Optimal truncation and Stokes lines in nonlinear ODEs

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Overview

Methodology

- Stokes phenomenon
- Singular perturbations and divergent series
- Stokes line smoothing
- Factorial/power ansatz

Examples

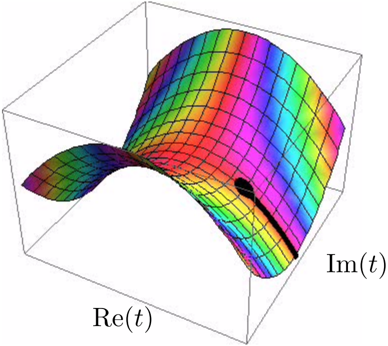
- Nonlinear eigenvalue problems (Kruskal-Segur)

Stokes Phenomenon

Complementary error function

$$\mathsf{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

For complex z deform to the steepest descent contour. As z crosses the imaginary axis the steepest descent contour goes to $-\infty$ rather than ∞.

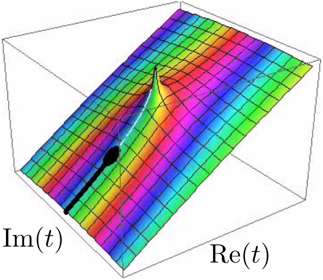


The relative size of each contribution is determined by the heights of the end point and the saddle. The new contribution is exponentially subdominant when it is turned on.

Stokes Phenomenon

Simplest example is the exponential integral.

$$\epsilon \frac{\mathrm{d}f}{\mathrm{d}z} + f = \frac{\epsilon}{z}, \qquad f \to 0 \text{ as } z \to -\infty.$$



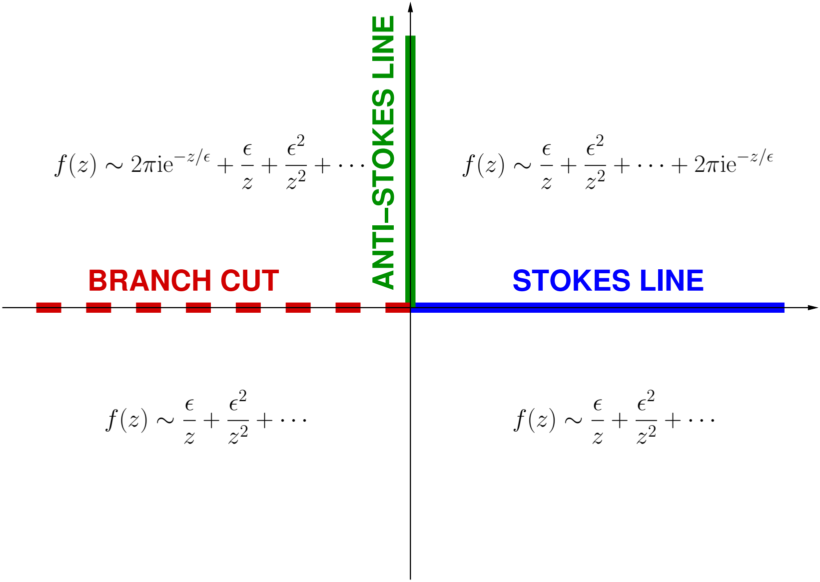
Using an integrating factor gives

$$f = \mathrm{e}^{-z/\epsilon} \int_{-\infty}^{z/\epsilon} \frac{\mathrm{e}^t}{t}\,\mathrm{d}t.$$

Asymptotic expansion as $\epsilon \to 0$ (integrate by parts):

$$f \sim \frac{\epsilon}{z} + \frac{\epsilon^2}{z^2} + \frac{2\epsilon^3}{z^3} + \cdots .$$

This time the extra contribution is $2\pi \mathrm{i} \mathrm{e}^{-z/\epsilon}$, and it is present for $\arg(z) > 0$. (NB f is not analytic. It has a log singularity at $z = 0$.)



ANTI–STOKES LINE
BRANCH CUT
STOKES LINE
$f(z) \sim 2\pi \mathrm{i} \mathrm{e}^{-z/\epsilon} + \frac{\epsilon}{z} + \frac{\epsilon^2}{z^2} + \cdots$
$f(z) \sim \frac{\epsilon}{z} + \frac{\epsilon^2}{z^2} + \cdots + 2\pi \mathrm{i} \mathrm{e}^{-z/\epsilon}$
$f(z) \sim \frac{\epsilon}{z} + \frac{\epsilon^2}{z^2} + \cdots$
$f(z) \sim \frac{\epsilon}{z} + \frac{\epsilon^2}{z^2} + \cdots$

Stokes phenomenon and asymptotic expansions

What if we don't have an integral representation?

$$\epsilon \frac{\mathrm{d}f}{\mathrm{d}z} + f = \frac{\epsilon}{z}, \qquad f \to 0 \text{ as } z \to -\infty.$$

Formally expanding f in powers of ϵ,

$$f \sim \sum_{n=1}^{\infty} \epsilon^n f_n,$$

substituting into the equation and equating coefficients gives

$$f_1 = \frac{1}{z}, \qquad \frac{\mathrm{d}f_{n-1}}{\mathrm{d}z} + f_n = 0, \quad n \geq 2.$$

To get the next term differentiate the previous one. Hence

$$f_2 = \frac{1}{z^2}, \quad f_3 = \frac{2}{z^3}, \quad f_4 = \frac{6}{z^4}, \quad \cdots, \quad f_n = \frac{(n-1)!}{z^n}.$$

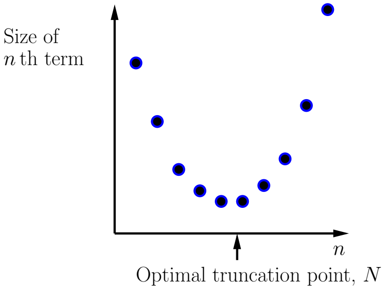
The factorial over power arises from continually differentiating $1/z$.

Smoothing the Stokes discontinuity

The full expansion of f is

$$f \sim \sum_{n=1}^{\infty} \frac{(n-1)!\epsilon^n}{z^n}.$$

This series diverges for all z. This divergence is a sign that an exponential is lurking behind the series.



For a good approximation we need to **truncate** the asymptotic expansion:

$$f \sim \sum_{n=1}^{N-1} \frac{(n-1)!\epsilon^n}{z^n} + R_N$$

For fixed N, $R_N \sim \epsilon^N$ as $\epsilon \to 0$. But if we truncate at the least term (that is, we truncate **optimally**) then R_N is exponentially small. The ratio between successive terms is $n\epsilon/z$. Hence the least term occurs for $N \sim |z|/\epsilon$.

Optimal truncation (Berry 89)

Let's truncate optimally and study the remainder. We find R_N satisfies

$$\epsilon \frac{\mathrm{d}R_N}{\mathrm{d}z} + R_N = \frac{(N-1)!\epsilon^N}{z^N}.$$

Let $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $N = r/\epsilon + \alpha$ where α bounded as $r \to \infty$. Then using Stirling's formula

$$n! \sim \sqrt{2\pi}\, n^{n+1/2}\, \mathrm{e}^{-n},$$

we find

$$\begin{aligned}
\frac{(N-1)!\epsilon^N}{z^N} &\sim \frac{\epsilon^N \sqrt{2\pi}(N-1)^{N-1/2}\mathrm{e}^{-(N-1)}}{(r\mathrm{e}^{\mathrm{i}\theta})^N} \\
&= \frac{\epsilon^{r/\epsilon+\alpha}\sqrt{2\pi}(r/\epsilon+\alpha-1)^{r/\epsilon+\alpha-1/2}\mathrm{e}^{-(r/\epsilon+\alpha-1)}}{r^{r/\epsilon+\alpha}\mathrm{e}^{\mathrm{i}\theta(r/\epsilon+\alpha)}} \\
&= \frac{\epsilon^{1/2}\sqrt{2\pi}}{r^{1/2}}\left(1 + \frac{\epsilon(\alpha-1)}{r}\right)^{r/\epsilon+\alpha-1/2} \frac{\mathrm{e}^{-r/\epsilon-\alpha+1}}{\mathrm{e}^{\mathrm{i}\theta(r/\epsilon+\alpha)}} \\
&\sim \frac{\epsilon^{1/2}\sqrt{2\pi}}{r^{1/2}}\mathrm{e}^{\alpha-1}\frac{\mathrm{e}^{-r/\epsilon-\alpha+1}}{\mathrm{e}^{\mathrm{i}\theta(r/\epsilon+\alpha)}} = \frac{\epsilon^{1/2}\sqrt{2\pi}}{r^{1/2}}\frac{\mathrm{e}^{-r/\epsilon}}{\mathrm{e}^{\mathrm{i}\theta(r/\epsilon+\alpha)}}.
\end{aligned}$$

as $r \to \infty$. Thus, as claimed, the remainder is exponentially small.

Now the homogeneous solution is

$$R_N = \mathrm{e}^{-z/\epsilon}.$$

Let us write $R_N = S(z)\mathrm{e}^{-z/\epsilon}$, where S is the **Stokes multiplier**. Then

$$\epsilon\frac{\mathsf{d}R_N}{\mathsf{d}z} + R_N = \epsilon\mathrm{e}^{-z/\epsilon}\frac{\mathrm{d}S}{\mathrm{d}z} \sim \frac{\epsilon^{1/2}\sqrt{2\pi}}{r^{1/2}}\frac{\mathrm{e}^{-r/\epsilon}}{\mathrm{e}^{\mathrm{i}\theta(r/\epsilon+\alpha)}}.$$

We also write $\mathrm{d}/\mathrm{d}z$ in terms of $\mathrm{d}/\mathrm{d}\theta$ because N is a function of r but not θ.

$$z = r\mathrm{e}^{\mathrm{i}\theta} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\theta} = r\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\frac{\mathrm{d}}{\mathrm{d}z} \quad \Rightarrow \quad \frac{\mathsf{d}}{\mathsf{d}z} = -\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}}{r}\frac{\mathsf{d}}{\mathsf{d}\theta}$$

Then

$$\frac{\mathsf{d}S}{\mathsf{d}\theta} \sim \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\sqrt{2\pi}\,r^{1/2}}{\epsilon^{1/2}}\frac{\mathrm{e}^{-r/\epsilon}\mathrm{e}^{r\mathrm{e}^{\mathrm{i}\theta}/\epsilon}}{\mathrm{e}^{\mathrm{i}\theta(r/\epsilon+\alpha)}}.$$

We see $\mathsf{d}S/\mathsf{d}\theta$ is **exponentially small** except at $\theta = 0$.

Optimal truncation (Berry 89)

$$\frac{\mathrm{d}S}{\mathrm{d}\theta} \sim \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\sqrt{2\pi}\,r^{1/2}}{\epsilon^{1/2}}\frac{\mathrm{e}^{-r/\epsilon}\mathrm{e}^{r\mathrm{e}^{\mathrm{i}\theta}/\epsilon}}{\mathrm{e}^{\mathrm{i}\theta(r/\epsilon+\alpha)}}.$$

There is a **boundary layer** in S near $\theta = 0$.
We examine the boundary layer using the method of matched asymptotic expansions by rescaling $\theta = \delta\bar{\theta}$. Since

$$\mathrm{e}^{r\mathrm{e}^{\mathrm{i}\theta}} \sim \mathrm{e}^{r+\mathrm{i}\theta r-\theta^2 r/2+\cdots}$$

we find

$$\frac{\mathrm{d}S}{\mathrm{d}\bar{\theta}} \sim \delta\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\delta\bar{\theta}}\sqrt{2\pi}\,r^{1/2}}{\epsilon^{1/2}}\frac{\mathrm{e}^{-r/\epsilon}\mathrm{e}^{r\mathrm{e}^{\mathrm{i}\delta\bar{\theta}}/\epsilon}}{\mathrm{e}^{\mathrm{i}\delta\bar{\theta}(r/\epsilon+\alpha)}} \sim \frac{\mathrm{i}\sqrt{2\pi}\,\delta r^{1/2}}{\epsilon^{1/2}}\mathrm{e}^{-\delta^2\bar{\theta}^2 r/2\epsilon}.$$

Thus we see that the right scaling is $\delta = \epsilon^{1/2}$, and

$$\frac{\mathrm{d}S}{\mathrm{d}\bar{\theta}} \sim \mathrm{i}\sqrt{2\pi}\,r^{1/2}\mathrm{e}^{-\bar{\theta}^2 r/2}.$$

The matching condition we have is $S \to 0$ as $\bar{\theta} \to -\infty$. Hence

$$S = \mathrm{i}\sqrt{2\pi}\int_{-\infty}^{r^{1/2}\bar{\theta}} \mathrm{e}^{-t^2/2}\,\mathrm{d}t.$$

As $\bar{\theta} \to +\infty$,

$$S \to \mathrm{i}\sqrt{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\,\mathsf{d}t = 2\pi\mathrm{i}.$$

Matching this inner boundary-layer expansion with the outer expansion away from the boundary layer gives the jump in S across the Stokes line as

$$[S]_{\theta=0-}^{\theta=0+} = 2\pi\mathrm{i}.$$

Thus we have seen explicitly the turning on of $2\pi\mathrm{i}\mathrm{e}^{-z/\epsilon}$ as we cross the Stokes line.
Error function smoothing of the Stokes discontinuity (Berry 89).

1. On the Stokes line successive terms in the asymptotic expansion have the same phase, i.e. the ratio of successive terms is **real and positive** (here z is real and positive) (Dingle).

2. On the Stokes line the phase of the dominant and subdominant exponentials are equal, so that the dominant exponential is maximally dominant (here $\mathrm{e}^{z/\epsilon}$ and $1 = \mathrm{e}^{0/\epsilon}$ have the same phase when z is real and positive).

3. To find R_N and the Stokes multiplier we only need to know f_n as $n \to \infty$.

Formal procedure

- Formally expand as $\phi \sim \sum_n \epsilon^n \phi_n$.
- Apply a factorial/power ansatz to the late terms

$$\phi_n \sim \frac{A\Gamma(n+\gamma)}{\chi^{n+\gamma}} \text{ as } n \to \infty.$$

- Singulant χ vanishes at one of the singularities of ϕ_0.
- Unknown constants determined by matching with an inner region near each singularity.
- Stokes lines where χ is real and positive, across which

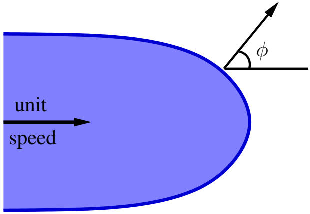
$$2\pi\mathrm{i}\epsilon^{-\gamma} A\mathrm{e}^{-\chi/\epsilon}$$

is turned on.

Nonlinear eigenvalue problems

Geometric model for crystal growth (Kruskal & Segur).

Equation of motion: $v_n = \kappa + \epsilon^2 \kappa''$ gives

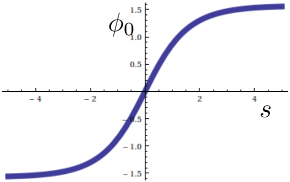


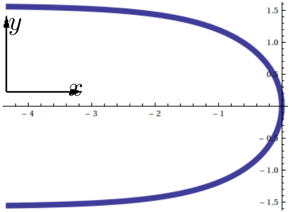
$$\epsilon^2 \phi''' + \phi' = \cos\phi$$

$$\phi \to \pm\pi/2 \quad \text{as } s \to \pm\infty.$$

For $\epsilon = 0$, the solution is $$\phi = \phi_0 = -\pi/2 + 2\tan^{-1} \mathrm{e}^s.$$

An asymptotic expansion $\phi \sim \sum_{n=0}^{\infty} \epsilon^{2n} \phi_n$ will satisfy the equation and boundary conditions at every order.





However, if we linearize about $-\pi/2$ as $s \to -\infty$, by setting $\phi \sim -\pi/2 + f$, then

$$\epsilon^2 f''' + f' = f$$

giving

$$f \sim \alpha \mathrm{e}^s + \beta \mathrm{e}^{-s/2+\mathrm{i}s/\epsilon} + \gamma \mathrm{e}^{-s/2-\mathrm{i}s/\epsilon} \text{ as } \epsilon \to 0,$$

for constants α, β and γ. Only the 1st term decays as $s \to -\infty$.
$\phi \to -\pi/2$ as $s \to -\infty$ is 2 boundary conditions ($\beta = \gamma = 0$).
$\phi \to \pi/2$ as $s \to \infty$ is also 2 boundary conditions.
This gives 4 boundary conditions on a 3rd order equation (infact we also have translational invariance).
It can be shown that no solution exists (Amick & McLeod).

If a solution were to exist it would be antisymmetric. Solving the equation subject to

$$\begin{aligned} \phi &\to -\pi/2 \text{ as } s \to -\infty, \\ \phi(0) &= 0, \end{aligned}$$

(3 boundary conditions - correctly specified) we will obtain a solution to the original problem if $\phi''(0)$ turns out to be zero. In fact, we will see that

$$\phi''(0) \sim C\epsilon^{-5/2}e^{-\pi/2\epsilon} \text{ as } \epsilon \to 0.$$

This explains the inability of the algebraic expansion in ϵ to detect that there is no solution.

$$\boxed{\epsilon^2\phi''' + \phi' = F(\phi), \qquad \phi \to \phi_\pm \quad \text{as } s \to \pm\infty}$$

Expand in powers of ϵ:
$$\phi \sim \sum_{n=0}^{\infty} \epsilon^{2n}\phi_n,$$

Leading order:
$$\phi_0' = F(\phi_0), \qquad \phi_0 \to \phi_\pm \quad \text{as } s \to \pm\infty,$$

First order:
$$\phi_0''' + \phi_1' = \phi_1 F'(\phi_0), \qquad \phi_1 \to 0 \quad \text{as } s \to \pm\infty,$$

In general:
$$\phi_{n-1}''' + \phi_n' = \phi_n F'(\phi_0) + (\phi_1\phi_{n-1} + \cdots)F''(\phi_0) + \cdots$$
$$\phi_n \to 0 \quad \text{as } s \to \pm\infty.$$

To approximate ϕ to exponential accuracy we wish to truncate optimally.
We need to know how ϕ_n behaves as $n \to \infty$.

Singular points of ϕ_n are same as those of ϕ_{n-1}.

But s^{-p} in ϕ_{n-1} becomes $p(p+1)s^{-(p+2)}$ in ϕ_n.

Hence we expect factorial/power divergence whenever ϕ_0 has (complex) singularities.

Factorial/power ansatz for large n: $\quad \phi_n \sim \dfrac{A\Gamma(2n+\gamma)}{\chi^{2n+\gamma}} \quad$ as $n \to \infty$.

This is very similar to a WKB ansatz. The singulant χ and the amplitude A are functions of s but are independent of n. The offset γ is a constant.

$$\underbrace{\phi'''_{n-1}}_{\Gamma(2n+\gamma+1)} + \underbrace{\phi'_n}_{\Gamma(2n+\gamma+1)} = \underbrace{\phi_n F'(\phi_0)}_{\Gamma(2n+\gamma)} + \underbrace{(\phi_1\phi_{n-1} + \cdots)F''(\phi_0)}_{\Gamma(2n+\gamma-2)} + \cdots$$

$$\phi_n \sim \frac{A\Gamma(2n+\gamma)}{\chi^{2n+\gamma}},$$

$$\phi_n' \sim -\frac{A\Gamma(2n+\gamma+1)\chi'}{\chi^{2n+\gamma+1}} + \frac{\Gamma(2n+\gamma)}{\chi^{2n+\gamma}}A' + \cdots,$$

$$\phi_{n-1}''' \sim -\frac{A\Gamma(2n+\gamma+1)(\chi')^3}{\chi^{2n+\gamma+1}} + \frac{3\Gamma(2n+\gamma)}{\chi^{2n+\gamma}}(A'(\chi')^2 + A\chi'\chi'') + \cdots$$

$$\phi_{n-1}''' + \phi_n' = \phi_n F'(\phi_0) + (\phi_1\phi_{n-1} + \cdots)F''(\phi_0) + \cdots$$

Leading order: $\quad (\chi')^3 + \chi' = 0 \quad \Rightarrow \quad \chi = \pm\mathrm{i}.$

Changing $\chi \to -\chi$ simply changes $A \to A(-1)^{-\gamma}$ so without loss of generality we can choose $\chi' = \mathrm{i}$.

Now ϕ_n has the same singular points as ϕ_0 (call these σ_k, $k = 1, 2, \ldots$).

Thus $\chi(\sigma_k) = 0$ for some k, so that $\chi = \mathrm{i}(s - \sigma_k)$.

In general there will be one factorial/power for each singularity σ_k.

For large n those with the smallest values of $|s - \sigma_k|$ will dominate, i.e. the singularities closest to the real axis.

$$
\begin{aligned}
\phi_n &\sim \frac{A\Gamma(2n+\gamma)}{\chi^{2n+\gamma}},\\
\phi_n' &\sim -\frac{A\Gamma(2n+\gamma+1)\chi'}{\chi^{2n+\gamma+1}} + \frac{\Gamma(2n+\gamma)}{\chi^{2n+\gamma}}A' + \cdots,\\
\phi_{n-1}''' &\sim -\frac{A\Gamma(2n+\gamma+1)(\chi')^3}{\chi^{2n+\gamma+1}} + \frac{3\Gamma(2n+\gamma)}{\chi^{2n+\gamma}}(A'(\chi')^2 + A\chi'\chi'') + \cdots
\end{aligned}
$$

$$
\phi_{n-1}''' + \phi_n' = \phi_n F'(\phi_0) + (\phi_1\phi_{n-1} + \cdots)F''(\phi_0) + \cdots
$$

Next order:

$$
3A'(\chi')^2 + A\chi'\chi'' + A' = AF'(\phi_0) \quad \Rightarrow \quad -2A' = AF'(\phi_0) = \frac{A\phi_0''}{\phi_0'},
$$

since $\phi_0' = F(\phi_0)$. Thus

$$
\log A = \int \frac{A'}{A}\,\mathrm{d}s = -\int \frac{\phi_0''\,\mathrm{d}s}{2\phi_0'} = -\frac{1}{2}\log(\phi_0') + \text{constant}.
$$

Hence

$$
A = \Lambda(\phi_0')^{-1/2}.
$$

Hence

$$\phi_n \sim \sum_k \frac{\Lambda_k \Gamma(2n+\gamma_k)}{(\mathrm{i}(s-\sigma_k))^{2n+\gamma_k}(\phi_0')^{1/2}} \quad \text{as } n \to \infty.$$

To determine γ_k, we make the singularity in ϕ_n consistent with that in ϕ_0. Suppose

$$\phi_0 \sim B(s-\sigma_k)^{\beta_k} \quad \text{as } s \to \sigma_k.$$

Then, putting $n = 0$ in our large-n expression, we need

$$(s-\sigma_k)^{\beta_k} \sim \frac{1}{(s-\sigma_k)^{\gamma_k}(s-\sigma_k)^{\frac{\beta_k-1}{2}}},$$

i.e.

$$\gamma_k = \frac{1}{2} - \frac{3\beta_k}{2}.$$

Only unknown is Λ_k. Return to this later.

We treat the factorial/power generated by each singularity separately. We expect there to be Stokes lines when ϕ_n and ϕ_{n+1} have the same phase, i.e. χ^2 is real and positive. Truncate after N terms:

$$\phi = \sum_{n=0}^{N-1} \epsilon^{2n}\phi_n + R_N.$$

The equation for the remainder is

$$\begin{aligned} \epsilon^2 R_N''' + R_N' - F'(\phi_0)R_N &= -\epsilon^{2N}\phi_{N-1}''' + \cdots \\ &\sim \epsilon^{2N}\phi_N' + \cdots \end{aligned}$$

as $\epsilon \to 0$, $N \to \infty$.
The right-hand side is dominated by the first term omitted: the equation for the next term would be

$$\phi_N' - F'(\phi_0)\phi_N = -\phi_{N-1}''' + \cdots .$$

Since we didn't include this term the right-hand side of this equation is the leading term in the remainder equation.

Homogeneous version of the remainder equation has solutions

$$R_N \sim \frac{\mathrm{e}^{-\mathrm{i}s/\epsilon}}{(\phi_0')^{1/2}}, \quad R_N \sim \frac{\mathrm{e}^{\mathrm{i}s/\epsilon}}{(\phi_0')^{1/2}}, \quad R_N \sim \phi_0',$$

as $\epsilon \to 0$ (WKB approximation). A multiple of the first is switched on across the Stokes line. Follow the same steps as before:

Switch to polar representation: $\quad \mathrm{i}(s-\sigma) = r\mathrm{e}^{\mathrm{i}\theta}$

Optimal truncation point is $\quad N \sim r/2\epsilon.$

Let $\quad N = r/2\epsilon + \alpha,$

Define Stokes multiplier S by

$$R_N = \frac{S\mathrm{e}^{-\mathrm{i}(s-\sigma)/\epsilon}}{(\phi_0')^{1/2}}.$$

Write

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\mathrm{e}^{-\mathrm{i}\theta}}{r}\frac{\mathrm{d}}{\mathrm{d}\theta}.$$

Use

$$\phi_N \sim \frac{\Lambda\Gamma(2N+\gamma)}{(\mathrm{i}(s-\sigma))^{2N+\gamma}(\phi_0')^{1/2}}$$

and Stirling to approximate the right-hand side $\epsilon^{2N}\phi_N'$.

Together this gives

$$-\frac{2\mathrm{e}^{-\mathrm{i}\theta}}{r}\mathrm{e}^{-r\mathrm{e}^{\mathrm{i}\theta}/\epsilon}\frac{\mathrm{d}S}{\mathrm{d}\theta} \sim -\frac{\mathrm{i}\Lambda\sqrt{2\pi}}{\sqrt{r}\,\epsilon^{\gamma+1/2}}\mathrm{e}^{-r/\epsilon}\mathrm{e}^{-\mathrm{i}\theta(r/\epsilon+2\alpha+\gamma+1)}.$$

Thus $\mathrm{d}S/\mathrm{d}\theta$ is exponentially small except at $\theta = 0$. Introducing the local coordinate $\theta = \delta\bar{\theta}$ gives

$$-\frac{2\mathrm{e}^{r\delta^2\bar{\theta}^2/2\epsilon}}{r\delta}\,\frac{\mathrm{d}S}{\mathrm{d}\bar{\theta}} \sim -\frac{\mathrm{i}\Lambda\sqrt{2\pi}}{\sqrt{r}\epsilon^{\gamma+1/2}}.$$

Correct inner scalings are $\delta = \sqrt{\epsilon}$, $S = \bar{S}/\epsilon^{\gamma}$ giving

$$\frac{\mathrm{d}\bar{S}}{\mathrm{d}\bar{\theta}} \sim \frac{\mathrm{i}\Lambda\sqrt{\pi r}}{\sqrt{2}}e^{-r\bar{\theta}^2/2} \quad \Rightarrow \quad \bar{S} \sim a + \frac{\mathrm{i}\Lambda\sqrt{\pi}}{\sqrt{2}}\int_{-\infty}^{\bar{\theta}\sqrt{r}} \mathrm{e}^{-t^2/2}\,\mathrm{d}t.$$

Hence for the outer solution (i.e. for $|\theta|$ order one) there is a jump in the coefficient S as we cross the Stokes line given by

$$[S]_{\theta=0-}^{\theta=0+} = \frac{\Lambda\pi\mathrm{i}}{\epsilon^{\gamma}}.$$

Let us now consider the implications of this jump for various forms of F.

Kruskal-Segur $F(\phi) = \cos(\phi)$

$\phi_0 = -\pi/2 + 2\tan^{-1} \mathrm{e}^s$ has logarithmic singularities at $s = \pm \mathrm{i}\pi/2$. Thus $\beta = 0$, $\gamma = 1/2$, $\phi_0' = \mathrm{sech} s$, and the late terms are

$$\phi_n \sim \frac{\Lambda(\cosh s)^{1/2}\Gamma(2n+1/2)}{(\mathrm{i}(s-\mathrm{i}\pi/2))^{2n+1/2}} + \frac{\Lambda^*(\cosh s)^{1/2}\Gamma(2n+1/2)}{(-\mathrm{i}(s+\mathrm{i}\pi/2))^{2n+1/2}}$$

There is a Stokes line down the imaginary axis from $s = \mathrm{i}\pi/2$ across which

$$\frac{\Lambda\pi\mathrm{i}(\cosh s)^{1/2}\mathrm{e}^{-\pi/2\epsilon}\mathrm{e}^{-\mathrm{i}s/\epsilon}}{\epsilon^{1/2}}$$

is turned on. Similarly there is a Stokes line up the imaginary axis from $s = -\mathrm{i}\pi/2$ across which

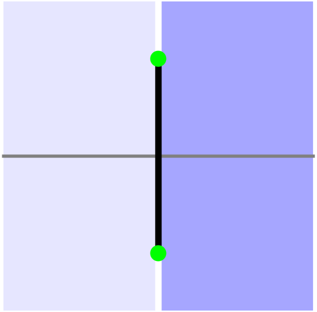
$$-\frac{\Lambda^*\pi\mathrm{i}(\cosh s)^{1/2}\mathrm{e}^{-\pi/2\epsilon}\mathrm{e}^{\mathrm{i}s/\epsilon}}{\epsilon^{1/2}}$$

is turned on.

On the real axis the two exponentials are comparable, so that across $s = 0$

$$\frac{2R\pi(\cosh s)^{1/2}}{\epsilon^{1/2}} \mathrm{e}^{-\pi/2\epsilon} \sin\left(\frac{s}{\epsilon} - \Phi\right)$$

is turned on, where $\Lambda = R\mathrm{e}^{\mathrm{i}\Phi}$.



This function is growing as $s \to \infty$, so the naive expansion ceases to be valid for $s = O(1/\epsilon)$ [where we cross the anti-Stokes line].

Suppose we impose $\phi(0) = 0$. Then, evaluating

$$\frac{2R\pi(\cosh s)^{1/2}}{\epsilon^{1/2}} \mathrm{e}^{-\pi/2\epsilon} \sin\left(\frac{s}{\epsilon} - \Phi\right)$$

at $s = 0$ gives

$$-\frac{R\pi e^{-\pi/2\epsilon}\sin(\Phi)}{\sqrt{\epsilon}}$$

(factor of 1/2 since we are exactly on the Stokes line).
To satisfy $\phi(0) = 0$ we must add

$$\frac{R\pi\phi_0'(s)e^{-\pi/2\epsilon}\sin(\Phi)}{\sqrt{\epsilon}}$$

to the optimally truncated expansion (remember this is a solution of the homogeneous version of the equation for R_N, and that $\phi_0'(0) = 1$). Now evaluating the second derivative gives

$$\phi''(0) \sim \frac{R\pi e^{-\pi/2\epsilon}\sin(\Phi)}{\epsilon^{5/2}}.$$

Kuramoto-Sivashinsky $F(\phi) = 1 - \phi^2$

Leading-order solution is $\phi_0 = \tanh s$, which has simple poles at $s = \pm \mathrm{i}\pi/2$. Thus $\beta = -1$, $\gamma = 2$, and the late terms are

$$\phi_n \sim \frac{\Lambda \cosh s\, \Gamma(2n+2)}{(\mathrm{i}(s - \mathrm{i}\pi/2))^{2n+2}} + \frac{\Lambda^* \cosh s\, \Gamma(2n+2)}{(\mathrm{i}(s + \mathrm{i}\pi/2))^{2n+2}}$$

There is a Stokes line down the imaginary axis from $s = \mathrm{i}\pi/2$ across which

$$\frac{\Lambda \pi \mathrm{i} \cosh s\, \mathrm{e}^{-\pi/2\epsilon} \mathrm{e}^{-\mathrm{i}s/\epsilon}}{\epsilon^2}$$

is turned on. Similarly, there is a Stokes line up the imaginary axis from $s = -\mathrm{i}\pi/2$ across which

$$-\frac{\Lambda^* \pi \mathrm{i} \cosh s\, \mathrm{e}^{-\pi/2\epsilon} \mathrm{e}^{\mathrm{i}s/\epsilon}}{\epsilon^2}$$

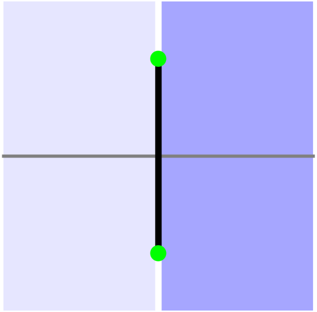
is turned on.

Kuramoto-Sivashinsky $F(\phi) = 1 - \phi^2$

On the real axis the two exponentials are comparable, so that across $s = 0$

$$\frac{2\pi R \cosh s}{\epsilon^2} \mathrm{e}^{-\pi/2\epsilon} \sin\left(\frac{s}{\epsilon} - \Phi\right)$$

is turned on, where $\Lambda = R\mathrm{e}^{\mathrm{i}\Phi}$.



Again, this function is growing as $s \to \infty$, so the naive expansion ceases to be valid for $s = O(1/\epsilon)$.

Anisotropic crystal growth $F(\phi) = \frac{\cos\phi}{1+\alpha\cos 4\phi}$

The leading-order solution is given implicitly by

$$s = (1+\alpha)\log(\sec\phi_0 + \tan\phi_0) - \frac{8}{3}\alpha(\sin\phi_0)^3.$$

Four square root singularities, one in each quadrant, at σ, $\bar{\sigma}$, $-\sigma$, $-\bar{\sigma}$.

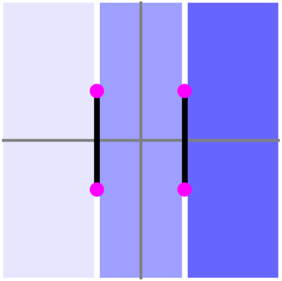
There are two Stokes lines across which

$$\frac{2\pi R\epsilon^{1/4}\mathrm{e}^{-\mathrm{Im}(\sigma)/\epsilon}}{(\phi_0')^{1/2}}\sin\left(\frac{s+\mathrm{Re}(\sigma)}{\epsilon}+\Phi\right)$$

is switched on at $s = -\mathrm{Re}(\sigma)$ and

$$\frac{2\pi R\epsilon^{1/4}\mathrm{e}^{-\mathrm{Im}(\sigma)/\epsilon}}{(\phi_0')^{1/2}}\sin\left(\frac{s-\mathrm{Re}(\sigma)}{\epsilon}-\Phi\right)$$

is switched on at $s = \mathrm{Re}(\sigma)$.



To get a solution to the BVP these must exactly cancel for $s > \mathrm{Re}(\sigma)$, i.e.

$$\sin\left(\frac{\mathrm{Re}(\sigma)}{\epsilon}+\Phi\right) = 0 \quad \Rightarrow \quad \epsilon = \epsilon_n = \frac{\mathrm{Re}(\sigma)}{n\pi - \Phi}.$$

Determining the constant Λ

Kuramoto-Sivashinksy equation $F(\phi) = 1 - \phi^2$.

$$\phi_0 = \tanh s \qquad \phi_0 \sim \frac{1}{s - \frac{\mathrm{i}\pi}{2}} \text{ as } s \to \frac{\mathrm{i}\pi}{2}.$$

$$\phi_n \sim \frac{\Lambda\Gamma(2n+2)\cosh s}{(\mathrm{i}(s - \frac{\mathrm{i}\pi}{2}))^{2n+2}} \sim \frac{\Lambda\Gamma(2n+2)}{(\mathrm{i}(s - \frac{\mathrm{i}\pi}{2}))^{2n+1}} \text{ as } s \to \frac{\mathrm{i}\pi}{2}$$

Inner region near $s = \mathrm{i}\pi/2$.

$$s = \frac{\mathrm{i}\pi}{2} + \epsilon\zeta, \quad \phi = \frac{\psi}{\epsilon} \qquad \Rightarrow \qquad \psi''' + \psi' = \epsilon^2 - \psi^2$$

Expand $\psi = \psi_0 + \epsilon^2\psi_1 + \cdots \qquad \Rightarrow \qquad \psi_0''' + \psi_0' = -\psi_0^2.$

$$\psi_0 \sim \sum_{n=0}^{\infty} \frac{A_n}{\zeta^{2n+1}}, \qquad A_0 = 1, \quad A_1 = -6,$$

$$A_n = -(2n+1)2nA_{n-1} + \frac{1}{2n-1}\sum_{m=1}^{n-1} A_{n-m}A_m, \quad n \geq 2.$$

Determining the constant Λ

Inner limit of the outer expansion:

$$\phi_n \sim \frac{\Lambda\Gamma(2n+2)\epsilon^{2n}}{(\mathrm{i}(s-\frac{\mathrm{i}\pi}{2}))^{2n+1}} = -\frac{\mathrm{i}\Lambda\Gamma(2n+2)}{(-1)^n\epsilon\zeta^{2n+1}}.$$

Outer limit of the inner expansion:

$$\phi = \frac{\psi}{\epsilon} = \frac{1}{\epsilon}\sum_{n=0}^{\infty}\frac{A_n}{\zeta^{2n+1}}.$$

Matching

$$\Lambda = \lim_{n\to\infty}\frac{\mathrm{i}(-1)^n A_n}{\Gamma(2n+2)}.$$

Formal procedure

- Formally expand as $\phi \sim \sum_n \epsilon^n \phi_n$.
- Apply a factorial/power ansatz to the late terms

$$\phi_n \sim \frac{A\Gamma(n+\gamma)}{\chi^{n+\gamma}} \text{ as } n \to \infty.$$

- Singulant χ vanishes at one of the singularities of ϕ_0.
- Unknown constants determined by matching with an inner region near each singularity.
- Stokes lines where χ is real and positive, across which

$$2\pi \mathrm{i} \epsilon^{-\gamma} A \mathrm{e}^{-\chi/\epsilon}$$

is turned on.