Introductory Lectures: Resurgence in Differential Equations, and Effective Summation Methods

Gerald Dunne

University of Connecticut

Isaac Newton Institute Spring School: *Asymptotic Methods and Applications*, March 22-26, 2021

Isaac Newton Institute Programme:
Applicable Resurgent Asymptotics, 2021/2022

Lecture 2

Basic Introduction to Resurgence: A Beginner's Guide

1. Lecture 1: Resurgence & Linear Differential Equations
   - Trans-series and Stokes Phenomenon in ODEs
   - Borel Summation basics
   - Recovering Non-perturbative Connection Formulas
2. Lecture 2: Resurgence & Nonlinear Differential Equations
   - Nonlinear Stokes Phenomenon
   - Painlevé Equation examples
   - Parametric Resurgence & Phase Transitions
3. Lecture 3: Effective Summation Methods
   - Probing the Borel Plane Numerically
   - The Physics of Padé Approximation
   - Optimal Summation and Extrapolation

Resurgence in Nonlinear Differential Equations

- trans-series for an nth order <u>linear</u> ODE has n non-perturbative exponential terms (n Borel singularities)

- new feature: trans-series for a <u>nonlinear</u> ODE has *infinitely many* non-perturbative exponential terms

- new feature: Borel singularities are *repeated in (certain) integer multiples*, due to the nonlinearity

- new feature: "resonance" of Borel singularities can lead to $\log(x)$ terms, which also become iterated $\Rightarrow$ sums over $(\log(x))^n$ terms in the trans-series $\Rightarrow$ more general trans-series

- new feature: <u>nonlinear</u> Stokes phenomenon

- Painlevé: "special functions of nonlinear ODE's" (dlmf) many applications: fluids, gravity, statistical physics, optics, random matrices, matrix models, QM, QFT, strings, ...

- N.B. integrability is NOT required for resurgence

Resurgence in Nonlinear ODEs: e.g. Painlevé II = "nonlinear Airy"

Painlevé II:

$$y'' = x\,y(x) + 2\,y^3(x)$$

- Tracy-Widom law for statistics of maximum eigenvalue for Gaussian random matrices
- correlators in polynuclear growth; directed polymers (KPZ)
- double-scaling limit in unitary matrix models
- double-scaling limit in 2d Yang-Mills
- double-scaling limit in 2d supergravity
- non-intersecting Brownian motions
- longest increasing subsequence in random permutations
- ... universal !

• here: illustrate differences from the linear Airy equation

Resurgence in Nonlinear ODEs: Painlevé II = "nonlinear Airy"

$$y''(x) = x\,y(x) + 2\,y^3(x)$$

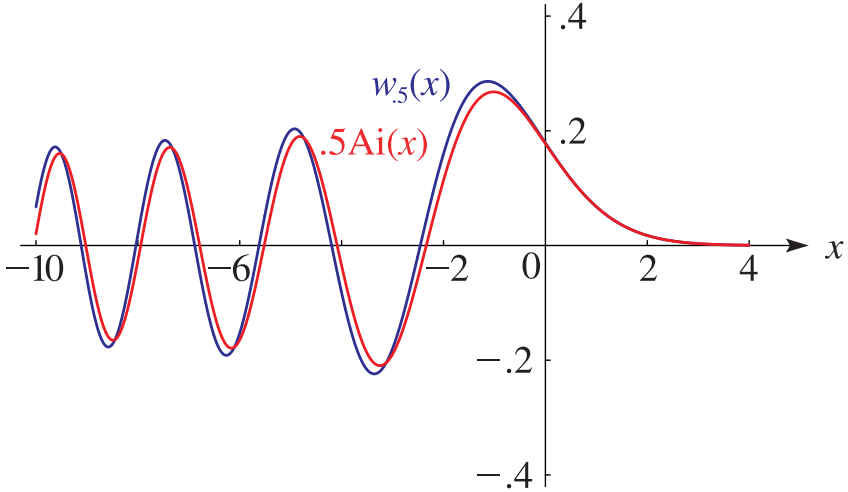
- $x \to +\infty$: $y''(x) \approx x\,y(x) + \ldots \Rightarrow \quad y(x) \sim \sigma_+\,\mathrm{Ai}(x) + \ldots$

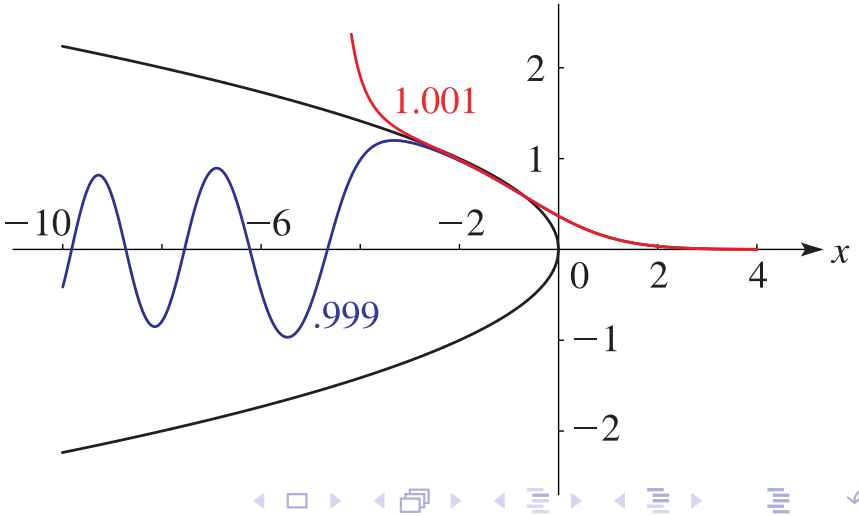
- $x \to -\infty$: $x\,y(x) + 2\,y^3(x) \approx 0 + \ldots \Rightarrow y(x) \sim \pm\sqrt{-\frac{x}{2}} + \ldots$

- general solution is meromorphic (Painlevé integrability)

$$y(x) = \frac{1}{x - x_0} - \frac{x_0}{6}(x - x_0) - \frac{1}{4}(x - x_0)^2 + h_0(x - x_0)^3 + \frac{x_0}{72}(x - x_0)^4 + .$$

- numerical instability: separatrix





$$y''(x) = x\,y(x) + 2\,y^3(x)$$

Exercise 2.1:

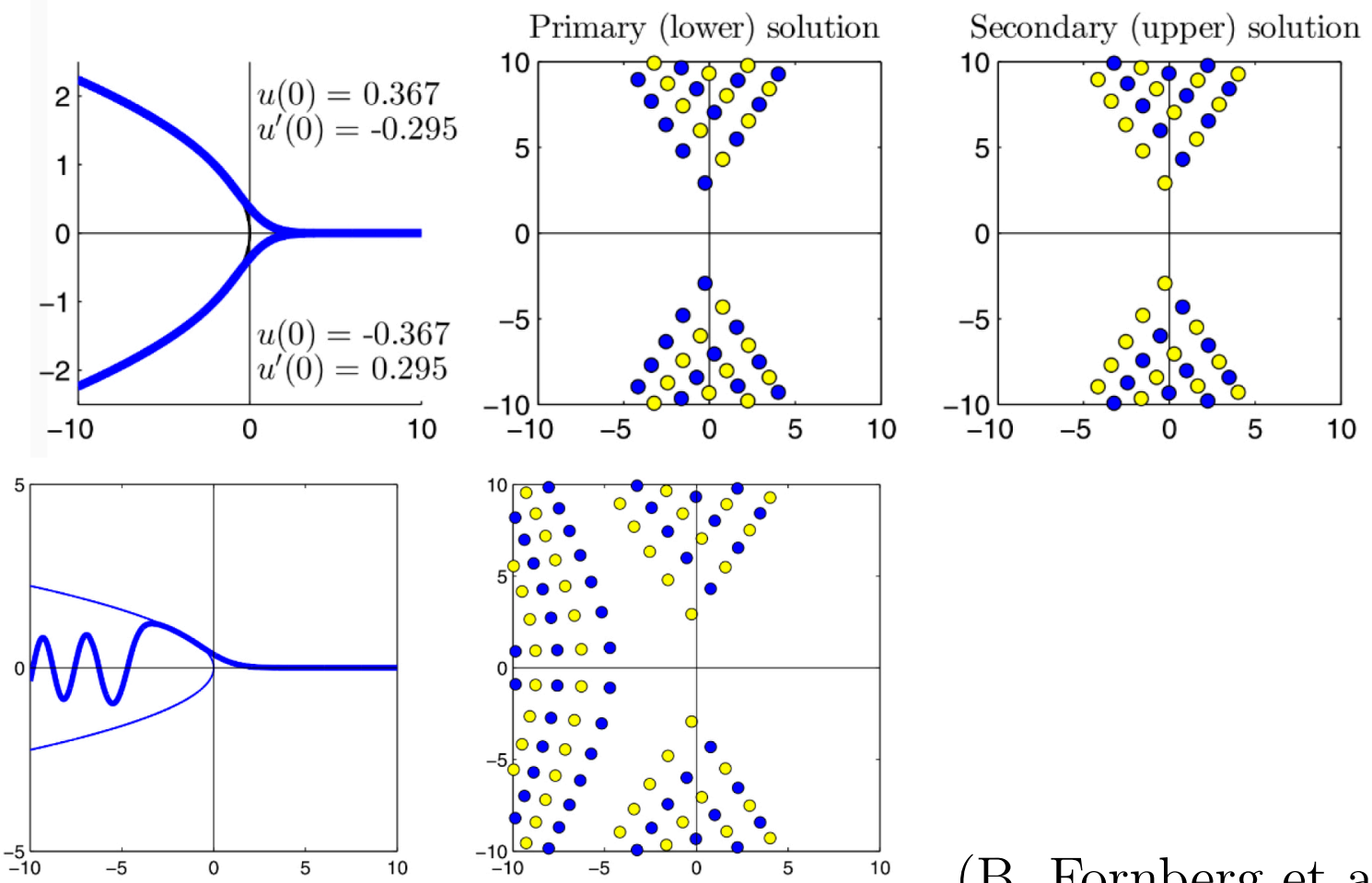
1. Show that the general Painlevé II solution has a meromorphic expansion with only poles for moveable singularities (those associated with boundary conditions):

$$y(x) = \frac{1}{x - x_0} - \frac{x_0}{6}(x - x_0) - \frac{1}{4}(x - x_0)^2 + h_0(x - x_0)^3 + \frac{x_0}{72}(x - x_0)^4 + \ldots$$

2. Change the nonlinearity of the equation from $y^3(x)$ to $y^4(x)$ and show that this Painlevé integrability condition fails (comment: nevertheless, despite being nonintegrable, all the subsequent resurgent trans-series analysis still holds for such an equation)

Resurgence in Nonlinear ODEs: Painlevé II = "nonlinear Airy"

• Hastings-McLeod: $\sigma_+ = 1$ unique real solution on $\mathbb{R}$ that matches $\sigma_+ \operatorname{Ai}(x)$ asymptotics as $x \to +\infty$ with $\sqrt{-\frac{x}{2}}$ asymptotics as $x \to -\infty$



(B. Fornberg et al)

Painlevé II: Boutroux's Asymptotic Weierstrass Form

$$y''(x) = 2\,y^3 + x\,y$$

- define a new function, $y(x) = \sqrt{x}\,w(x)$:

$$w'' + \frac{1}{x}\,w' - \frac{1}{4x^2}\,w(x) = 2x\,w^3(x) + x\,w(x)$$

- change variable to "Écalle variable", $z = \frac{2}{3}x^{3/2}$:

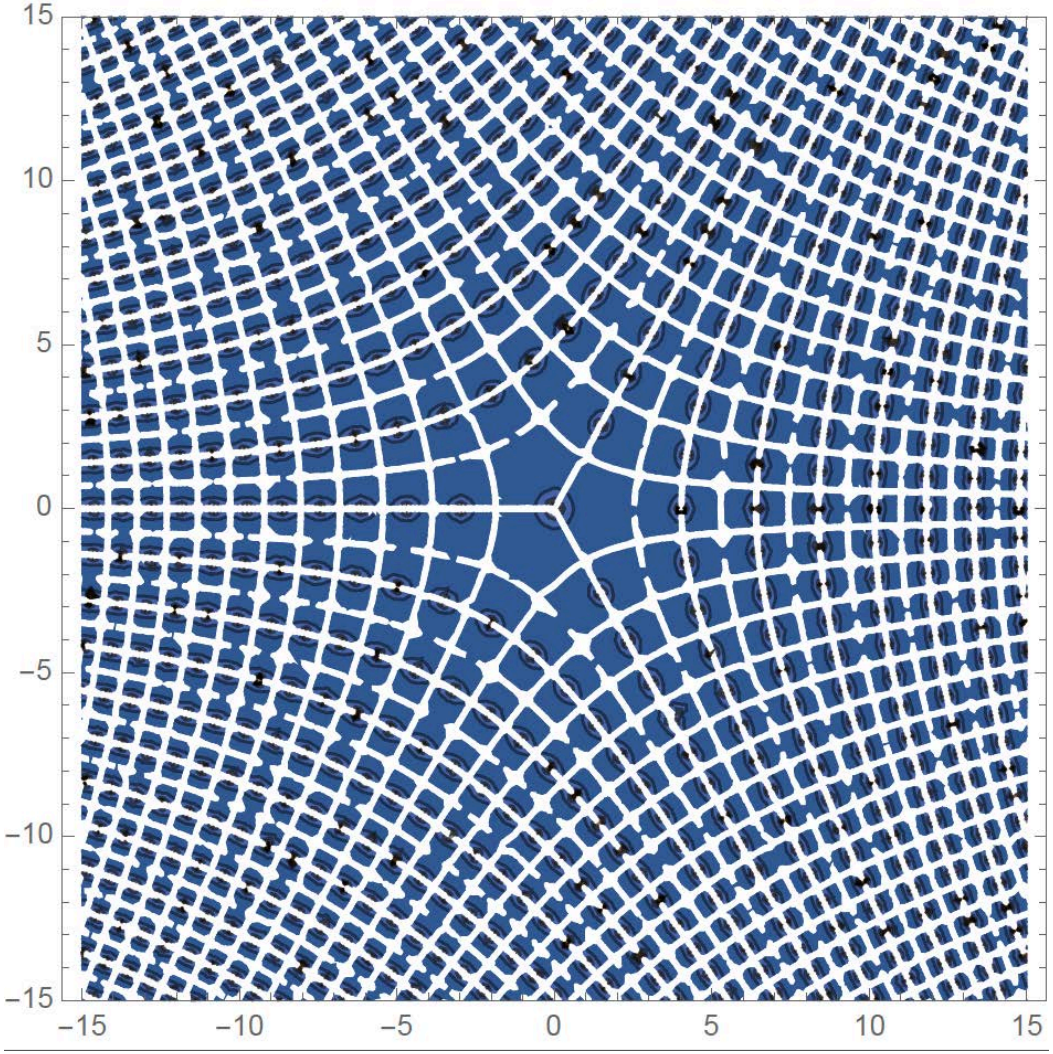
$$w''(z) = \left[2\,w^3(z) + w(z)\right] + \left\{\frac{1}{9z^2}w(z) - \frac{w'(z)}{z}\right\}$$

- approximate solution in terms of Weierstrass $\mathcal{P}$ function

$$y(x) \approx \sqrt{x}\sqrt{\mathcal{P}\left(\frac{2}{3}x^{3/2}; \left\{g_2, \frac{4-9g_2}{27}\right\}\right) - \frac{1}{3}}$$

- delicate tuning removes groups of poles to infinity

Painlevé II: Boutroux's Asymptotic Weierstrass Form



- asymptotic $\frac{2\pi}{3}$ structure

Painlevé II: trans-series analysis as $x \to +\infty$: $\quad y''(x) = 2\,y^3 + x\,y$

- exact integral equation:

$$y(x) = \sigma_+ \,\mathrm{Ai}(x) + 2\pi \int_x^\infty dz\, y^3(z)\,[\mathrm{Ai}(x)\,\mathrm{Bi}(z) - \mathrm{Ai}(z)\,\mathrm{Bi}(x)]$$

- iterate $\to$ trans-series (all odd powers of σ_+)

$$y_+(x) \sim \sum_{n=0}^{\infty} \sigma_+^{2n+1}\, Y_{[2n+1]}(x) \qquad , \quad x \to +\infty$$

$$\begin{aligned}
Y_{[1]}(x) &= \mathrm{Ai}(x) \\
Y_{[3]}(x) &= 2\pi \left(\mathrm{Ai}(x) \int_x^\infty \mathrm{Ai}^3(z)\mathrm{Bi}(z)\, dz - \mathrm{Bi}(x) \int_x^\infty \mathrm{Ai}^4(z)\, dz \right) \\
Y_{[5]}(x) &= 6\pi \left(\mathrm{Ai}(x) \int_x^\infty Y_{[3]}(z) \left(Y_{[1]}(z)\right)^2 \mathrm{Bi}(z)\, dz \right. \\
&\qquad \left. -\mathrm{Bi}(x) \int_x^\infty Y_{[3]}(z) \left(Y_{[1]}(z)\right)^2 \mathrm{Ai}(z)\, dz \right)
\end{aligned}$$

Painlevé II: trans-series analysis as $x \to +\infty$

- trans-series with all odd powers of Airy exponentials

$$y_+(x) \sim \sum_{n=0}^{\infty} \left(\frac{\sigma_+ \, e^{-\frac{2}{3} x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}} \right)^{2n+1} \mathcal{F}_{[2n+1]}(x)$$

- instanton fluctuation series of the form

$$\mathcal{F}_{[2n+1]}(x) \sim \frac{1}{x^n} \sum_{m=0}^{\infty} \frac{d_m^{[2n+1]}}{x^{3m/2}}$$

- evaluate $y(0)$ and $y'(0)$ for Hastings-McLeod $\sigma_+ = 1$:

$$y_+(0) \approx \mathrm{Ai}(0) + 2\pi \left(\frac{1}{24\pi} \mathrm{Ai}(0) - \frac{\ln 3}{24\pi^2} \mathrm{Bi}(0) \right) = 0.366693782...$$

$$y_+'(0) \approx \mathrm{Ai}'(0) + 2\pi \left(\frac{1}{24\pi} \mathrm{Ai}'(0) - \frac{\ln 3}{24\pi^2} \mathrm{Bi}'(0) \right) = -0.293451526...$$

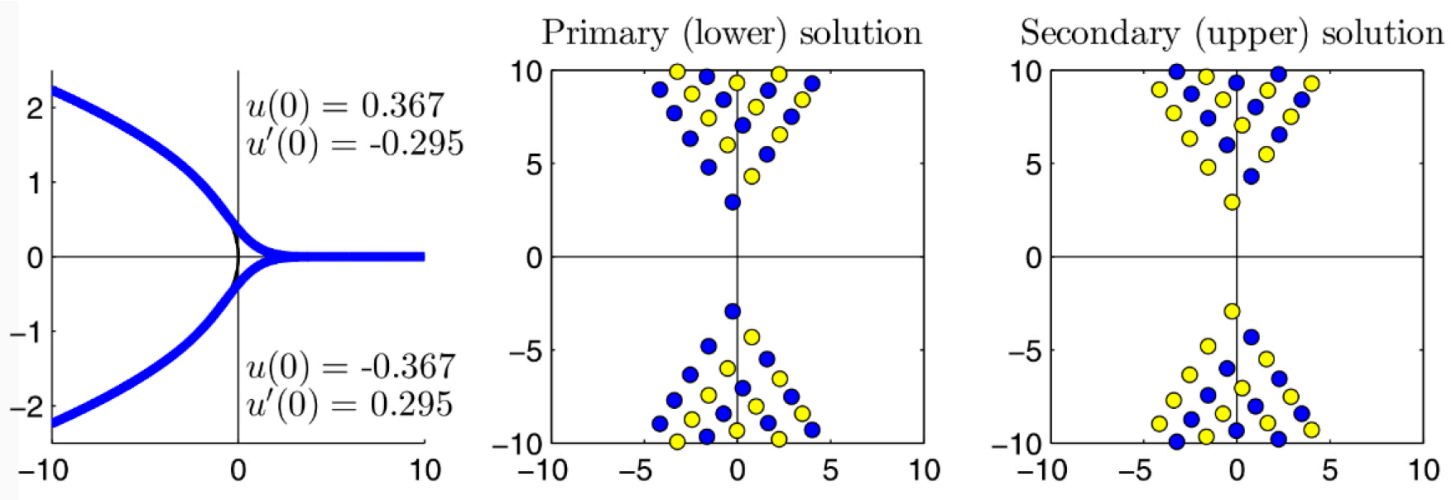
note: $\int_0^\infty \mathrm{Ai}^4(z)\, dz = \frac{\ln 3}{24\pi^2}$ and $\int_0^\infty \mathrm{Bi}(z)\mathrm{Ai}^3(z)\, dz = \frac{1}{24\pi}$

Painlevé II: trans-series analysis as $x \to +\infty$

$$y_+(0) \approx \mathrm{Ai}(0) + 2\pi\left(\frac{1}{24\pi}\mathrm{Ai}(0) - \frac{\ln 3}{24\pi^2}\mathrm{Bi}(0)\right) = 0.366693782...$$

$$y_+'(0) \approx \mathrm{Ai}'(0) + 2\pi\left(\frac{1}{24\pi}\mathrm{Ai}'(0) - \frac{\ln 3}{24\pi^2}\mathrm{Bi}'(0)\right) = -0.293451526...$$

- compare with numerics:



Painlevé II: trans-series analysis as $x \to -\infty$

- now consider the opposite direction: $x \to -\infty$

$$y''(x) = x\,y(x) + 2\,y^3(x)$$

- separatrix tracking as $x \to -\infty \Rightarrow$

$$0 \approx x\,y(x) + 2\,y^3(x) \quad , \quad x \to -\infty$$

- formal series solution

$$y_-(x) \sim \sqrt{\frac{-x}{2}}\left(1 - \frac{1}{8(-x)^3} - \frac{73}{128(-x)^6} - \frac{10567}{1024(-x)^9} - \dots\right)$$

- no parameter! $\Rightarrow$ something is missing (non-perturbative corrections)

- <u>non-alternating</u> factorially divergent $\Rightarrow$ something is missing (non-perturbative corrections)

Trans-series Ansatz for Nonlinear ODEs

- finding the building blocks of the trans-series

$$\mathcal{D}_{\text{linear}}\, y(x) = F_{\text{nonlinear}}(y(x))$$

- formal perturbative series $y_{[0]}(x)$ satisfies this equation term-by-term

- extend to include first non-pert. trans-series correction

$$y(x) \approx y_{[0]}(x) + y_{[\text{np}]}(x) + \ldots$$

$$\rightarrow \quad \mathcal{D}_{\text{linear}}\, y_{[0]} + \mathcal{D}_{\text{linear}}\, y_{[\text{np}]} = F_{\text{nonlinear}}(y_{[0]}) + F'_{\text{nonlinear}}(y_{[0]})\, y_{[\text{np}]}$$

- linear and homogeneous equation for $y_{[\text{np}]}(x)$

- solve by ansatz, where $x_c =$ Écalle critical variable

$$y_{[\text{np}]}(x) \sim \sigma\, x_c^{\beta}\, e^{-\gamma\, x_c} \sum_n \frac{d_n}{x_c^n}$$

- all higher trans-series terms solve linear equations

Painlevé II: trans-series analysis as $x \to -\infty$

- formal perturbative series solves $y'' = x\,y(x) + 2\,y^3(x)$

$$y_{[0]}(x) \sim \left(\frac{-x}{2}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{c_n}{(-x)^{3n}}$$

- trans-series ansatz for $x \to -\infty$

$$y(x) \sim \left(\frac{-x}{2}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \sigma_-^k\, y_{[k]}(x)$$

- $\Rightarrow$ tower of *linear* equations for $y_{[k]}$
- $y_{[1]}$: linear & homogeneous equation

$$y''_{[1]} = \left(6\,y_{[0]}^2 + x\right) y_{[1]}$$

- $y_{[2]}$: linear & inhomogeneous equation

$$y''_{[2]} = \left(6\,y_{[0]}^2 + x\right) y_{[2]} + 6\,y_{[0]}\,y_{[1]}^2$$

Painlevé II: trans-series analysis as $x \to -\infty$

- first non-perturbative correction "beyond all orders":

$$y''_{[1]} = \left(6\, y_{[0]}^2 + x\right) y_{[1]} \sim \left(-\,2\,x - \frac{3}{4x^2} + \dots\right) y_{[1]}$$

- exponential ansatz (using Écalle critical variable):

$$y_{[1]}(x) \sim (-x)^{\beta} e^{-\gamma(-x)^{\frac{3}{2}}}\,(1 + \dots)$$

- matching terms $\Rightarrow$

$$y_{[1]}(x) \sim \frac{\sigma_-}{(-x)^{1/4}} e^{-\sqrt{2}\,\frac{2}{3}(-x)^{3/2}} \left(1 - \frac{\frac{17}{72}}{\sqrt{2}\,\frac{2}{3}(-x)^{3/2}} + \frac{\frac{1513}{10368}}{(\sqrt{2}\,\frac{2}{3}(-x)^{3/2})^2} - \dots\right)$$

- the $\sqrt{2}$ factor is not a misprint !

Resurgence Relation for Painlevé II

- recall formal perturbative series as $x \to -\infty$

$$y_{[0]}(x) \sim \sqrt{\frac{-x}{2}}\left(1 - \frac{1}{8(-x)^3} - \frac{73}{128(-x)^6} - \frac{10567}{1024(-x)^9} - \dots\right)$$

- large order growth of coefficients as $n \to \infty$

$$c_n \sim -(0.1466323\dots)\frac{\Gamma\left(2n - \frac{1}{2}\right)}{\left(\frac{2\sqrt{2}}{3}\right)^{2n}}\left(1 - \frac{\frac{17}{72}}{(2n - \frac{3}{2})} + \frac{\frac{1513}{10368}}{(2n - \frac{3}{2})(2n - \frac{5}{2})} - \dots\right)$$

- compare with fluctuations around the first exponential:

$$y_{[1]}(x) \sim \frac{\sigma_-}{(-x)^{1/4}} e^{-\sqrt{2}\frac{2}{3}(-x)^{\frac{3}{2}}}\left(1 - \frac{\frac{17}{72}}{\sqrt{2}\,\frac{2}{3}(-x)^{\frac{3}{2}}} + \frac{\frac{1513}{10368}}{(\sqrt{2}\,\frac{2}{3}(-x)^{\frac{3}{2}})^2} - \dots\right)$$

- large-order/low-order resurgence relation (*cf.* Airy)
- continues to all orders of trans-series (bridge equations)

Exercise 2.2:

1. Generate many terms, and verify the large order behavior of the coefficients of the formal $x \to -\infty$ series for the Painlevé II Hastings-McLeod solution
2. Explain why you should have been able to predict that the factorial growth involves $\Gamma(2n - ...)$ instead of $\Gamma(n - ...)$.
3. Numerically identify the Stokes constant to high precision

$$0.1466323... = \frac{1}{\pi}\sqrt{\frac{2}{3\pi}}$$

4. Confirm the large-order/low-order resurgence relation

Resurgence in Nonlinear ODEs: e.g. Painlevé II

- Hastings-McLeod solution
- different trans-series solutions for $x \to \pm\infty$
- $x \to +\infty$

$$y_+(x) \sim \sum_{k=0}^{\infty} \left(\frac{\sigma_+ \, e^{-\frac{2}{3}\, x^{3/2}}}{2\,\sqrt{\pi}\, x^{1/4}} \right)^{2k+1} \mathcal{F}_{[2k+1]}(x)$$

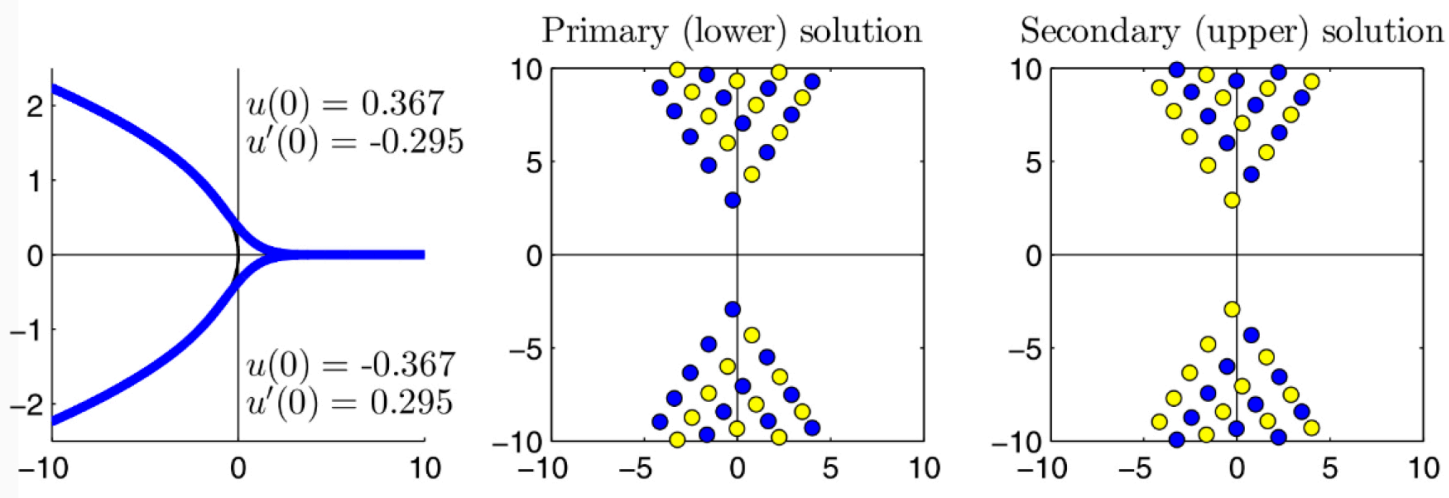
- $x \to -\infty$

$$y_-(x) \sim \sqrt{\frac{-x}{2}} \sum_{k=0}^{\infty} \left(\frac{\sigma_- \, e^{-\frac{2\sqrt{2}}{3}(-x)^{3/2}}}{2\sqrt{\pi}\,(-x)^{1/4}} \right)^{k} \mathcal{Y}_{[k]}(x)$$

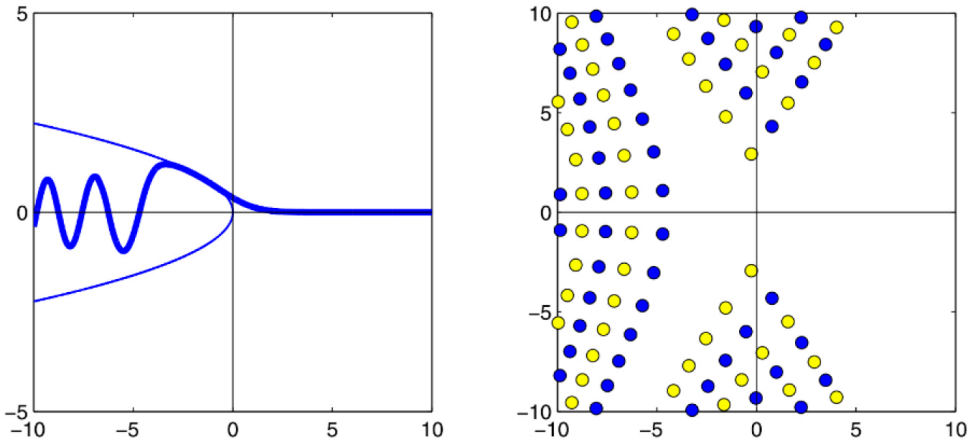
- non-linear Stokes phenomenon
- "condensation of instantons" across the transition
- trans-asymptotic analysis describes the transition

Resurgence in Nonlinear ODEs: Painlevé II = "nonlinear Airy"

- recall the interesting structure in the complex plane ...
- Hastings-McLeod: $\mathrm{Ai}(x)$ asymptotics as $x \to +\infty$, $\sqrt{-\frac{x}{2}}$ asymptotics as $x \to -\infty$



- there are other more general solutions ...



Gross-Witten-Wadia Matrix Model and Painlevé III

- partition function

$$Z(t,N) = \int_{U(N)} DU \, \exp\left[\frac{N}{t}\mathrm{tr}\left(U + U^\dagger\right)\right]$$

- two variables: 't Hooft coupling t, and matrix size N
- 3rd order phase transition at $N = \infty$, $t = 1$ (universal)

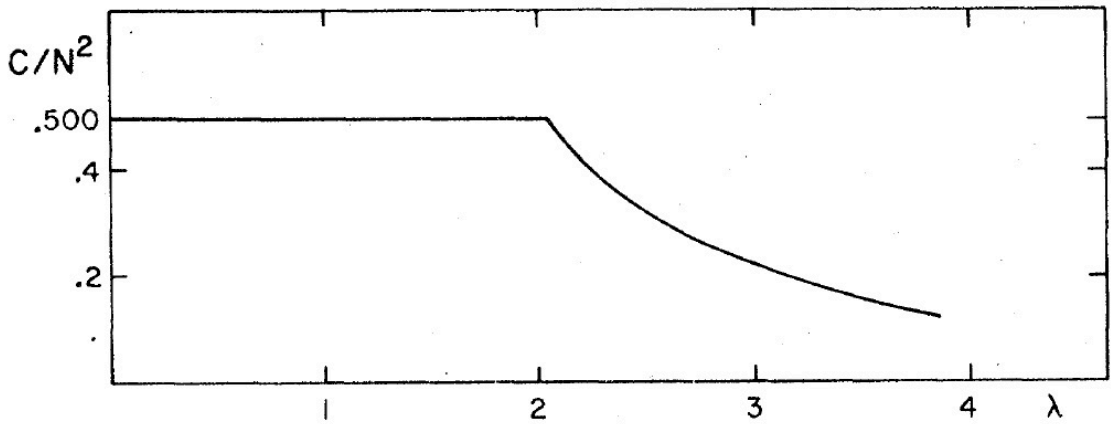


FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

- random matrix theory: $Z(t,N) = \det\left[I_{j-k}\left(\frac{N}{t}\right)\right]_{j,k=1,\ldots N}$
- large N asymptotics? Beyond the double-scaling region

Gross-Witten-Wadia Matrix Model and Painlevé III

- "order parameter" $\Delta(t,N) \equiv \langle \det U \rangle$ satisfies a Painlevé III eqn

$$t^2\Delta'' + t\Delta' + \frac{N^2\Delta}{t^2}\left(1-\Delta^2\right) = \frac{\Delta}{1-\Delta^2}\left(N^2 - t^2\left(\Delta'\right)^2\right)$$

- $' \equiv \frac{\partial}{\partial t}$; N appears only as a parameter
- direct relation to the partition function

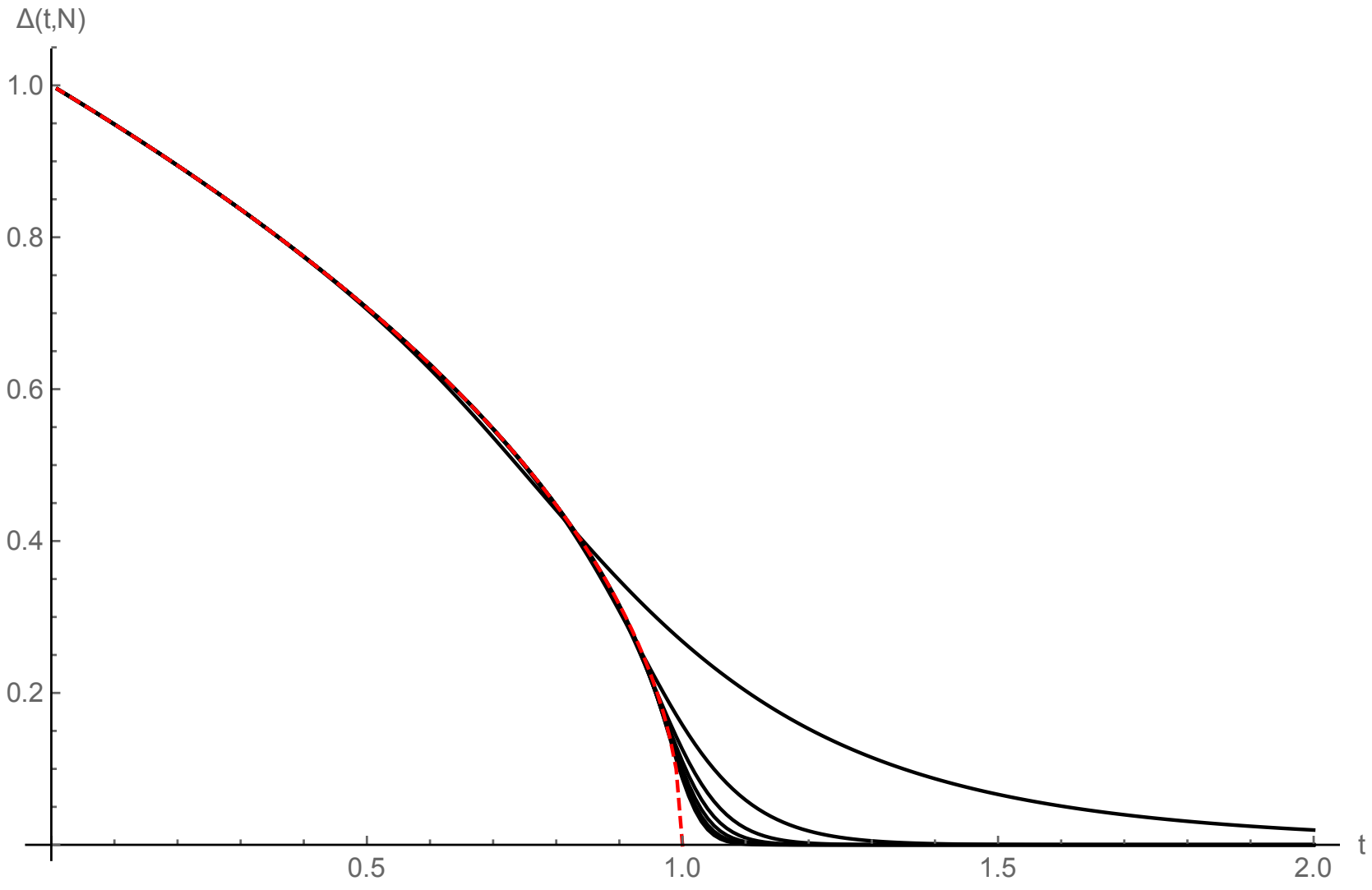
$$\Delta^2(t,N) = 1 - \frac{Z(t,N-1)\,Z(t,N+1)}{Z^2(t,N)}$$

- non-perturbative large N effects from the ODE

$$\Delta(t,N) = \sum_n \frac{c_n^{(0)}(t)}{N^n} + e^{-N\,S(t)} \sum_n \frac{c_n^{(1)}(t)}{N^n} + e^{-2N\,S(t)} \sum_n \frac{c_n^{(2)}(t)}{N^n} + \ldots$$

- all physical observables inherit this trans-series structure
- trans-series changes from strong to weak coupling: Stokes jump from real to complex saddles of eigenvalue integrals

Gross-Witten-Wadia Matrix Model and Painlevé III



Exercise 2.3: Consider the Painlevé III (Okamoto form) for the GWW model:

$$t^2\Delta'' + t\Delta' + \frac{N^2\Delta}{t^2}\left(1-\Delta^2\right) = \frac{\Delta}{1-\Delta^2}\left(N^2 - t^2\left(\Delta'\right)^2\right)$$

1. Show that in the $t > 1$ region this equation linearizes to

$$t^2\Delta'' + t\Delta' + \frac{N^2}{t^2}\left(1-t^2\right)\Delta \approx 0$$

   and this is solved by the Bessel functions $J_N\left(\frac{N}{t}\right)$, $Y_N\left(\frac{N}{t}\right)$.

2. Hence show that a solution decreasing at large t can be written as an exact integral equation, which can be iterated to generate the $t > 1$ large N trans-series.

3. Show that for $t < 1$ the dominant large N solution is algebraic, $\Delta(t) \sim \sqrt{1-t}$, from which the formal large N series solution can be generated.

Resurgence: Large N 't Hooft limit at Weak Coupling

- large N trans-series at weak-coupling ($t < 1$)

$$\Delta(t,N) \sim \sqrt{1-t}\sum_{n=0}^{\infty}\frac{d_n^{(0)}(t)}{N^{2n}} - \frac{\sigma_{\text{weak}}}{2\sqrt{2\pi N}}\,\frac{t\,e^{-NS_{\text{weak}}(t)}}{(1-t)^{1/4}}\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} + \dots$$

- weak-coupling large N instanton action

$$S_{\text{weak}}(t) = \frac{2\sqrt{1-t}}{t} - 2\,\text{arctanh}\left(\sqrt{1-t}\right)$$

- fluctuation around first exponential from the ODE

$$\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{1}{N} + \dots$$

- large-order growth of perturbative coefficients ($\forall\, t < 1$):

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}}\,\frac{\Gamma(2n-\frac{5}{2})}{(S_{\text{weak}}(t))^{2n-\frac{5}{2}}}\left[1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{S_{\text{weak}}(t)}{(2n-\frac{7}{2})} + \dots\right.$$

- (parametric) resurgence relation, for all t

Resurgence: Large N 't Hooft limit at Strong Coupling

- large N trans-series at strong-coupling ($t > 1$)

$$\Delta(t,N) \approx J_N\left(\frac{N}{t}\right) \sim \frac{\sqrt{t}\, e^{-N S_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n} + \dots$$

- strong-coupling large N instanton action

$$S_{\text{strong}}(t) = \text{arccosh(t)} - \sqrt{1-\frac{1}{t^2}}$$

- complex saddles of eigenvalue integrals (*cf.* Stokes)
- more accurate uniform large N instanton expansion

$$\Delta(t,N) \sim \left(\frac{4\left(\frac{3}{2}S_{\text{strong}}(t)\right)^{2/3}}{1-1/t^2}\right)^{\frac{1}{4}} \frac{\text{Ai}\left(N^{\frac{2}{3}}\left(\frac{3}{2}S_{\text{strong}}(t)\right)^{2/3}\right)}{N^{\frac{1}{3}}} + \dots$$

- analogous to uniform WKB
- double-scaling limit = Painlevé II = non-linear Airy eqn

Resurgence in GWW: double-scaling limit = Painlevé II

- reduction cascade of Painlevé equations
- "zoom in" on vicinity of phase transition:

$$\kappa \equiv N^{2/3}(t-1) \quad ; \quad \Delta(t,N) = \frac{t^{1/3}}{N^{1/3}} y(\kappa)$$

- $N \to \infty$ with κ fixed:

$$\Delta \quad \text{PIII equation} \quad \longrightarrow \quad \frac{d^2 y}{d\kappa^2} = 2\, y^3(\kappa) + 2\kappa\, y(\kappa) \quad \text{(PII)}$$

- e.g. on strong-coupling side:

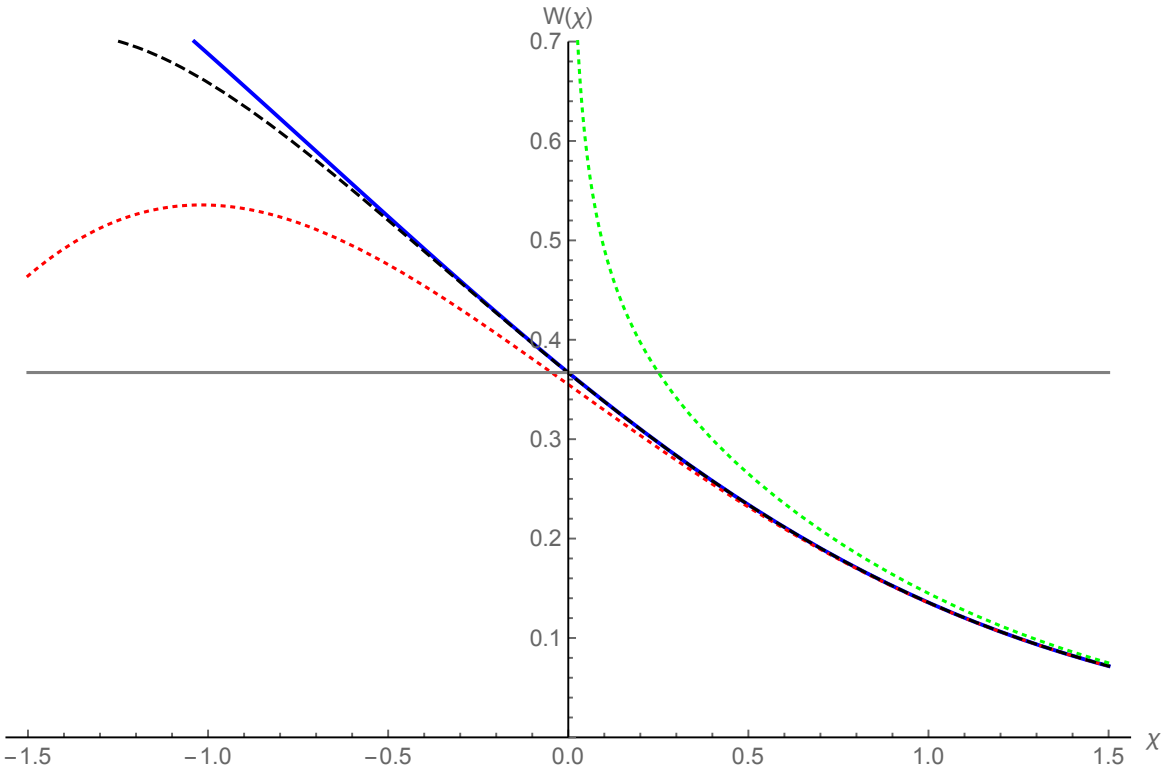
$$\lim_{N\to\infty} J_N(N - N^{1/3}\kappa) = \left(\frac{2}{N}\right)^{1/3} \mathrm{Ai}\left(2^{1/3}\kappa\right)$$

- the immediate vicinity of the physical phase transition region is described by the Hastings-McLeod PII solution

Resurgence in GWW: double-scaling limit = Painlevé II

- "zoom in" on vicinity of phase transition:
- integral equation form of PII:

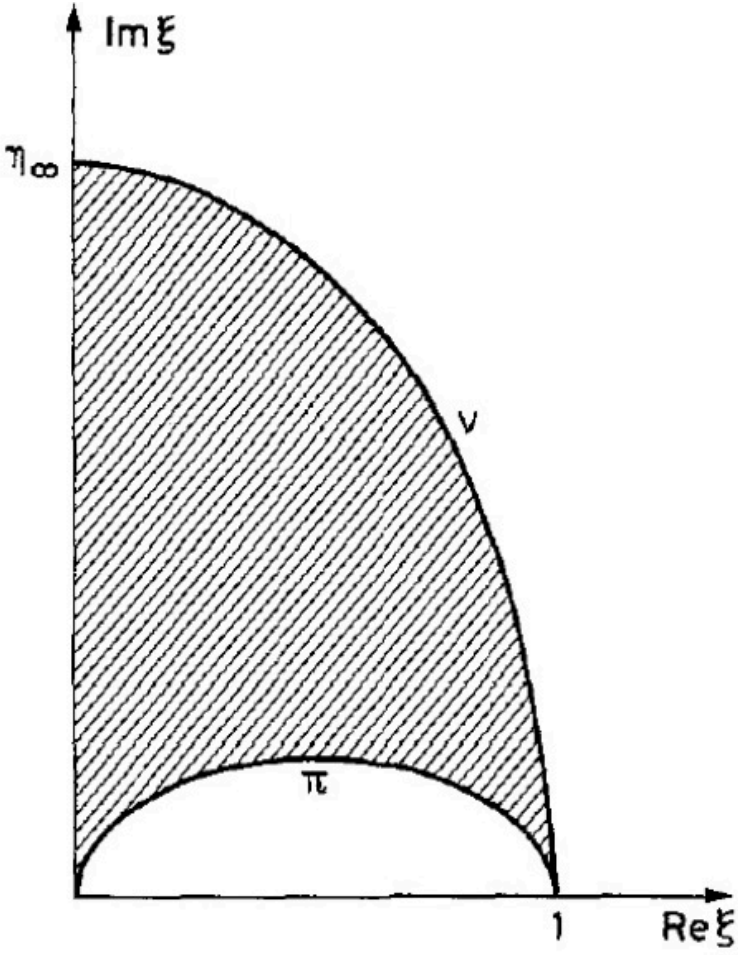
$$y(\chi) = \sigma\,\mathrm{Ai}(\chi) + 2\pi \int_{\chi}^{\infty} \left[\mathrm{Ai}(\chi)\mathrm{Bi}(\chi') - \mathrm{Ai}(\chi')\mathrm{Bi}(\chi)\right] y^3(\chi')\, d\chi'$$

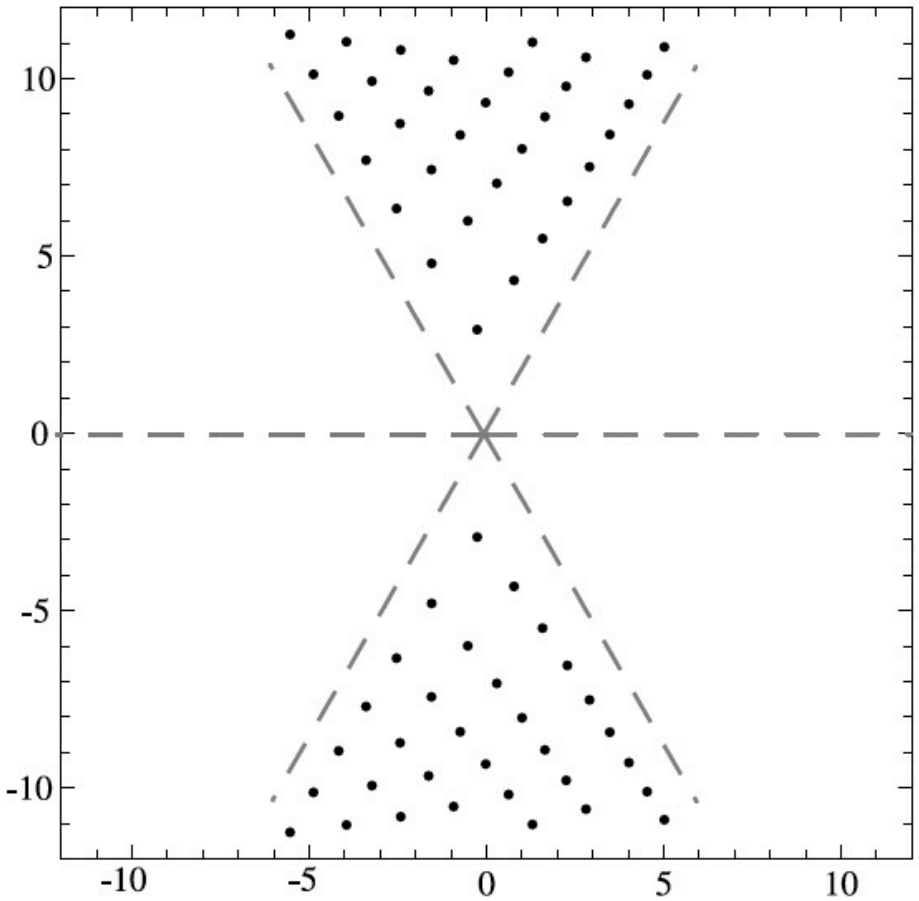


iterate $\longrightarrow$ resummed trans-series instanton expansion
blue: exact
red: leading uniform large N
dashed: sub-leading uniform large N
green dashed: usual large N

Gross-Witten-Wadia Phase Transition and Lee-Yang zeros

Lee-Yang: complex zeros of Z pinch the real axis at the phase transition point in the thermodynamic limit





GWW zeros (Kolbig)

Painlevé II (Novokshenov; Huang)

Resurgence for Painlevé VI

• Painlevé VI reduces to the other Painlevé equations by a well-defined cascade of coalescence limits, analogous to what happens for the familiar (linear) special functions

• remarkable recent result (Gamayun, Iorgov, Lisovyy, 2013): the Jimbo expansion of the Painlevé VI tau function has an all orders convergent conformal block expansion (*cf.* CFT)

$$\tau(t) \sim \sum_{n=-\infty}^{\infty} s^n \, C(\vec{\theta}, \sigma+n) \, \mathcal{B}(\vec{\theta}, \sigma+n; t)$$

$$\mathcal{B}(\vec{\theta}, \sigma; t) \propto t^{\sigma^2} \sum_{\lambda,\mu \in \mathcal{Y}} \mathcal{B}_{\lambda,\mu}(\vec{\theta}, \sigma) t^{|\lambda|+|\mu|}$$

$\vec{\theta}$: monodromy parameters; $C(\vec{\theta}, \sigma)$: known combinatorial functions; s & σ are b.c parameters; the t-dependent term $\mathcal{B}(\vec{\theta}, \sigma; t)$ is written as a sum over partitions.

• expression is manifestly resurgent: the sum over n is an instanton sum, encoded in the zero-instanton sector

A Few Selected References

- Digital Library of Mathematical Functions: https://dlmf.nist.gov/
- C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*
- O. Costin, *Asymptotics and Borel Summability*
- M. Mariño, *Instantons and Large N : An Introduction to Non-Perturbative Methods in Quantum Field Theory*
- M. V. Berry and C. J. Howls, "Hyperasymptotics for integrals with saddles", Proc. R. Soc. A **434**, 657 (1991)
- G. Edgar, "Trans-series for beginners", arXiv:0801.4877
- P. A. Clarkson, "Painlevé Equations: Nonlinear Special Functions"
- D. Dorigoni, "An Introduction to Resurgence, Trans-Series and Alien Calculus," arXiv:1411.3585.
- I. Aniceto, G. Basar and R. Schiappa, "A Primer on Resurgent Transseries and Their Asymptotics," arXiv:1802.10441.
- O.Costin, G. V. Dunne, "Physical Resurgent Extrapolation", arXiv:2003.07451, "Resurgent Extrapolation: Rebuilding a Function from Asymptotic Data. Painleve I", arXiv:1904.11593