\mathbb{R}}/\mathbb{Z}(n)$-coefficients)

- For $\mathcal{Y} = \mathcal{X}, \overline{\mathcal{X}}, \mathcal{X}_\infty$ there are exact triangles in D^b(l.c.a.grps)

$$R\Gamma_{\mathrm{ar},?}(\mathcal{Y}, \mathbb{Z}(n)) \to R\Gamma_{\mathrm{ar},?}(\mathcal{Y}, \tilde{\mathbb{R}}(n)) \to R\Gamma_{\mathrm{ar},?}(\mathcal{Y}, \tilde{\mathbb{R}}/\mathbb{Z}(n)) \to$$

- $H^{2n}_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}(n)) \cong CH(\overline{\mathcal{X}})_{\mathbb{R}}$ (Gillet-Soule Arakelov Chow group)
- **Proposition**
  - There are dualities of finite-dimensional $\mathbb{R}$-vector spaces

$$H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \times H^{2d+1-i}_{\mathrm{ar}}(\mathcal{X}, \tilde{\mathbb{R}}(d-n)) \to H^{2d+1}_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(d)) \to \mathbb{R}$$

and

$$H^i_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}(n)) \times H^{2d+1-i}_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}(d-n)) \to H^{2d+1}_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}(d)) \to \mathbb{R}$$

  - There are Pontryagin dualities

$$R\,\mathrm{Hom}(R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \mathbb{Z}(n)), \mathbb{R}/\mathbb{Z}) \cong R\Gamma(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(d-n))[-2d-1]$$

and

$$H^i_{\mathrm{ar}}(\overline{\mathcal{X}}, \mathbb{Z}(n)) \times H^{2d+1-i}_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}/\mathbb{Z}(d-n)) \to H^{2d+1}_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}$$

Weil-Arakelov cohomology (Special values of $\zeta(\mathcal{X}, s)$)

For any $n \in \mathbb{Z}$ the exact triangle

$$R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \mathbb{Z}(n)) \to R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \to R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n)) \to \qquad (1)$$

has the following properties

- The groups $H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n))$ are finite dimensional vector spaces over $\mathbb{R}$ for all i and there is an exact sequence

$$\cdots \xrightarrow{\cup\theta} H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \xrightarrow{\cup\theta} H^{i+1}_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \xrightarrow{\cup\theta} \cdots \qquad (2)$$

  In particular, the complex $R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n))$ has vanishing Euler characteristic:

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) = 0.$$

- The groups $H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n))$ are compact, commutative Lie groups for all i, i.e. isomorphic to

$$S^1 \times \cdots \times S^1 \times \text{finite}.$$

Conjectural relation to $\zeta(\mathcal{X}, s)$

- The function $\zeta(\mathcal{X}, s)$ has a meromorphic continuation to $s = n$ and

$$\operatorname{ord}_{s=n} \zeta(\mathcal{X}, s) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)).$$

- If $\zeta^*(\mathcal{X}, n) \in \mathbb{R}$ denotes the leading Taylor-coefficient of $\zeta(\mathcal{X}, n)$ at $s = n$ then

$$|\zeta^*(\mathcal{X}, n)|^{-1} = \prod_{i \in \mathbb{Z}} \left(\mathrm{vol}(H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n))) \right)^{(-1)^i}. \qquad (3)$$

Meaning of the volume

If G is a locally compact abelian group, define its **tangent space** $T_\infty G$ by

$$T_\infty G = \mathrm{Hom}_{cts}(\mathrm{Hom}_{cts}(G, \mathbb{R}/\mathbb{Z}), \mathbb{R})$$

which comes with an exponential map

$$\exp : T_\infty G \to \mathrm{Hom}_{cts}(\mathrm{Hom}_{cts}(G, \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z}) = G$$

T_∞ gives the expected answer for $G = \mathbb{Z}, \mathbb{R}, \mathbb{R}/\mathbb{Z}$ and extends to $D^b(l.c.a.)$.

Weil-etale cohomology is the perfect complex of abelian groups $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))$ defined as the mapping fibre of the exponential map

$$R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) \to T_\infty R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n)) \xrightarrow{\exp} R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n))$$

A **volume form** is a nonzero section $v \in det_{\mathbb{R}} T_\infty R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n))$
Given v, the volume in (3) is the unique $\mu \in \mathbb{R}^{>0}$ with

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) = \mu \cdot v$$

Definition of the volume form

Applying T_∞ to (1) we get an exact triangle in $D^b(\mathbb{R})$

$$R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})_{\mathbb{R}}[-2] \to R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \to R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))_{\mathbb{R}} \to \tag{4}$$

where $R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})$ (**derived de Rham cohomology** modulo Fil^n as defined by Illusie) is also a perfect complex of abelian groups.

Taking determinants of (4) gives an isomorphism

$$\begin{aligned}
&\det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))_{\mathbb{R}} \\
\cong&\det_{\mathbb{R}} R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \otimes_{\mathbb{R}} \det_{\mathbb{R}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})_{\mathbb{R}}[-1] \\
\cong&\det_{\mathbb{R}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})_{\mathbb{R}}[-1]
\end{aligned}$$

where the trivialization $\det_{\mathbb{R}} R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \cong \mathbb{R}$ is induced by the exact sequence (2).

The volume in (3) is the unique $\mu \in \mathbb{R}^{>0}$ so that

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) = \mu \cdot C(\mathcal{X}, n) \cdot \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})[-1] \tag{5}$$

for a certain correction factor $C(\mathcal{X}, n) \in \mathbb{Q}^\times$.

Concerning the correction factor $C(\mathcal{X}, n)$

Definition of $C(\mathcal{X}, n)$ is forced by compatibility with Tamagawa number conjecture and involves p-adic Hodge theory.

Theorem

a) *One has* $C(\mathcal{X}, n) = 1$ *if* $n \leq 0$ *(trivial).*

b) *One has* $C(\mathcal{X}, 1) = 1$ *(nontrivial)*

c) *One has* $C(\mathcal{X}, n) = 1$ *if* $\mathcal{X} \to \mathrm{Spec}(\mathbb{F}_p)$ *is smooth, proper over a finite field (Thm of Morin)*

d) *For* $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ *and* $n \geq 1$ *one has* $C(\mathcal{X}, n) = (n-1)!^{-[F:\mathbb{Q}]}$

In general we expect

$$C(\mathcal{X}, n)^{-1} = \prod_{i \leq n-1;\, j} (n-1-i)!^{(-1)^{i+j} \dim_{\mathbb{Q}} H^j(\mathcal{X}_{\mathbb{Q}}, \Omega^i)} \qquad (6)$$

A correction factor free reformulation

For a free $\mathbb{Z}$-algebra P, consider the subcomplex

$$\tilde{\Omega}^{<n}_{P/\mathbb{Z}} := [(n-1)!\Omega^0_{P/\mathbb{Z}} \to (n-2)!\Omega^1_{P/\mathbb{Z}} \to \cdots \to 0!\Omega^{n-1}_{P/\mathbb{Z}}]$$

of the truncated de Rham complex

$$\Omega^{<n}_{P/\mathbb{Z}} := [\Omega^0_{P/\mathbb{Z}} \to \Omega^1_{P/\mathbb{Z}} \to \cdots \to \Omega^{n-1}_{P/\mathbb{Z}}].$$

The complex $\tilde{\Omega}^{<n}_{P/\mathbb{Z}}$ is functorial in the free algebra P. For an arbitrary $\mathbb{Z}$-algebra A, let $P_\bullet(A) \to A$ be the standard simplicial resolution of A by free $\mathbb{Z}$-algebras. Consider the simplicial complex $\tilde{\Omega}^{<n}_{P_\bullet(A)/\mathbb{Z}}$ and its total complex

$$L\tilde{\Omega}^{<n}_{A/\mathbb{Z}} := \mathrm{Tot}(\tilde{\Omega}^{<n}_{P_\bullet(A)/\mathbb{Z}}),$$

and define

$$\tilde{R\Gamma}_{dR}(\mathcal{X}/\mathbb{Z})/F^n := R\Gamma(\mathcal{X}_{Zar}, L\tilde{\Omega}^{<n}_{\mathcal{X}/\mathbb{Z}})$$

Conjecture

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) = \mu \cdot \det_{\mathbb{Z}} \tilde{R\Gamma}_{dR}(\mathcal{X}/\mathbb{Z})/F^n[-1] \qquad (7)$$

Compatibility with functional equation

Theorem

Assume $\zeta(\mathcal{X}, s)$ satisfies the functional equation

$$A(\mathcal{X})^{(d-s)/2}\zeta(\overline{\mathcal{X}}, d-s) = A(\mathcal{X})^{s/2}\zeta(\overline{\mathcal{X}}, s)$$

where $\zeta(\overline{\mathcal{X}}, s) = \zeta(\mathcal{X}_\infty, s)\zeta(\mathcal{X}, s)$ is the completed Zeta-function and $A(\mathcal{X})$ is the Bloch conductor of $\mathcal{X}$. Then (7) for $(\mathcal{X}, n)$ is equivalent to (7) for $(\mathcal{X}, d-n)$ for any $n \in \mathbb{Z}$.

Proof uses $\chi R\Gamma(\mathcal{X}, C^r_{\mathcal{X}/\mathbb{Z}}[r]) = A(\mathcal{X})$ for any $r \in \mathbb{Z}$ where

$$L\wedge^r \Omega_{\mathcal{X}/\mathbb{Z}} \to \underline{R\operatorname{Hom}}(L\wedge^{d-1-r}\Omega_{\mathcal{X}/\mathbb{Z}}, \omega_{\mathcal{X}/\mathbb{Z}}) \to C^r_{\mathcal{X}/\mathbb{Z}}$$

Compatibility with BSD

Theorem

If $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ is an arithmetic surface with finite $\mathrm{Br}(\mathcal{X})$ then (7) for $(\mathcal{X}, 1)$ is equivalent to the Birch and Swinnerton-Dyer conjecture for $\mathrm{Jac}(\mathcal{X}_F)$.

Compatibility with the Tamagawa number conjecture

If $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ is smooth proper over a number ring it turns out that

$$\Delta(\mathcal{X}_{\mathbb{Q}}, n) := \det_{\mathbb{Q}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))_{\mathbb{Q}} \otimes_{\mathbb{Q}} \det_{\mathbb{Q}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})_{\mathbb{Q}}$$

is the fundamental line of Fontaine and Perrin-Riou for the motive

$$h(\mathcal{X}_{\mathbb{Q}})(n) = \bigoplus_{i=0}^{2d-2} h^i(\mathcal{X}_{\mathbb{Q}})(n)[-i]$$

of the generic fibre of $\mathcal{X}$ with trivialization

$$\lambda_\infty : \mathbb{R} \xrightarrow{\sim} \Delta(\mathcal{X}_{\mathbb{Q}}, n)_{\mathbb{R}} \tag{8}$$

induced by (4) and (2). An element $\mu \in \mathbb{R}$ maps to $\Delta(\mathcal{X}_{\mathbb{Q}}, n)$ under this trivialization if and only if it satisfies (5) up to factors in $\mathbb{Q}^\times$. Our conjecture can be restated in terms of the integral fundamental line

$$\Delta(\mathcal{X}/\mathbb{Z}, n) := \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})$$

which is attached to the entire arithmetic scheme $\mathcal{X}$ rather than to individual motivic summands. Equation (3) is equivalent to

$$\lambda_\infty(\zeta^*(\mathcal{X}, n)^{-1} \cdot C(\mathcal{X}, n) \cdot \mathbb{Z}) = \Delta(\mathcal{X}/\mathbb{Z}, n)$$

The example $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$

F number field with r_1 real and r_2 complex places.

$$\zeta(\mathcal{X}, s) = \zeta_F(s)$$

Dedekind Zeta function

All assumptions going into the definition of our groups are satisfied, in particular for $i = 1, 2$

$$H^i(\mathcal{X}_{\mathrm{et}}, \mathbb{Z}(n)) \sim_2 K_{2n-i}(\mathcal{O}_F)$$

is finitely generated. Note

$$0 \to \mathbb{Z}/2\mathbb{Z} \to K_3(\mathbb{Z}) \to H^1(\mathrm{Spec}(\mathbb{Z})_{et}, \mathbb{Z}(2)) \to 0$$

The conjectures on the vanishing order hold true (Borel 1975)

$$\mathrm{ord}_{s=n}\, \zeta_F(s) = \begin{cases} r_2 & n < 0 \text{ odd} \\ r_1 + r_2 & n < 0 \text{ even} \\ r_1 + r_2 - 1 & n = 0 \\ -1 & n = 1 \\ 0 & n > 1 \end{cases}$$

The Beilinson regulator map

$$H^1(\mathcal{X}_{\mathrm{et}},\mathbb{Z}(n)) \xrightarrow{r_n} H^1_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}},\mathbb{R}(n)) \cong \prod_{v|\infty} H^0(F_v,(2\pi i)^{n-1}\mathbb{R})$$

induces isomorphisms

$$r_{n,\mathbb{R}}: H^1(\mathcal{X}_{\mathrm{et}},\mathbb{Z}(n))_{\mathbb{R}} \xrightarrow{\sim} \prod_{v|\infty} H^0(F_v,(2\pi i)^{n-1}\mathbb{R})$$

for $n>1$ and

$$r_{1,\mathbb{R}}: H^1(\mathcal{X}_{\mathrm{et}},\mathbb{Z}(1))_{\mathbb{R}} \cong \Big(\prod_{v|\infty}\mathbb{R}\Big)^{\Sigma=0}$$

for $n=1$. For $n\geq 1$ we set

$$\begin{aligned} h_n &:= |H^2(\mathcal{X}_{\mathrm{et}},\mathbb{Z}(n))| \sim_2 |K_{2n-2}(\mathcal{O}_F)| \\ w_n &:= |H^1(\mathcal{X}_{\mathrm{et}},\mathbb{Z}(n))_{tor}| \sim_2 |K_{2n-1}(\mathcal{O}_F)_{tor}| \\ R_n &:= \mathrm{vol}(\mathrm{coker}(r_n)) \end{aligned}$$

where the volume is taken with respect to the $\mathbb{Z}$-structure $\prod_{v|\infty} H^0(F_v,(2\pi i)^{n-1}\mathbb{Z})$, resp. $(\prod_{v|\infty}\mathbb{Z})^{\Sigma=0}$, of the target.

Our conjecture is equivalent to

$$\zeta_F^*(n) = \pm \frac{h_{1-n} \cdot R_{1-n}}{w_{1-n}} \tag{9}$$

for $n \leq 0$ and to

$$\zeta_F^*(n) = \zeta_F(n) =$$
$$= (n-1)!^{-[F:\mathbb{Q}]} \cdot \frac{2^{r_1 \cdot (-1)^{n-1}} (2\pi)^{[F:\mathbb{Q}] \cdot n - r_2 - r_1 \cdot (((-1)^n - 1)/2)} h_n R_n}{|D_F|^{n-1} \sqrt{|D_F|} \cdot w_n} \tag{10}$$

for $n \geq 1$.

Proposition

Equations (9) and (10) hold for $n = 0, 1$ if F is arbitrary and for any $n \in \mathbb{Z}$ if $F/\mathbb{Q}$ is abelian.

This follows from known cases of the Tamagawa number conjecture.

Additive K-theory (Cyclic Homology)

There is a spectral sequence from derived de Rham cohomology to (derived) cyclic homology, an additive analogue of algebraic K-theory.

$$H_p(R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})[n-1]) \Rightarrow HC^L_{p+n-1}(\mathcal{X}/\mathbb{Z})$$

For $n \geq 1$ and $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ one has

$$R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{O}_F/\mathbb{Z}}) \cong \left(\mathcal{O}_F \xrightarrow{d(n)} \Omega_{\mathcal{O}_F/\mathbb{Z}}(n)\right)$$

where $\Omega_{\mathcal{O}_F/\mathbb{Z}}(n)$ is a certain finite abelian group of order $|D_F|^{n-1}$.

$$K^{add}_{2n-1}(\mathcal{O}_F) := HC_{2n-2}(\mathcal{O}_F) = \ker\left(\mathcal{O}_F \xrightarrow{d(n)} \Omega_{\mathcal{O}_F/\mathbb{Z}}(n)\right)$$

$$K^{add}_{2n-2}(\mathcal{O}_F) := HC_{2n-3}(\mathcal{O}_F) = \mathrm{coker}\left(\mathcal{O}_F \xrightarrow{d(n)} \Omega_{\mathcal{O}_F/\mathbb{Z}}(n)\right)$$

$$|D_F|^{n-1}\sqrt{|D_F|} = h^{add}_n R^{add}_n$$

where $R^{add}_n = \mathrm{covol}(K^{add}_{2n-1}(\mathcal{O}_F))$ and $h^{add}_n = |K^{add}_{2n-2}(\mathcal{O}_F)|$.

Additive K-theory(Topological Positive Cyclic Homology)

How to explain $C(\mathrm{Spec}(\mathcal{O}_F), n) = (n-1)!^{-[F:\mathbb{Q}]}$?
Recall

$$HC^L(\mathcal{X}/\mathbb{Z}) = HH^L(\mathcal{X}/\mathbb{Z})_{hS^1}$$

Topological Hochschild homology (Bökstedt,...)

$$THH(\mathcal{X}) := HH(\mathcal{X}/\mathbb{S})$$

where $\mathbb{S}$ is the sphere spectrum.

Definition

Topological positive cyclic homology

$$TC^+(\mathcal{X}) := THH(\mathcal{X})_{hS^1}$$

Theorem

(Madsen, Lindenstrauss, 2000)

$$THH_i(\mathcal{O}_F) = \begin{cases} \mathcal{O}_F & i = 0 \\ \mathcal{D}_F^{-1}/j \cdot \mathcal{O}_F & i = 2j-1 \\ 0 & else \end{cases}$$

Additive K-theory ctd.

The spectral sequence

$$H_i(BS^1, THH_j(\mathcal{O}_F)) \Rightarrow TC^+_{i+j}(\mathcal{O}_F)$$

shows that $TC^+_{2n-3}(\mathcal{O}_F)$ is finite and $TC^+_{2n-2}(\mathcal{O}_F) \subseteq \mathcal{O}_F$ is a sublattice so that

$$(n-1)!^{[F:\mathbb{Q}]}|D_F|^{n-1}\sqrt{|D_F|} = h_n^{add} R_n^{add}$$

where $R_n^{add} = \mathrm{covol}(TC^+_{2n-2}(\mathcal{O}_F))$ and $h_n^{add} = |TC^+_{2n-3}(\mathcal{O}_F)|$.

Additive K-theory ctd.

The spectral sequence

$$H_i(BS^1, THH_j(\mathcal{O}_F)) \Rightarrow TC^+_{i+j}(\mathcal{O}_F)$$

shows that $TC^+_{2n-3}(\mathcal{O}_F)$ is finite and $TC^+_{2n-2}(\mathcal{O}_F) \subseteq \mathcal{O}_F$ is a sublattice so that

$$(n-1)!^{[F:\mathbb{Q}]}|D_F|^{n-1}\sqrt{|D_F|} = h_n^{add} R_n^{add}$$

where $R_n^{add} = \mathrm{covol}(TC^+_{2n-2}(\mathcal{O}_F))$ and $h_n^{add} = |TC^+_{2n-3}(\mathcal{O}_F)|$.
In general there should be a (motivic) filtration on $TC^+(\mathcal{X})$ with graded pieces

$$R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}})[n-1]$$

which for a polynomial $\mathbb{Z}$-algebra P are given by

$$\tilde{\Omega}^{<n}_{P/\mathbb{Z}} := [(n-1)!\Omega^0_{P/\mathbb{Z}} \to (n-2)!\Omega^1_{P/\mathbb{Z}} \to \cdots \to 0!\Omega^{n-1}_{P/\mathbb{Z}}]$$