Case study

Where probability, geometry and analysis collide

Case Study: Random Geometry

We are all familiar with the Platonically perfect objects of ordinary geometry: circles, lines, spheres, planes, and so forth. But are there Platonically perfect ways to describe random geometric structures? In other words, is there a Platonically perfect probability measure on the set of loops, or the set of curves, or the set of surfaces, etc.? And if so, can one show that these perfect probability measures are limits of simple discrete probability measures (the way the perfect “bell shaped curve” is the limit of the probability measure describing the number of heads in \( n \) tosses of a fair coin)? How do these perfect probability measures somehow appear in the mathematical models inspired by string theory, statistical physics, finance, biology, and so forth? What interesting results and predictions does one obtain from this theory?

These were some of the questions posed by the organisers of the highly successful 2015 programme on Random Geometry (RGM). Working at the interface between probability, geometry and analysis, and with motivations coming from theoretical physics and in particular the work on Liouville quantum gravity, they brought together specialists in these and other connected areas, including the Fields medalist Wendelin Werner (ETH Zürich). They studied random geometries that appear random at every scale (fractal) alongside other types of random geometries that somehow “average out” at large scales.

As it turns out, many of the objects considered (random surfaces, random curves, random graphs, etc.) have very counterintuitive properties and hence such questions have driven the theory far beyond traditional boundaries, leading to the development of radically new concepts and intuitions. For example, a natural notion of random surface displays properties that are characteristic of manifolds in both two and four dimensions. A holy grail of this subject, and a key goal of the RGM programme, was the unification of the diverse perspectives on random planar geometry that have emerged from disciplines such as conformal field theory, Gaussian multiplicative...
chaos, combinatorics, random planar map theory, statistical physics and string theory. These approaches have led to a plethora of highly sophisticated and powerful tools which describe distinct aspects of what, at a different level, might be perceived as one and the same theory. Thanks to the RGM programme, and through a coordinated effort by many programme participants, much of this unification has been achieved. For instance:

- Programme Organiser Scott Sheffield (MIT) and participant Jason Miller (Cambridge) have recently announced a landmark result unifying Liouville quantum gravity and the Brownian map, two distinct models of measure-endowed random surfaces. Specifically, they proved that the Brownian Map and the $\sqrt{8/3}$Liouville quantum gravity sphere are equivalent: that is, they encode the same structure (a topological sphere with a measure, a metric, and a conformal structure) and have the same law.

- Several younger participants (including Benoist (MIT), Dumaz (Paris Dauphine), Gwynne (MIT), Holden (MIT), Laslier (Cambridge), Mao (MIT), Ray (British Columbia) and Sun (Columbia)) proved new results about discrete random surfaces decorated by trees or loops, while establishing precise relationships to continuum versions of these decorated surfaces.

- Participants Remi Rhodes (Université Paris-Est) and Vincent Vargas (ENS - Paris) explored higher genus random surfaces (a.k.a. surfaces with “handles”) and developed techniques that have since enabled them to rigorously derive some fundamental explicit constants from the theory of Liouville quantum gravity (including the celebrated DOZZ formula).

- Curien (Université Paris-Sud), Miermont (ENS - Lyon), Le Gall (Université Paris-Sud), Bettini (École polytechnique) and others have produced many new results on discrete random geometries and the metric space structure of their continuum analogs.

The programme, which included a School for Young Researchers, gave the high proportion of early career researchers the opportunity to interact with field leaders and to learn from them first hand and in a unified way about the state of the art of the subject. The impact of this was immediate and a team of more junior participants, Juhan Aru (ENS-Lyon), Yichao Huang (ENS-Paris) and Xin Sun (MIT), all PhD students at the time, rigorously proved during the course of the programme the equivalence of two existing random sphere constructions.

With much to capitalise on, in July 2018 a Follow-On Workshop to the programme, including a combination of intensive mini-courses and seminars, allowed high-level researchers at all careers stages and from across all disciplines involved, to catch up on developments and to plan future research directions for the subject. For example, Loève Prize winner Sourav Chatterjee (Stanford), who had participated in the original programme via video conference, came in person to present a mini-course on the Yang-Mills problem (one of the Clay Institute’s famous million dollar problems), and many of the researchers are actively exploring analogies between the string trajectories of Chatterjee’s theory and the analogous trajectories developed in Liouville quantum gravity. Many of the researchers who were just getting started (as students and postdocs) in 2015 had substantial achievements to present at the Follow-On Workshop. Wendelin Werner said of the programme, “The Random Geometry semester has had a transformative effect on this research area. Many research directions of today find their roots in the interactions and discussions that took place then. One big success has been the way in which it enabled the younger participants (PhD students and postdocs) to define their own research agendas. At the follow up event it was truly remarkable to see the talks and witness the achievements of this impressive group.”