Tate systems & the BSD conjecture

I. Background

Conjecture (BSD, '65): E/k elliptic curve, L(E,s) L-function

\[ \text{rank } E(k) = \text{ord}_{s=1} L(E, s) \]

\text{analytic rank}

Generalisations:
- for A/k abelian variety (e.g. Jacobians of alg var/k)
- abelian var/number fields

Known results for E/k:
- Coates-Wiles, '77: E complex mult, 
  \[ L(E,1) \neq 0 \]
- Kolyvagin '89, Kato '94: E (modular)
  elliptic curve, \[ \text{ord}_{s=1} L(E, s) = 0, 1 \]

*Bartoli-H-Darmon-Rotger, '15:
  equivariant BSD in analytic rk 0,
  E/k twisted by 2-dim' odd Artin rep

Remark: E/k, analytic rank \geq 1
today: case of ab. surfaces/elliptic curves
over number fields

I. Kato's strategy in analytic rk 0

Def: \( p \)-adic rep of \( G_k = \text{Gal}(\bar{k}/k) \) =
fin-dim \( \mathbb{Q}_p \)-vector space \( V \) with ct.
action of \( G_k \), invariant outside finite set
of primes
\[ L(V,s) \] conject. to have numer/aux.
cont' to \( L \)
\( \exists x : V = C_f \quad L(V,s) = \zeta(s) \)

\( \forall x : g = C_f \quad L(V,s) = L(E,s) \quad L(V,s) = L(E,s+1) \)

\( V = \text{modular form, } \omega \geq 2 \)

- \( \text{p-adic } \mathbb{Q}_p^* \to \text{Sel}(E,V) \subset H^1(C_f,V) \)

- \( \text{Ex: } E(\mathbb{Q}) \otimes \mathbb{Q}_p \to \text{Sel}(E,V) \quad (*) \)

- \( \text{Similar for abelian varieties} \)

Conjecture (Beilinson-Kato):

\[ \text{ord}_{s=1} L(V,s) = \dim_{\mathbb{Q}_p} \text{Sel}(E,V) - \dim_{\mathbb{Q}_p} V^* \]

- Usually 0

Kato: proved BK Conjecture for \( V \in E \) when \( L(E,1) \neq 0 \)

\( \Rightarrow \text{hyp } (*) \quad \text{This implies BSD when analytic } \text{rk } = 0. \)

Main tool: \( \text{Euler system} \) (discovered by Kolyvagin)

Def (Kato): ES for \( V = \) collection of cohomology classes \((Z_n)_{n \geq 1}, Z_n \in H^1(C_f/E_n), V)\)

which satisfy the so-called ES norm relations as \( n \) varies

\( \text{Euler system modules } \) (Kato, Rubin):

\[ \text{If } Z_n \neq 0, \quad \text{then } \dim H^1(\mathbb{Q}_p/V_n) = 0 \]

Main steps in Kato's proof:

(i) Construct ES \((Z_n)\) for \( V = V_f \)

(ii) Evaluation \( x_1 \) to \( L(E,1) \) (explicit reciprocity law)

Essential ingredient. \( E \) is modular

\[ E \leftrightarrow \text{mod. form of weight } \]

\[ \Rightarrow V_f \otimes \text{is a direct summand of } \]

\[ H^1(Y_{\mathcal{O}_E}) \quad \text{of modular curve} \]
III. The case of abelian varieties

Paraniodular conjecture (Brumer-Kramer):

\{ abelian surface \} \leftrightarrow \{ certain genus 2 \\
\text{of general type} \} \leftrightarrow \{ \text{Siegel modular forms}, \text{w} (2,2) \}

\[ V_{\mathbb{F}} \leftrightarrow V_{\mathbb{F}_{\ell}, \text{smooth}} \]

Genus 2 Siegel \( K \)-: analogue of \( K \)- for \( g \) \text{APq}
\( \Rightarrow \) global sections of certain line/vector bundles on \( \text{Siegel} \) 3-folds

Remarks:
* Bajo-Klosin: proved conj. for \( F \) of Saito-Kurokawa type
* Klaar-Thorne: many cases of conj. for \( E/k \), \( K \) imag. quad.
* Boxto: Calegari-fee-Pilloni: potential modularity

Problem: even assuming paraniodular conj., \( V_{\mathbb{F}} \) is \textbf{not} a direct summand of \( H_{\mu}^0(S) \)

5 Sierpinski 3-fold!

But: \( V_{\mathbb{F}} \otimes V_{\mathbb{F}_{\ell}, \text{smooth}} \) is the \( \ell \)-adic limit

\[ \text{of } V_{\mathbb{F}_{\ell}, \text{smooth}} \]

\( \Rightarrow \) do appear in \( H_{\mu}^0(S) \)

11. Problem: \( (2,2) \) is too small
\( \Rightarrow \) \( w \) \{ \text{modular forms} \}

Strategy for constructing ES for \( V_{\mathbb{F}} \):
1. Use group methods to construct ES \( (\mathbb{Z}_\ell)^m \)
   for \( V_{\mathbb{F}_{\ell}, \text{smooth}} \), \( F \) \text{w} \( (k_1,k_2), k_1 > k_2 > 3 \)

2. Use \( \ell \)-adic limiting process to obtain ES \( (\mathbb{Z}_\ell)^n \) for \( V_{\mathbb{F}} \)

Thus (Zaimi, 3, \text{11}): (1)
(11) in progress (Z)

Problem: to get at $\text{BSD}$, need tell

\[ Z_{\mathbb{F}}^{(n)} \leftrightarrow L(A,1) \]
Solve: use $p$-adic variation

(1) prove $\mathcal{L}^{m}$ of $Z_{k}^{*}$, if $k \geq 2$, $k \geq 3$, to values of $L_{p, \text{spin}}(3,5)$

(2) use $p$-adic limiting process to deduce $Z_{l}^{A} \leftarrow \mathcal{L}(A,1)$

Problem: need a relation (Euler systems) $\leftrightarrow$ (values of $L$-functions)
which deforms $p$-adically

Right tool: $p$-adic $L$-function.


Let $F^{k}(L, k, k)$ be a Coleman family through $F^{k}(5,5) = F^{5(k)}$, $k$ odd.

Then $F$ is $p$-adic $L$-function.

Lemma: from above, $p$-adic deformation, get

$Z_{l}^{(k)} \leftarrow \mathcal{L}(A,1)$

$\Rightarrow$ deduce, if $L(A,1) \neq 0$, then $Z_{l}^{(k)} 
eq 0$

Thus $Z_{l}^{(k)}$ above, $Z_{l}^{(k)} \neq 0$

$\Rightarrow$ (ES moduli) $Sel(8, V_{4}) = 0$

$\Rightarrow$ rank $A(k) = 0$, since $A(8) \not\in Sel$. □
Remarks: (1) if $A$ arises from elliptic curve over $\mathbb{Q}$, then smoothness of eigenvariety is implied by conjecture of Calegari-Hajir.

(2) (in progress) BSD & elliptic analytic $rk 0$ for a twisted by $2$-dim odd Artin rep.

(3) $(L, (2))$: many cases of BK conjecture via analytic $rk 0$ for $V_{\text{spin}}(F)$, $F$ wt $(k_1, k_2)$, $k_1, k_2 \geq 3$. 